Multi-componets Data, Signal and Image Processing for Biological and Medical Applications

Ali Mohammad-Djafari

Laboratoire des Signaux et Systèmes
UMR 8506 CNRS - CS - Univ Paris Sud
CentraleSupélec, Gif-sur-Yvette.

djafari@lss.supelec.fr
http://djafari.free.fr

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Summary 2.1: Time series

- Time serie and Fourier representation
  - Continuous / Discrete
  - Correlation, Inter-correlation, Inter-dependance
  - Stationarity / non-stationarity
  - Convolution and Deconvolution

- Filtering and Denoising

- Modelling and Prediction

- Parametric and Non parametric models

- Parametric models:
  - Least Squares
  - Maximum Likehood
  - Bayesian estimation
Summary 2.2: Images

- Continuous / Discrete
- Gray and Color images
  - 2D FT and FFT
  - 2D Correlation and inter-correlation
  - Stationarity / non-stationarity
  - 2D Convolution
- Filtering and Denoising
- Modelling and Prediction
- Simple Markovian models
- Contours and Regions
- Hierarchical Markov models
1D signals: Time series

- 1D Signal: Time series: \( x_i = f(t_i) \)
- In general no exchangeable.
- Time representation \( f(t) \), Fourier Transform and Fourier representation \( F(\nu) \), Auto Correlation function \( R(\tau) \), Power Spectral Density \( S(\nu) \)
- Stationary and non stationary
- STFT, Time-Frequency, Time-Scale, Wavelets, ...
- Smoothing, Noise removing, Filtering
- Periodic signals, estimation of the period, Fourier series
- Modeling:
  - Sum of sinusoids model and parameter estimation
  - Moving average (MA) model
  - Autoregressive (AR) model
Representation of signals and images

- **Signal:** $f(t), f(x), f(\nu)$
  - $f(t)$ Variation of temperature in a given position as a function of time $t$
  - $f(x)$ Variation of temperature as a function of the position $x$ on a line
  - $f(\nu)$ Variation of temperature as a function of the frequency $\nu$

- **Image:** $f(x, y), f(x, t), f(\nu, t), f(\nu_1, \nu_2)$
  - $f(x, y)$ Distribution of temperature as a function of the position $(x, y)$
  - $f(x, t)$ Variation of temperature as a function of $x$ and $t$
  - ...

- **3D, 3D+t, 3D+\nu, ... signals:** $f(x, y, z), f(x, y, t), f(x, y, z, t)$
  - $f(x, y, z)$ Distribution of temperature as a function of the position $(x, y, z)$
  - $f(x, y, z, t)$ Variation of temperature as a function of $(x, y, z)$ and $t$
  - ...

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Representation of signals

1D signal

2D signal = image

3D signal
Linear Transformations

\[ g(s) = \int_D f(r) h(r, s) \, dr \]

\[ f(r) \rightarrow \boxed{h(r, s)} \rightarrow g(s) \]

- **1–D**: 
  \[ g(t) = \int_D f(t') h(t, t') \, dt' \]
  \[ g(x) = \int_D f(x') h(x, x') \, dx' \]

- **2–D**: 
  \[ g(x, y) = \iint_D f(x', y') h(x, y; x', y') \, dx' \, dy' \]
  \[ g(r, \phi) = \iint_D f(x, y) h(x, y; r, \phi) \, dx \, dy \]
Linear and Invariant systems: convolution

\[ h(r, r') = h(r - r') \]

\[ f(r) \rightarrow [h(r)] \rightarrow g(r) = h(r) \ast f(r) \]

- 1–D:
  \[ g(t) = \int_D f(t') h(t - t') \, dt' \]
  \[ g(x) = \int_D f(x') h(x - x') \, dx' \]

- 2–D:
  \[ g(x, y) = \iint_D f(x, y) h(x - x', y - y') \, dx' \, dy' \]

- \( h(t) \) impulse response
- \( h(x, y) \) Point Spread Function
Linear Transformations: Separable systems

\[ g(s) = \int_D f(r) h(r, s) \, dr \]

\[ h(r, s) = \prod_j h_j(r_j, s_j) \]

Examples:

