Kalman Filtering and smoothing

- Input-Output and State space modelling
- State space modelling examples
  - Electrical circuit examples
  - Radar tracking of object
- Kalman filtering basics
- Kalman Filtering as recursive Bayesian estimation
- Kalman Filtering extensions: Adaptive signal processing
- Kalman Filtering extensions: Fast Kalman filtering
- Kalman Filtering for signal deconvolution

Input-Output model

- Linear Systems
  - Single Input Single Output (SISO) systems

$$y(t) = \int h(t,\tau)\, u(\tau)\, \mathrm{d}\tau$$

  - Multi Input Multi Output (MIMO) systems

$$\mathbf{y}(t) = \int \mathbf{H}(t,\tau)\, \mathbf{u}(\tau)\, \mathrm{d}\tau$$

- Linear Time Invariant System
  - SISO Convolution

$$y(t) = h(t) * u(t) = \int h(t-\tau)\, u(\tau)\, \mathrm{d}\tau$$

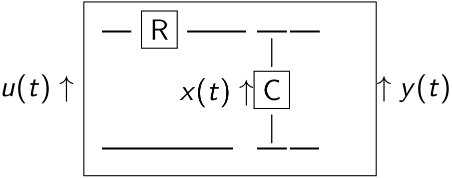
  - MIMO Convolution

$$\mathbf{y}(t) = \int \mathbf{H}(t-\tau)\, \mathbf{u}(\tau)\, \mathrm{d}\tau$$

  - Impulse response $h(t)$ or $\mathbf{H}(t) = \left[\begin{array}{ccc} . & . & . \\ . & h_{ij}(t) & . \\ . & . & . \end{array}\right]$

State space modelling

- A simple electric system



- State space model

$$u(t) = R\,i(t) + v_c(t) = RC\,\dot{v}_c(t) + v_c(t) = RC\,\dot{x}(t) + x(t)$$

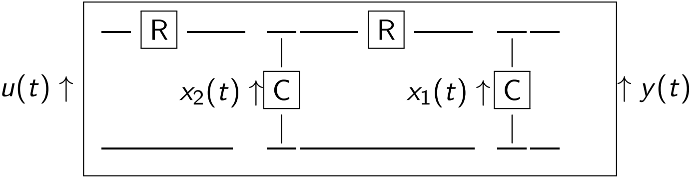
$$\begin{cases} \dot{x}(t) & = \left(\frac{-1}{RC}\right) x(t) + \left(\frac{1}{RC}\right) u(t) \\ y(t) & = x(t) \end{cases}$$

- $RC = 1$:

$$\begin{cases} \dot{x}(t) = -x(t) + u(t) \\ y(t) = x(t) \end{cases} \rightarrow LT \rightarrow \begin{cases} pX(p) + X(p) = U(p) \rightarrow X(p) = \frac{1}{p+1} U(p) \\ y(t) = e^{-t} * u(t) = h(t) * u(t) \end{cases}$$

State space modelling

- A more complex electric system example



- State space model

$$u(t) = RC\,\dot{x}_2(t) + x_2(t), \quad x_2(t) = RC\,\dot{x}_1(t) + x_1(t)$$

- $RC = 1$:

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1(t) \end{cases}$$

State space model: Continuous case

Dynamic systems:

- Single Input Single Output (SISO) system:

$$\begin{cases} \dot{x}(t) & = F\,x(t) + G\,u(t) \quad \text{State equation} \\ y(t) & = C\,x(t) + D\,v(t) \quad \text{Observation equation} \end{cases}$$

- Multiple Input Multiple Output (MIMO) system:

$$\begin{cases} \dot{\mathbf{x}}(t) & = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t) \quad \text{State equation} \\ \mathbf{y}(t) & = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{v}(t) \quad \text{Observation equation} \end{cases}$$

- MIMO discrete-time system:

$$\begin{cases} \mathbf{x}(n+1) & = \mathbf{F}\,\mathbf{x}(n) + \mathbf{G}\,\mathbf{u}(n) \quad \text{State equation} \\ \mathbf{y}(n) & = \mathbf{C}\,\mathbf{x}(n) + \mathbf{D}\,\mathbf{v}(n) \quad \text{Observation equation} \end{cases}$$

F, **G**, **C** and **D** are the matrices of the system.