- **2D Fourier Transform**

  \[ g(\omega_x, \omega_y) = \iint f(x, y) \exp \left[ -j(\omega_x x + \omega_y y) \right] \, dx \, dy \]

  \[ h(x, y, \omega_x, \omega_y) = h_1(\omega_x x) \, h_2(\omega_y y) \exp \left[ -j(\omega_x x + \omega_y y) \right] = \exp \left[ -j(\omega_x x) \right] \exp \left[ -j(\omega_y y) \right] \]

- **nD Fourier Transform**

  \[ g(\omega) = \int f(x) \exp \left[ -j\omega' x \right] \, dx \]
Fourier Transform

[Joseph Fourier, French Mathematicien (1768-1830)]

- **1D Fourier:** $\mathcal{F}_1$

\[
\begin{align*}
\begin{cases}
  g(\omega) &= \int f(t) \exp[-j\omega t] \, dt \\
  f(t) &= \frac{1}{2\pi} \int g(\omega) \exp[j\omega t] \, d\omega
\end{cases}
\end{align*}
\]

- **2D Fourier:** $\mathcal{F}_2$

\[
\begin{align*}
\begin{cases}
  g(\omega_x, \omega_y) &= \iint f(x, y) \exp[-j(\omega_x x + \omega_y y)] \, dx \, dy \\
  f(x, y) &= \left(\frac{1}{2\pi}\right)^2 \iint g(\omega_x, \omega_y) \exp[j(\omega_x x + \omega_y y)] \, d\omega_x \, d\omega_y
\end{cases}
\end{align*}
\]

- **nD Fourier:** $\mathcal{F}_n$

\[
\begin{align*}
\begin{cases}
  g(\omega) &= \int f(x) \exp[-j\omega' x] \, dx \\
  f(x) &= \left(\frac{1}{2\pi}\right)^n \int g(\omega) \exp[j\omega' x] \, d\omega
\end{cases}
\end{align*}
\]
1D Fourier Transform $\mathcal{F}_1$

\[
\begin{align*}
  g(\omega) &= \int f(t) \exp[-j\omega t] \, dt \\
  f(t) &= \frac{1}{2\pi} \int g(\omega) \exp[+j\omega t] \, d\omega
\end{align*}
\]

- $|g(\omega)|^2$ is called the spectrum of the signal $f(t)$
- For real valued signals $f(t)$, $|g(\omega)|$ is symmetric

Examples:

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$g(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\exp[-j\omega_0 t]$</td>
<td>$\delta(\omega - \omega_0)$</td>
</tr>
<tr>
<td>$\sin(\omega_0 t)$</td>
<td>?</td>
</tr>
<tr>
<td>$\cos(\omega_0 t)$</td>
<td>?</td>
</tr>
<tr>
<td>$\exp[-t^2]$</td>
<td>?</td>
</tr>
<tr>
<td>$\exp\left[-\frac{1}{2}(t - m)^2/\sigma^2\right]$</td>
<td>?</td>
</tr>
<tr>
<td>$\exp[-t/\tau], , t &gt; 0$</td>
<td>?</td>
</tr>
<tr>
<td>$1$ if $</td>
<td>t</td>
</tr>
</tbody>
</table>
2D Fourier Transform: $\mathcal{F}_2$

\[
\begin{align*}
g(\omega_x, \omega_y) &= \iiint f(x, y) \exp \left[-j(\omega_x x + \omega_y y)\right] \, dx \, dy \\
f(x, y) &= \left(\frac{1}{2\pi}\right)^2 \iiint g(\omega_x, \omega_y) \exp \left[j(\omega_x x + \omega_y y)\right] \, d\omega_x \, d\omega_y
\end{align*}
\]

$|g(\omega_x, \omega_y)|^2$ is called the spectrum of the image $f(x, y)$

- For real valued image $f(x, y)$, $|g(\omega_x, \omega_y)|$ is symmetric with respect of the two axes $\omega_x$ and $\omega_y$.

**Examples:**

<table>
<thead>
<tr>
<th>$f(x, y)$</th>
<th>$g(\omega_x, \omega_y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\exp \left[-j(\omega_{x0} x + \omega_{y0} y)\right]$</td>
<td>$\delta(\omega_x - \omega_{x0})\delta(\omega_y - \omega_{y0})$</td>
</tr>
<tr>
<td>$\exp \left[-(x^2 + y^2)\right]$</td>
<td>?</td>
</tr>
<tr>
<td>$\exp \left[-\frac{1}{2}[(x - m_x)^2/\sigma_x^2 + (y - m_y)^2/\sigma_y^2]\right]$</td>
<td>?</td>
</tr>
<tr>
<td>$\exp \left[-(</td>
<td>x</td>
</tr>
<tr>
<td>$1 \text{ if }</td>
<td>x</td>
</tr>
<tr>
<td>$1 \text{ if } (x^2 + y^2) &lt; a$</td>
<td>?</td>
</tr>
</tbody>
</table>
**nD Fourier Transform: \( \mathcal{F}_n \)**

\[
\begin{align*}
g(\omega) &= \int f(x) \exp \left[-j\omega'x\right] \, dx \\
f(x) &= \left(\frac{1}{2\pi}\right)^n \int g(\omega) \exp \left[+j\omega'x\right] \, d\omega
\end{align*}
\]

- \(|g(\omega)|^2\) is called the spectrum of \(f(x)\)
- For real valued image \(f(x)\), \(|g(\omega)|\) is symmetric with respect of all the axis \(\omega_j\).

**Examples:**

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( g(\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exp \left[-j(\omega_0'x)\right] )</td>
<td>( \delta(\omega - \omega_0) )</td>
</tr>
<tr>
<td>( \exp \left[-x'x\right] = \exp \left[-</td>
<td></td>
</tr>
<tr>
<td>( \exp \left[-</td>
<td></td>
</tr>
<tr>
<td>( \exp \left[-\frac{1}{2}[(x - m)'\Sigma^{-1}(x - m)]\right] )</td>
<td>?</td>
</tr>
<tr>
<td>1 if (</td>
<td></td>
</tr>
<tr>
<td>1 if (</td>
<td>x_j</td>
</tr>
</tbody>
</table>
Hilbert Transform: \( \mathcal{H} \)

[David Hilbert, German mathematicien (1862-1943)]

- **Definition:** If \( f \in L_2 \) on \((-\infty, \infty)\),

\[
  g(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{t-x} \, dt
\]

\[
  f(t) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{g(x)}{x-t} \, dx
\]

The integrals are interpreted in the Cauchy principal value (CPV) sense at \( t = x \).

- **Alternate expression useful in signal processing:**

\[
  g(t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{f(t + \tau) - f(t - \tau)}{\tau} \, d\tau
\]
Hilbert Transform: $\mathcal{H}$

- If $f \in L_2$
  - $\mathcal{H}(\mathcal{H}(f)) = f$
  - $f$ and $\mathcal{H}(f)$ are orthogonal, i.e.,
    \[
    \lim_{r \to \infty} \int_{-r}^{r} [f \mathcal{H}(f)](u) \, du = 0
    \]

- The Hilbert transform of a constant is zero.

- Hilbert and Fourier Transforms
  \[
  \mathcal{H}(f) = f \ast \frac{-1}{\pi t} \quad \longrightarrow \quad \mathcal{F}\{\mathcal{H}(f)\} = j\text{sgn}(\omega)F(\omega)
  \]
Radon Transform (RT): $\mathcal{R}$

- **Definition:**
  This transform is defined for the functions in 2 or more dimensions. Here we give the relations only in the 2–D case.

$$
\begin{align*}
g(r, \phi) &= \int_{L_{r,\phi}} f(x, y) \, dl \\
&= \int \int f(x, y) \delta(r - x \cos(\phi) - y \sin(\phi)) \, dx \, dy
\end{align*}
$$

- The Radon transform maps the spatial domain $(x, y) \in \mathbb{R}^2$ to the domain $(r, \phi) \in \mathbb{R} \times [0, \pi]$. Each point in the $(r, \phi)$ space corresponds to a line in the spatial domain $(x, y)$.