State space modelling examples

- One-dimensional motion:
  Track-While-Scan (TWS) Radar
  X_t, V_t, A_t: Position, Speed, Acceleration

$$\left\{\begin{array}{l} V_t = \frac{\partial X_t}{\partial t} \\ A_t = \frac{\partial V_t}{\partial t} \end{array}\right. \longrightarrow \left\{\begin{array}{l} X_{n+1} \simeq X_n + T\, V_n \\ V_{n+1} \simeq V_n + T\, A_n \end{array}\right.$$

$$\left\{\begin{array}{l} \mathbf{x} = \begin{bmatrix} X \\ V \end{bmatrix} \\ u = A \\ Y = X + v \end{array}\right. \longrightarrow \left\{\begin{array}{l} \mathbf{x}_n = \begin{bmatrix} X_n \\ V_n \end{bmatrix} \\ u_n = A_n \\ y_n = X_n + v_n \end{array}\right.$$

$$\left\{\begin{array}{ll} \mathbf{x}_{n+1} & = \mathbf{F}\mathbf{x}_n + \mathbf{G} u_n \\ y_n & = \mathbf{H}\mathbf{x}_n + v_n \end{array}\right. \quad \text{with} \quad \mathbf{F} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 0 \\ T \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\left\{\begin{array}{l} \mathbf{f}_n(\mathbf{x}_n, \mathbf{u}_n) = \mathbf{F}\,\mathbf{x}_n + \mathbf{G}\,\mathbf{u}_n \\ \mathbf{h}_n(\mathbf{x}_n, \mathbf{v}_n) = \mathbf{H}\,\mathbf{x}_n + \mathbf{v}_n \end{array}\right.$$

State space modelling examples

- 1D motion of heavy targets
  Track-While-Scan (TWS) Radar with dependent acceleration sequences
- heavy target:

$$A_{n+1} = \rho A_n + W_n, \quad n = 0, 1, \ldots$$

- ρ near to 0: low inertia target
- ρ near to 1: high inertia target.

$$\begin{aligned} \begin{bmatrix} X_{n+1} \\ V_{n+1} \\ A_{n+1} \end{bmatrix} &= \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} X_n \\ V_n \\ A_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} W_n \\ Y_n &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_n \\ V_n \\ A_n \end{bmatrix} + e_n \end{aligned}$$

Kalman Filtering: Recursive Linear Filtering

$$\underset{\text{non observable}}{x(n)} \quad \longrightarrow \boxed{\text{Linear System}} \longrightarrow \quad \underset{\text{observable}}{y(n)}$$

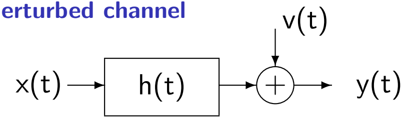
- Objective: Estimate $x(n)$ from the observed values of $\{y(n), n = 1, \ldots, k\}$.
- $\widehat{x}(n)$ is then a function of the data $\{y(n), n = 1, \ldots, k\}$

$$\widehat{x}(n \mid y(1), y(2), \ldots, y(k)) \overset{\triangle}{=} \widehat{x}(n \mid k)$$

- $\widehat{x}(n+k|n)$ is called the k-th *order prediction* of $y(n)$ and the estimation procedure is called *prediction*.
- $\widehat{x}(n|n)$ is the *filtered value* of $y(n)$ and the estimation procedure is called *filtering*.
- $\widehat{x}(n|n+l)$ is the *smoothed value* of $y(n)$ and the estimation procedure is called *smoothing*.

State space and input-Output modelling

Transmission of a AR1 signal through a perturbed channel



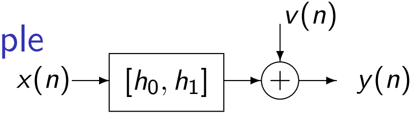
$$x(t) = a\,x(t-1) + u(t)$$
$$X(p) = a\,p\,X(p) + U(p)$$
$$X(p) = \frac{1}{a-p} U(p)$$
$$X(\omega) = \frac{1}{1-a(j\omega)} U(\omega)$$
$$x(n) = a\,x(n-1) + u(n)$$

$$y(t) = \int_0^t h(\tau)\,x(t-\tau)\,\mathrm{d}\tau + v(t)$$
$$Y(p) = H(p)\,X(p) + V(p)$$
$$Y(p) = H(p)\,X(p) + V(p)$$
$$Y(\omega) = H(\omega)\,X(\omega) + V(\omega)$$
$$y(n) = \sum_{k=0}^{p} h_k\,x(n-k) + v(n)$$

$$\left\{\begin{array}{lcl} x_{n+1} & = & a\,x_n + u_n \\ y_n & = & \displaystyle\sum_{k=0}^{p} h_n\,x_{n-k} + v_n \end{array}\right. \rightarrow ? \rightarrow \left\{\begin{array}{lcl} \mathbf{x}_{n+1} & = & \mathbf{F}\,\mathbf{x}_n + \mathbf{G}u_n \\ y_n & = & \mathbf{H}\,\mathbf{x}_n + v_n \end{array}\right.$$

State space modelling example

$$u(n) \longrightarrow \boxed{\text{AR1}} \longrightarrow x(n)$$



$$x(n) = a\,x(n-1) + u(n) \qquad y(n) = h_0 x(n) + h_1 x(n-1) + v(n)$$

$$\mathbf{x}_n = [x(n), x(n-1)]', \quad \mathbf{x}_{n+1} = [x(n+1), x(n)]'$$

$$\begin{bmatrix} x(n+1) \\ x(n) \end{bmatrix} = \begin{bmatrix} a & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(n)$$

$$y(n) = \begin{bmatrix} h_0 & h_1 \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} + v(n)$$

$$\mathbf{F} = \begin{bmatrix} a & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{H} = [h_0, h_1] \longrightarrow$$

$$\left\{ \begin{array}{lcl} \mathbf{x}_{n+1} & = & \mathbf{F}\,\mathbf{x}_n + \mathbf{G} u_n \\ y_n & = & \mathbf{H}\,\mathbf{x}_n + v_n \end{array} \right.$$

State space and input-Output modelling

A FIR channel $y(n) = \sum_{k=0}^{p} h_k x(n-k) + v(n)$

$$\begin{aligned}
\mathbf{x}_n &= [x(n), x(n-1), \cdots, x(n-p)]' \\
\mathbf{x}_{n+1} &= [x(n+1), x(n), \cdots, x(n-p+1)]'
\end{aligned}$$

$$\begin{bmatrix} x(n+1) \\ x(n) \\ x(n-1) \\ \vdots \\ \vdots \\ x(n-p+1) \end{bmatrix} = \begin{bmatrix} a & 0 & \ldots & \ldots & 0 \\ 1 & 0 & \ldots & \ldots & 0 \\ & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ \vdots \\ \vdots \\ x(n-p+1) \\ x(n-p) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \end{bmatrix} u(n)$$

$$\begin{aligned} \mathbf{G} &= [1, 0, \cdots, 0]' \\ \mathbf{H} &= [h_0, h_1, \cdots, h_p]' \end{aligned} \longrightarrow \left\{ \begin{array}{lcl} \mathbf{x}_{n+1} & = & \mathbf{F}\,\mathbf{x}_n + \mathbf{G} u_n \\ y_n & = & \mathbf{H}\,\mathbf{x}_n + v_n \end{array} \right.$$

State space and input-Output modelling

- A FIR channel $y(n) = \sum_{k=0}^{p} h_k x(n-k) + v(n)$

$$\left\{ \begin{array}{lcl} \mathbf{x}_{n+1} & = & \mathbf{F}\,\mathbf{x}_n + \mathbf{G} u_n \\ y_n & = & \mathbf{H}\,\mathbf{x}_n + v_n \end{array} \right.$$

$$\begin{array}{l} \mathbf{x}_n \;\;= [x_n, x_{n-1}, \cdots, x_{n-p+1}]' \\ \mathbf{x}_{n+1} = [x_{n+1}, x_n, \cdots, x_{n-p+2}]' \\ \mathbf{G} = [1, 0, \cdots, 0]' \\ \mathbf{H} = [h_0, h_1, \cdots, h_n]' \end{array} \qquad \mathbf{F} = \begin{bmatrix} a & 0 & \ldots & \ldots & 0 \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & 1 & 0 \end{bmatrix}$$

- A perfect but noisy channel $h(t) = h_0\,\delta(t) \longrightarrow p = 1$

$$\left\{ \begin{array}{lcl} x_{n+1} & = & a\,x_n + u_n \\ y_n & = & h_0\,x_n + v_n \end{array} \right.$$

State space modelling: Examples

$$\begin{cases} x_{n+1} & = & a\,x_n + u_n \\ y_n & = & h\,x_n + v_n \end{cases}$$

$$\begin{cases} u_n & \sim & \mathcal{N}(0, q) \\ v_n & \sim & \mathcal{N}(0, r) \\ x_0 & \sim & \mathcal{N}(m_0, p_0) \end{cases}$$

- Try to obtain $\widehat{x}_{n+1|n}$ as a function of $\widehat{x}_{n|n}$ recursively

$$\begin{cases} \widehat{x}_{n+1|n} & = & a\,\widehat{x}_{n|n} \\ \widehat{x}_{n|n} & = & \widehat{x}_{n|n-1} + k_n\,(y_n - h\,\widehat{x}_{n|n-1}) \\ k_n & = & \dfrac{p_{n|n-1}\,h}{h^2 p_{n|n-1} + r} = \dfrac{1}{h}\,\dfrac{p_{n|n-1}}{p_{n|n-1} + r/h^2} \end{cases}$$

$$\begin{cases} p_{n+1|n} & = & a^2\,p_{n|n} + q \\ p_{n|n} & = & \dfrac{1}{h}\,\dfrac{p_{n|n-1}}{\frac{r}{h^2}\,p_{n|n-1+1}} \end{cases}$$