- Note that $(r, \phi)$ are not the polar coordinates of $(x, y)$. In fact if we note the polar coordinates corresponding to the $(x, y)$ point $(\rho, \theta)$, then we have

$$
\begin{align*}
x &= \rho \cos \theta, \quad y = \rho \sin \theta, \quad r = \rho \cos(\phi - \theta)
\end{align*}
$$
X ray Tomography

\[
I = I_0 \exp \left[ - \int f(x) \, dx \right]
\]

\[
- \ln \frac{I}{I_0} = \int f(x, y) \, dl
\]

\[
g(r, \phi) = \int f(x, y) \, dl
\]

\[
f(x, y) \rightarrow \text{Radon Transform} \rightarrow g(r, \phi)
\]

\[
g(r, \phi) \rightarrow \text{Image Reconstruction} \rightarrow f(x, y)
\]

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Radon Transform: Some properties

[Johann K.A. Radon, Austrian mathematician (1887-1956)]

- Definition in cartesian coordinate system:

\[
f(x, y) \xrightarrow{\mathcal{R}} g(r, \phi) = \iint f(x, y) \delta(r - x \cos(\phi) - y \sin(\phi)) \, dx \, dy
\]

- Definition in polar coordinate system:

\[
f(\rho, \theta) \xrightarrow{\mathcal{R}} g(r, \phi) = \int_0^\infty \int_0^{2\pi} f(\rho, \theta) \delta(r - \rho \cos(\phi - \theta) \rho) \, d\rho \, d\theta
\]

- Inversion

\[
f(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{\partial g(r, \phi)}{\partial r} \cdot \frac{1}{r - x \cos(\phi) - y \sin(\phi)} \, dr \, d\phi
\]
Radon Transform: Inversion

Direct Inverse Radon Transform

\[ g(r, \phi) \xrightarrow{\text{Differentiate}} \frac{1}{2\pi} D \xrightarrow{\text{Hilbert Transform}} \tilde{g}(r, \phi) \xrightarrow{\text{Back–projection}} f(x, y) \]

Convolution Back–projection method

\[ g(r, \phi) \xrightarrow{\text{1–D Filter}} \tilde{g}(r, \phi) \xrightarrow{\text{Back–projection}} f(x, y) \]

Filter Back–projection method

\[ g(r, \phi) \xrightarrow{\text{FT}} \mathcal{F}_1 \xrightarrow{\text{Filter}} \Omega \xrightarrow{\text{IFT}} \tilde{g}(r, \phi) \xrightarrow{\text{Back–projection}} f(x, y) \]
Radon Transform (RT) and Filtered Back-Projection (FBP) image reconstruction

a) original \( f(x, y) \)

b) RT \( g(r, \phi) \)

c) FBP reconstruction \( \hat{f}(x, y) \)
Radon Transform (RT), Back-Projection (BP) and Filtered Back-Projection (FBP) image reconstruction

a) $f(x, y)$

b) $\hat{f}(x, y)$ by BP

$\hat{f}(x, y)$ by FBP

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1D signals $f(t)$ and its FT $F(\nu)$

signal $f(t)$  
Modulus of its FT $|F(\nu)|^2$
1D signals: Noise removing using FT

\[ g(t) = f(t) + \epsilon(t) \]

\[ g(t) \rightarrow \text{FFT} \rightarrow \text{Threshold} \rightarrow \text{IFFT} \rightarrow \hat{f}(t) \]

\[ g(t) = f(t) + \epsilon(t) \rightarrow \text{FFT} \rightarrow \ \overline{G(\omega)} = \overline{F(\omega)} + \overline{E(\omega)} \]

\[ \hat{f} \leftarrow \text{IFFT} \leftarrow \text{Thresholded} \overline{G(\omega)} \]
1D signals: Noise removing using FT

\[ g(t) = f(t) + \epsilon(t) \quad \rightarrow \text{FFT} \rightarrow \quad G(\omega) = F(\omega) + E(\omega) \]
1D signals: Noise removing using FT

\[ g(t) = f(t) + \epsilon(t) \]

\[ \rightarrow \text{FFT} \rightarrow \]

\[ G(\omega) = F(\omega) + E(\omega) \]

\[ \hat{f} \]

\[ \leftarrow \text{IFFT} \leftarrow \text{Thresholded } G(\omega) \]
Images denoising using FFT

\[ f(x, y) \]

\[ g(x, y) = f(x, y) + \epsilon(x, y) \]

\[ F(u, v) \]

\[ \hat{f}(x, y) \]