State space modelling of the systems

- Time Varying systems:

$$\begin{cases} \mathbf{x}_{n+1} & = & \mathbf{f}_n(\mathbf{x}_n, \mathbf{u}_n) & \text{state equation} \\ \mathbf{y}_n & = & \mathbf{h}_n(\mathbf{x}_n, \mathbf{v}_n) & \text{observation equation} \end{cases}$$

- Time Variying but Linear system

$$\begin{cases} \mathbf{x}_{n+1} & = & \mathbf{F}_n\,\mathbf{x}_n + \mathbf{G}_n\,\mathbf{u}_n \\ \mathbf{y}_n & = & \mathbf{H}_n\,\mathbf{x}_n + \mathbf{v}_n \end{cases}$$

- Time Invariant and Linear system

$$\begin{cases} \mathbf{x}_{n+1} & = & \mathbf{F}\,\mathbf{x}_n + \mathbf{G}\,\mathbf{u}_n \\ \mathbf{y}_n & = & \mathbf{H}\,\mathbf{x}_n + \mathbf{v}_n \end{cases}$$

State space modelling: General case

$$\begin{cases} \mathbf{x}_{k+1} & = \mathbf{F}_k\, \mathbf{x}_k + \mathbf{G}_k\, \mathbf{u}_k & \text{state equation,} \\ \mathbf{y}_k & = \mathbf{H}_k\, \mathbf{x}_k + \mathbf{v}_k & \text{observation equation} \end{cases}$$

- $\mathbf{x}_k$ N-dimensional *state vector*
- $\mathbf{y}_k$ P-dimensional *observations vector*
- $\mathbf{v}_k$ P-dimensional *observations error vector*
- $\mathbf{u}_k$ M-dimensional *state representation error*
- $\mathbf{F}_k$, $\mathbf{G}_k$ and $\mathbf{H}_k$ with respective dimensions of (N, N), (N, M) and (P, N) are the *state transition*, the *state input* and the *observation matrices* and are assumed to be known.
- The noise sequences $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$ and the *initial state* $\mathbf{x}_0$ are assumed to be centered, white and jointly Gaussian.

$$\mathrm{E}\left\{ \begin{bmatrix} \mathbf{v}_k \\ \mathbf{x}_0 \\ \mathbf{u}_k \end{bmatrix} \left[\mathbf{v}_l^t, \mathbf{x}_0^t, \mathbf{u}_l^t\right] \right\} = \begin{bmatrix} \mathbf{R}_k & 0 & 0 \\ 0 & \mathbf{P}_0 & 0 \\ 0 & 0 & \mathbf{Q}_k \end{bmatrix} \delta_{kl}$$

Kalman Filtering: Prediction, Filtering and Smoothing

Objective: Find the best estimate $\widehat{\mathbf{x}}_{k|l}$ of $\mathbf{x}_k$ from the observations $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_l$.

$$\widehat{\mathbf{x}}_k(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_l) \overset{\triangle}{=} \widehat{\mathbf{x}}(k \mid l)$$

- If $k > l$ prediction. For example $l = n, k = n + 1$:

$$\widehat{\mathbf{x}}_{n+1}(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n) \overset{\triangle}{=} \widehat{\mathbf{x}}(n + 1 \mid n)$$

- If $k = l$ filtering. For example $l = n, k = n$:

$$\widehat{\mathbf{x}}_n(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n) \overset{\triangle}{=} \widehat{\mathbf{x}}(n \mid n)$$

- If $k < l$ smoothing. For example $l = n + 1, k = n$:

$$\widehat{\mathbf{x}}_n(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n+1}) \overset{\triangle}{=} \widehat{\mathbf{x}}(n \mid n + 1)$$

Kalman Filtering: 3 Interpretations

Three different approaches can be used to obtain the Kalman filtering equations:

- Least Mean Square (LMS) estimation:

  $$\widehat{\mathbf{x}}_{k|l} \overset{\triangle}{=} \mathrm{LMS}(\mathbf{x}_k \mid \mathbf{y}_1, \ldots, \mathbf{y}_l)$$

  which minimizes

  $$\mathrm{E}\left\{[\mathbf{x}_k - \widehat{\mathbf{x}}_{k|l}]^t\, \mathbf{W}_k\, [\mathbf{x}_k - \widehat{\mathbf{x}}_{k|l}]\right\}$$

- Maximum A posteriori (MAP) estimate:

  $$\widehat{\mathbf{x}}_{k|l} = \arg\max_{\mathbf{x}} \left\{p(\mathbf{x}_k \mid \mathbf{y}_1, \ldots, \mathbf{y}_l)\right\}$$

- Bayesian MSE estimate:

  $$\widehat{\mathbf{x}}_{k|l} = \mathrm{E}\left\{\mathbf{x}_k \mid \mathbf{y}_1, \ldots, \mathbf{y}_k\right\}$$

Basic Bayes

- Two related events A and B with probabilities $P(A,B)$, $P(A|B)$ and $P(A|B)$.
- Product rule: $P(A,B) = P(A|B)P(B) = P(B|A)P(A)$
- Sum rule: $P(A) = \sum_B P(A|B)P(B)$
- Bayes rule: $P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{P(A|B)P(B)}{\sum_B P(A|B)P(B)}$

- Two related variables X and Y with probability distributions: $P(X,Y)$, $P(Y|X)$ and $P(X|Y)$.
- Bayes rule: $P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)} = \frac{P(Y|X)P(X)}{\sum_X P(Y|X)P(X)}$

- Two related continuous variables X and Y with probability density functions: $p(x,y)$, $p(y|x)$ and $p(x|y)$.
- Bayes rule: $p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int p(y|x)p(x)\,\mathrm{d}x}$

Basic Bayes

- Two related vector variables $\mathbf{x}$ and $\mathbf{y}$ with probability density functions: $p(\mathbf{x},\mathbf{y})$, $p(\mathbf{y}|\mathbf{x})$ and $p(\mathbf{x}|\mathbf{y})$.
- Bayes rule:

$$p(\mathbf{x}|\mathbf{y}) = \frac{p(\mathbf{y}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{y})} = \frac{p(\mathbf{y}|\mathbf{x})p(\mathbf{x})}{\int p(\mathbf{y}|\mathbf{x})p(\mathbf{x})\,\mathrm{d}\mathbf{x}}$$

- Example: Gaussian case $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{u}$

$$p(\mathbf{u}) = \mathcal{N}(\mathbf{u}|0,\mathbf{R}_u) \longrightarrow p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{y}|\mathbf{H}\mathbf{x},\mathbf{R}_u)$$

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|0,\mathbf{R}_x),\ \mathbf{x} \text{ and } \mathbf{u} \text{ independent}$$

$$p(\mathbf{y}) = \mathcal{N}(\mathbf{y}|0,\mathbf{H}\mathbf{R}_x\mathbf{H}' + \mathbf{R}_u)$$

- Bayes rule:

$$p(\mathbf{x}|\mathbf{y}) = \frac{p(\mathbf{y}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{y})} = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})$$

with

$$\boldsymbol{\mu} = \mathbf{R}_x\mathbf{H}'[\mathbf{H}\mathbf{R}_x\mathbf{H}' + \mathbf{R}_u]^{-1}\mathbf{y}, \quad \boldsymbol{\Sigma} = [\mathbf{H}'\mathbf{R}_u^{-1}\mathbf{H} + \mathbf{R}_x^{-1}]^{-1}$$

Basic Bayes

- More general Gaussian case $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{u}$, $p(\mathbf{u}) = \mathcal{N}(\mathbf{u}|0, \mathbf{R}_u)$

$$p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{y}|\mathbf{H}\mathbf{x}, \mathbf{R}_u) \propto \exp\left\{-\frac{1}{2}(\mathbf{y} - \mathbf{H}\mathbf{x})'\mathbf{R}_u^{-1}(\mathbf{y} - \mathbf{H}\mathbf{x})\right\}$$

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\mathbf{x}_0, \mathbf{R}_x) \propto \exp\left\{-\frac{1}{2}(\mathbf{x} - \mathbf{x}_0)'\mathbf{R}_x^{-1}(\mathbf{x} - \mathbf{x}_0)\right\}$$

- Bayes rule: $p(\mathbf{x}|\mathbf{y}) \propto p(\mathbf{y}|\mathbf{x})p(\mathbf{x})$:

$$p(\mathbf{x}|\mathbf{y}) \propto \exp\left\{-\frac{1}{2}[(\mathbf{y} - \mathbf{H}\mathbf{x})'\mathbf{R}_u^{-1}(\mathbf{y} - \mathbf{H}\mathbf{x}) + (\mathbf{x} - \mathbf{x}_0)'\mathbf{R}_x^{-1}(\mathbf{x} - \mathbf{x}_0)]\right\}$$

$$p(\mathbf{x}|\mathbf{y}) \propto \exp\left\{-\frac{1}{2}[\mathbf{x}'[\mathbf{H}'\mathbf{R}_u^{-1}\mathbf{H} + \mathbf{R}_x^{-1}]\mathbf{x} + 2\cdots\mathbf{x} + \cdots\right\}$$

$$p(\mathbf{x}|\mathbf{y}) \propto \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})' + ...\right\}$$

with $\boldsymbol{\Sigma} = [\mathbf{H}\mathbf{R}_u^{-1}\mathbf{H} + \mathbf{R}_x^{-1}]^{-1}$, $\boldsymbol{\mu} = \mathbf{x}_0 + \mathbf{R}_x\mathbf{H}'[\mathbf{H}\mathbf{R}_x\mathbf{H}' + \mathbf{R}_u]^{-1}(\mathbf{y} - \mathbf{H}\mathbf{x}_0)$