Thresholded \( G(u, v) \)

\[ G(u, v) \]
Images: Pyramidal and Wavelet Transform

Pyramidal representation:
- Invertible Linear Transform
- image $\rightarrow$ Coarse and Fine images
- $B$: Band Pass Filtering

![Diagram]

- Number of samples: $(1 + 1/2 + 1/4 + \ldots)$
- Overcomplete representation
Images: Pyramidal representations

Original

Pyramid

Pyramid
Images: QMF and Wavelet Transform

- Invertible Linear Transform
- image $\rightarrow$ Coarse and Fine images
- Quadrature Mirror Filtering

![Diagram of QMF and Wavelet Transform]

- Number of samples: 1
- Complete representation
Images: Pyramidal representations

Original

Pyramid

Pyramid

WT 1

WT 2

WT 3
Images compression using WT
Images compression using WT

Original

WT 1

WT 2

WT 1-2

IWT 1-1

IWT 1-2
Images: Different representations

Original

FFT

WT 2

Pyramid

Pyramid

Laplacien

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Images: Space, Fourier and Wavelets representations
Sparse images (Fourier and Wavelets domain)

Image

Fourier

Wavelets

Image hist.

Fourier coeff. hist.

Wavelet coeff. hist.

bands 1-3

bands 4-6

bands 7-9
Temperature and activity Time series before, during and some treatment

BOD: before

BOD: during

BOD: after

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Temperatures, before, during and after changes
Simple questions for 1D time series

► Main question: Has something changed during and some medical action?
► In this study: effects of circadian cycle on cancer cells.

1. Is there any periodic components in these signals? Yes/No (Detection)? Confidence ?
2. If Yes, How many?
3. What are those components (Periods \( p_i \) or Frequencies \( \nu_i \), Amplitudes \( a_i \) )?

► When questions 1 and 2 are answered, the problem becomes easier: Parameter estimation

► Trying to answer all the three questions at the same time: semi- or Non-Parametric modelling

► Biologists always need uncertainties \( \rightarrow \) Bayesian inference
Simple Analysis tools may not be successful even in very simple cases

Case of 1 sinusoid

Case of 2 sinusoids+noise

Case of 3 sinusoids+noise
Classical methods: Spectral estimation $S(\omega)$?

- **Fast Fourier Transform (FFT):**
  
  $g(t) \rightarrow FFT \rightarrow f(\omega) \rightarrow S(\omega) = |f(\omega)|^2$
  
  - **Advantages:** Well-known and understood, fast
  - **Drawbacks:** linear in frequencies $\nu$, but not equidistance in periods
    
    $\nu = [0, \cdots, N - 1] \rightarrow p = [\infty, 1, \cdots, 1/(N - 1)]$

- **Autocorrelation function:** $\gamma(\tau)$
  
  - If $g(t)$ is periodic, then $\gamma(\tau)$ is also periodic, but much smoother
    
    $\gamma(0) = 1 \gamma(\tau) \leq \gamma(0), \forall \tau$

- **Power spectral density:** $\gamma(\tau) \rightarrow FFT \rightarrow S(\omega)$

- **Autoregressive (AR), Moving Average (MA) and ARMA models**

- **Non-stationary GARCH models**

- **Sum of sinusoidal components**
Parametric, Semi- and Non-Parametric models

► Parametric:

\[ g(t) = \sum_{k=1}^{K} a_k \sin(2\pi \nu_k t + \phi_k) + \epsilon(t), \quad \theta = \{a_k, \phi_k, \nu_k\} \]

\[ g(t) = \sum_{k=1}^{K} a_k \cos(2\pi \nu_k t) + b_k \sin(2\pi \nu_k t) + \epsilon(t), \quad \theta = \{a_k, b_k, \nu_k\} \]

\[ g(t) = \sum_{k=1}^{K} c_k \exp[j2\pi \nu_k t] + \epsilon(t), \quad \theta = \{c_k, \nu_k\}, \quad t = 0, \ldots, T \]

► Semi-Parametric: \( \nu_k = k\nu_0, \nu_0 = 1/T, K = T \rightarrow \text{DFT} \)

► Non-Parametric: \( \nu_k \) fixed in a given interval with given precision, so \( K \) is fixed but can be as large as necessary.

\[ g(t) = \sum_{k=1}^{K} c_k \exp[j2\pi \nu_k t] + \epsilon(t), \quad \theta = \{c_k\} \quad \text{Linear model} \]
Can we propose a unifying approach for all these problems?