Recursive Bayes

- Direct

$$\begin{tikzcd} & p(\mathbf{y}|\mathbf{x}) \arrow[d] & \\ p(\mathbf{x}) \arrow[r] & \boxed{\text{Bayes}} \arrow[r] & p(\mathbf{x}|\mathbf{y}) \end{tikzcd}$$

- Recursive

$$p(\mathbf{x}|\mathbf{y}) \propto \prod_i p(y_i|\mathbf{x})p(\mathbf{x}) \propto \left[\left[[p(\mathbf{x})p(y_1|\mathbf{x})]\; p(y_2|\mathbf{x})\right] \cdots p(y_n|\mathbf{x})\right]$$

$$\begin{tikzcd} & p(y_1|\mathbf{x}) \arrow[d] & & p(y_2|\mathbf{x}) \arrow[d] & \ldots\ldots & p(y_n|\mathbf{x}) \arrow[d] & \\ p(\mathbf{x}) \arrow[r] & \boxed{\text{Bayes}} \arrow[r] & p(\mathbf{x}|y_1) \arrow[r] & \boxed{\text{Bayes}} \arrow[r] & p(\mathbf{x}|y_1, y_2)\ldots \arrow[r] & \boxed{\text{Bayes}} \arrow[r] & p(\mathbf{x}|\mathbf{y}) \end{tikzcd}$$

Kalman Filtering: Bayesian approach

The main procedure is to apply the Bayes rule recursively to find the expression of the posterior law $p(\mathbf{x}_k \mid \mathbf{y}_1, \ldots, \mathbf{y}_l)$.

1. All variables are assumed Gaussian ;
2. All the conditional laws such as $p(\mathbf{y}_{k+1}|\mathbf{x}_{k+1})$ and $p(\mathbf{y}_{k+1}|\mathbf{y}_{1:k})$ are Gaussian. So, the posterior law
$$p(\mathbf{x}_{k+1}|\mathbf{y}_{1:k+1}) = p(\mathbf{x}_{k+1}|\mathbf{y}_{1:k}) \, \frac{p(\mathbf{y}_{k+1}|\mathbf{x}_{k+1})}{p(\mathbf{y}_{k+1}|\mathbf{y}_{1:k})}$$
is also Gaussian.

To obtain the equations in the general case we note

- $\widehat{\mathbf{x}}_{k|k}$ the estimate of the state vector at time k from the observations up to time k ;
- $\widehat{\mathbf{x}}_{k+1|k}$ the estimate of the state vector at time $k+1$ from the observations up to the instant k ;
- $\mathbf{e}_{k+1} = \mathbf{y}_{k+1} - H_{k+1} \, \widehat{\mathbf{x}}_{k+1|k}$ the *innovation process* of the observations at the instant $k+1$

Kalman Filtering: Bayesian approach

- The covariance matrix of the innovation by

$$\mathbf{R}^e_{k+1} = \mathrm{E}\left\{\mathbf{e}_{k+1|k}\,\mathbf{e}^t_{k+1|k}\right\}$$

which is diagonal;

- The covariance matrix of the prediction error by

$$\mathbf{P}_{k+1|k} = \mathrm{E}\left\{\left[\mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1|k}\right]\left[\mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1|k}\right]^t\right\},$$

- The posterior covariance matrix of the estimation error by

$$\mathbf{P}_{k+1|k+1} = \mathrm{E}\left\{\left[\mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1|k+1}\right]\left[\mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1|k+1}\right]^t\right\}$$

which is also called the covariance matrix of the filtering error.