My answer is Yes:

- Identify what you are looking for. (red color $f$)
- Identify what are the data: (blue color $g$)
- Consider the errors (modeling and measurement $\epsilon$)
- Write the Forward model relating them: $g = Hf + \epsilon$
- Write the expression of the likelihood $p(g|f)$
- Translate your prior knowledge on the unknowns in $p(f)$
- Use the Bayes rule:

$$p(f|g) = \frac{p(g|f)p(f)}{p(g)} \propto p(g|f)p(f)$$

- Infer on $f$ using the posterior $p(f|g)$:
  - Maximum A Posteriori (MAP): $\hat{f} = \arg\max_f \{p(f|g)\}$
  - Posterior Mean (PM): $\hat{f} = \int f p(f|g) \, df$
Estimating Periodic Components: Inverse Problems Approach

\[ g(t) = \sum_{k=1}^{K} c_k \exp[j2\pi \nu_k t] + \epsilon(t), \quad \theta = \{c_k\} \text{ Linear model} \]

Slight changes of notations: use of periods \( p_n \) in place of frequencies \( \nu_k \) and \( f_n \) in place of \( c_k \):

\[ g(t) = \sum_{n=1}^{N} f_n \exp[j2\pi/p_m t] + \epsilon(t), \quad t = m\Delta t, \quad m = 1, \ldots, M \]

Defining the vectors: \( g = [g_1, \ldots, g_M]' \), \( \epsilon = [\epsilon_1, \ldots, \epsilon_M]' \), \( f = [f_1, \ldots, f_N]' \) and the matrix \( H: H_{m,n} = \exp[j2\pi/p_m m\Delta t] \), we obtain:

\[ g = Hf + \epsilon \]

The objective is to infer on \( f \).

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Inverse Problems Approach

\[ g = Hf + \epsilon \]

Bayesian approach:

- Assign the Likelihood: \( p(g|f) \)
- Assign the prior law: \( p(f) \)
- Use the Bayes rule: \( p(f|g) \propto p(g|f)p(f) \)
- MAP:

\[ \hat{f} = \arg \max_f \{ p(f|g) \} = \arg \min_f \{ J(f) \} \]

- Assuming Gaussian noise and Gaussian prior

\[ p(f|g) = \mathcal{N}(f|\hat{f}, \Sigma) \text{ with } \hat{\Sigma} = (H'H + \lambda I)^{-1} \text{ and } \hat{f} = \arg \max_f \{ J(f) \} \]

\[ J(f) = \|g - Hf\|^2 + \lambda \|f\|^2 \]

- Other priors (Generalized Gaussian, Student-t or Cauchy)

\[ J(f) = \|g - Hf\|^2 + \lambda \Omega(f) \]
Bayesian estimation with priors enforcing sparsity

- Sparsity: For any periodic signal, the spectrum is a set of Diracs
- Biological signals related to clock genes: a few independent oscillators
- Spectrum has a few non zero elements in any given interval

\[ g(m\Delta t) = \sum_{n=1}^{N} f_n \exp\left[-j\frac{2\pi}{p_m} m\Delta t\right] + \epsilon(t), \ m = 1, \ldots, M \]

\[ g = Hf + \epsilon \quad \text{with} \ f \ \text{sparse} \]

- The question is now: How to translate sparsity?
- Two solutions: L1 regularization and Bayesian sparsity enforcing priors.
- Three main options in Bayesian: Generalized Gaussian, Student-t, mixtures models
Bayesian estimation with priors enforcing sparsity

- \( g = Hf + \epsilon \) with \( f \) sparse
- To translate this information use the heavy tailed prior law Student-t with its hierarchical structure and hidden variables

\[
St(f_j|\nu) \propto \exp \left[ -\frac{\nu + 1}{2} \log \left( 1 + \frac{f_j^2}{\nu} \right) \right]
\]