Kalman Filtering: Bayesian approach

$$\begin{aligned}
\mathrm{E}\left\{\mathbf{x}_k|\mathbf{y}_{1:k}\right\} &\overset{\triangle}{=} \widehat{\mathbf{x}}_{k|k} \\
\mathrm{cov}[\mathbf{x}_k|\mathbf{y}_{1:k}] &\overset{\triangle}{=} \mathbf{P}_{k|k} \\
\mathrm{E}\left\{\mathbf{x}_{k+1}|\mathbf{y}_{1:k}\right\} &\overset{\triangle}{=} \widehat{\mathbf{x}}_{k+1|k} \\
\mathrm{cov}[\mathbf{x}_{k+1}|\mathbf{y}_{1:k}] &\overset{\triangle}{=} \mathbf{P}_{k+1|k}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left\{\mathbf{y}_{k+1}|\mathbf{x}_{k+1}\right\} &= \mathbf{H}_{k+1}\,\widehat{\mathbf{x}}_{k+1} \\
\mathrm{cov}[\mathbf{y}_{k+1}|\mathbf{x}_{k+1}] &= \mathbf{R}_{k+1}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left\{\mathbf{x}_{k+1}|\mathbf{y}_{1:k}\right\} &= \mathbf{F}_k\,\widehat{\mathbf{x}}_{k|k} \\
\mathrm{cov}[\mathbf{x}_{k+1}|\mathbf{y}_{1:k}] &= \mathbf{F}_k\,\mathbf{P}_k\mathbf{F}_k^t + \mathbf{G}_k\,\mathbf{Q}_k\,\mathbf{G}_k^t
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left\{\mathbf{y}_{k+1}|\mathbf{y}_{1:k}\right\} &= \mathbf{H}_{k+1}\,\mathbf{F}_k\,\widehat{\mathbf{x}}_{k|k} \\
\mathrm{cov}[\mathbf{y}_{k+1}|\mathbf{y}_{1:k}] &= \mathbf{H}_{k+1}\,\mathbf{P}_{k+1|k}\mathbf{H}_{k+1}^t + \mathbf{R}_{k+1}
\end{aligned}$$

Kalman Filtering: Bayesian approach

Replacing the expressions of

$$p(\mathbf{y}_{k+1}|\mathbf{x}_{k+1}) = \mathcal{N}\left(\mathbf{y}_{k+1}|\mathbf{H}_{k+1}\,\widehat{\mathbf{x}}_{k+1},\mathbf{R}_{k+1}\right)$$

and

$$p(\mathbf{y}_{k+1}|\mathbf{y}_{1:k}) = \mathcal{N}\left(\mathbf{y}_{k+1}|\mathbf{H}_{k+1}\,\mathbf{F}_k\,\widehat{\mathbf{x}}_{k|k},\mathbf{H}_{k+1}\,\mathbf{P}_{k+1|k}\mathbf{H}_{k+1}^t+\mathbf{R}_{k+1}\right)$$

in

$$p(\mathbf{x}_{k+1}|\mathbf{y}_{1:k+1}) = p(\mathbf{x}_{k+1}|\mathbf{y}_{1:k})\,\frac{p(\mathbf{y}_{k+1}|\mathbf{x}_{k+1})}{p(\mathbf{y}_{k+1}|\mathbf{y}_{1:k})}$$

we obtain

$$p(\mathbf{x}_{k+1}|\mathbf{y}_{1:k+1}) = A\exp\left\{-\frac{1}{2}[\mathbf{x}_{k+1}-\widehat{\mathbf{x}}_{k+1|k}]^t\mathbf{P}_{k+1|k+1}^{-1}[\mathbf{x}_{k+1}-\widehat{\mathbf{x}}_{k+1|k}]\right\}$$

with

$$A=\frac{1}{(2\pi)^{n/2}}\left\{\left|\mathbf{H}_{k+1}\,\mathbf{P}_{k+1|k}\,\mathbf{H}_{k+1}^t+\mathbf{R}_k\,\right|^{1/2}\left|\mathbf{R}_k\,\right|^{-1/2}\left|\mathbf{P}_{k+1|k}\,\right|\right\}^{-1/2}$$

Kalman Filtering: Bayesian approach

$$p(\mathbf{x}_{k+1}|\mathbf{y}_{1:k+1}) = A \exp\left\{-\frac{1}{2}[\mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1|k}]^t \mathbf{P}_{k+1|k+1}^{-1} [\mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1|k}]\right\}$$

$$\begin{cases} \widehat{\mathbf{x}}_{k+1|k} = \mathbf{F}_k \, \widehat{\mathbf{x}}_{k|k} + \\ \quad \mathbf{P}_{k+1|k} \mathbf{H}_{k+1}^t \left[\mathbf{R}_{k+1} + \mathbf{H}_{k+1} \mathbf{P}_{k+1|k} \mathbf{H}_{k+1}^t\right]^{-1} \left[\mathbf{y}_{k+1} - \mathbf{H}_{k+1} \mathbf{F}_k \widehat{\mathbf{x}}_{k|k}\right] \\ \\ \mathbf{P}_{k+1|k} = \mathbf{F}_k \, \mathbf{P}_{k|k} \, \mathbf{F}_k^t + \mathbf{G}_k \, \mathbf{Q}_k \, \mathbf{G}_k^t \\ \mathbf{P}_{k+1|k+1}^{-1} = \mathbf{P}_{k+1|k}^{-1} + \mathbf{H}_{k+1}^t \mathbf{R}_{k+1}^{-1} \, \mathbf{H}_{k+1} \end{cases}$$