- Infinite Gaussian Scaled Mixture (IGSM) property:

\[
St(f_j|\nu) = \int_0^\infty \mathcal{N}(f_j|0,1/z_j) \mathcal{G}(z_j|\alpha,\beta) \, dz_j, \quad \text{with } \alpha = \beta = \nu/2
\]

- Hierarchical prior model:

\[
p(f_j|z_j) = \mathcal{N}(f_j|0,1/z_j), \quad p(z_j) = \mathcal{G}(z_j|\alpha,\beta)
\]

\[
\begin{cases}
p(f|z) = \prod_j p(f_j|z_j) \\
p(z) = \prod_j p(z_j) \quad \Rightarrow \quad p(f, z|g) \propto p(g|f)p(f|z)p(z)
\end{cases}
\]

\[
p(g|f) = \mathcal{N}(g|Hf, \sigma_\epsilon^2 I)
\]
Results on simulated and real activity data

Data BEFORE

Proposed method

FFT

Data DURING

Proposed method

FFT

Data AFTER

Proposed method

FFT

A. Mohammad-Djafari, Data, signals, images in Biological and medical application. Master ATSI, UPSa, 2015-2016, Gif, France.
Computed Tomography as Linear Inverse Problem

- $f$: all the pixels of the object put together in a vector;
- $g$: all the projection data put together in a vector;
- $g_k$: the projection at angle $\phi_k$, so that $g = [g_1; g_2; \cdots, g_K]$;
- $H$: the forward problem matrix: $H_{i,j}$ is the length of the ray $i$ in the pixel $j$;
- $H_k$: the forward problem matrix for the projection at angle $\phi_k$, so that $H = [H_1; H_2; \cdots, H_K]$;
- $\epsilon$: the noise over all the projection data;
- $\epsilon_k$: the noise over the projection at angle $\phi_k$ so that $\epsilon = [\epsilon_1; \epsilon_2; \cdots, \epsilon_K]$.

$$g_k = H_k f + \epsilon_k \longrightarrow g = H f + \epsilon$$

with $H^t = [H_1^t, \cdots, H_K^t]$, $g^t = [g_1^t, \cdots, g_K^t]$, $\epsilon^t = [\epsilon_1^t, \cdots, \epsilon_K^t]$, where $K$ is the number of projections at angles
$\phi = [\phi_1, \cdots, \phi_K]$. 
Computed Tomography: BP and FBP

Back-Projection (BP)

\[ \hat{f} = H^t g = \sum_{k=1}^{K} H_k^t g_k. \]

Filtered Back-Projection (FBP)

\[ \hat{f} = H^t g = \sum_{k=1}^{K} H_k^t \tilde{g}_k. \]

where \( \tilde{g}_k \) is the \( |\Omega| \) filtered of \( g_k \).

It can also be regarded as the minimum norm solution of \( H f = g \):

\[
\begin{align*}
\text{minimize} & \quad \| \hat{f} \|_2^2 \\
\text{subject to} & \quad H \hat{f} = g
\end{align*}
\]

\[ \hat{f} = H^t (H H^t)^{-1} g = \sum_{k=1}^{K} H_k^t (H_k H_k^t)^{-1} g_k. \]
Computed Tomography: LS and Regularization

Least-Square (LS) and Quadratic Regularization (QR) methods

\[ \hat{f} = \arg \min_f \{ J(f) \} \]

with

\[ J(f) = \| g - Hf \|^2_2 + \lambda \| Df \|^2_2 \]

which is given by

\[ \hat{f} = (H^tH + \lambda D^tD)^{-1} H^tg. \]

\[ J(f) = \sum_{k=1}^{K} \| g_k - H_kf \|^2_2 + \lambda \| Df \|^2_2 \]

\[ \hat{f} = \sum_{k=1}^{K} (H_k^tH_k + \lambda D^tD)^{-1} H_k^tg. \]
Computed Tomography: Images, Contours, Regions

\[ f(x, y) \]

\[ z(x, y) \]

\[ q(x, y) \]

Next week: Markovian and Hierarchical models