or

$$\begin{cases} \widehat{\mathbf{x}}_{k+1|k} & = \mathbf{F}_k \, \widehat{\mathbf{x}}_{k|k} + \mathbf{K}_{k+1} \left[\mathbf{y}_{k+1} - \mathbf{H}_{k+1} \mathbf{F}_k \widehat{\mathbf{x}}_{k|k}\right] \\ \mathbf{K}_{k+1} & = \mathbf{P}_{k+1|k} \mathbf{H}_{k+1}^t \left[\mathbf{R}_{k+1} + \mathbf{H}_{k+1} \mathbf{P}_{k+1|k} \mathbf{H}_{k+1}^t\right]^{-1} \\ \mathbf{P}_{k+1|k} & = \mathbf{F}_k \, \mathbf{P}_{k|k} \, \mathbf{F}_k^t + \mathbf{G}_k \, \mathbf{Q}_k \, \mathbf{G}_k^t \\ \mathbf{P}_{k+1|k+1}^{-1} & = \mathbf{P}_{k+1|k}^{-1} + \mathbf{H}_{k+1}^t \mathbf{R}_{k+1}^{-1} \, \mathbf{H}_{k+1} \end{cases}$$

Kalman Filtering: Different forms of equation

- **Prediction-Correction form :**
  - **Prediction (Time update) :**

$$\begin{array}{rcl} \widehat{\mathbf{x}}_{k+1|k} & = & \mathbf{F}_k \, \widehat{\mathbf{x}}_{k|k} \\ \mathbf{P}_{k+1|k} & = & \mathbf{F}_k \, \mathbf{P}_{k|k} \, \mathbf{F}_k^t + \mathbf{G}_k \, \mathbf{Q}_k \, \mathbf{G}_k^t \end{array}$$

  - **Correction (measurement update) :**

$$\begin{array}{rcl} \widehat{\mathbf{x}}_{k+1|k+1} & = & \widehat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}^g [\mathbf{y}_{k+1} - \mathbf{H}_{k+1} \, \widehat{\mathbf{x}}_{k+1|k}] \\ \mathbf{K}_{k+1}^g & = & \mathbf{P}_{k+1|k} \, \mathbf{H}_{k+1}^t (\mathbf{R}_{k+1}^e)^{-1} \\ \mathbf{R}_{k+1}^e & = & \mathbf{R}_{k+1} + \mathbf{H}_{k+1} \, \mathbf{P}_{k+1|k} \, \mathbf{H}_{k+1}^t \\ \mathbf{P}_{k+1|k+1} & = & [\mathbf{I} - \mathbf{K}_{k+1}^f \mathbf{H}_{k+1}] \mathbf{P}_{k+1|k} \end{array}$$

Kalman Filtering: Different forms of equation

- **Compact form for prediction :**

$$
\begin{aligned}
\widehat{\mathbf{x}}_{k+1|k} &= \mathbf{F}_k\,\widehat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{R}^e_k)^{-1}[\mathbf{y}_k - \mathbf{H}_k\,\widehat{\mathbf{x}}_{k|k-1}] \\
\mathbf{R}^e_k &= \mathbf{R}_k + \mathbf{H}_k\,\mathbf{P}_{k|k-1}\,\mathbf{H}^t_k \\
\mathbf{K}_k &= \mathbf{F}_k\,\mathbf{P}_{k|k-1}\,\mathbf{H}^t_k \\
\mathbf{P}_{k+1|k} &= \mathbf{F}_k\,\mathbf{P}_{k|k-1}\,\mathbf{F}^t_k + \mathbf{G}_k\,\mathbf{Q}_k\,\mathbf{G}^t_k - \mathbf{K}_k(\mathbf{R}^e_k)^{-1}\,\mathbf{K}^t_k
\end{aligned}
$$

- In all cases the initialization is :

$$
\widehat{\mathbf{x}}_{0|-1} = 0 \quad \mathbf{P}_{0|-1} = \mathbf{P}_0
$$

Kalman Filtering: Different forms of equation

- **Compact form for filtering :**

$$\begin{aligned}
\widehat{\mathbf{x}}_{k|k} &= \widehat{\mathbf{x}}_{k|k-1} + \mathbf{K}^g_k[\mathbf{y}_k - \mathbf{H}_k\,\widehat{\mathbf{x}}_{k|k-1}] \\
\mathbf{K}^g_k &= \mathbf{P}_{k|k-1}\,\mathbf{H}^t_k[\mathbf{R}_k + \mathbf{H}_k\,\mathbf{P}_{k|k-1}\,\mathbf{H}^t_k]^{-1} \\
\mathbf{P}_{k|k} &= \mathbf{P}_{k|k-1} - \mathbf{K}^g_k\mathbf{H}_k\mathbf{P}_{k|k-1} \\
\mathbf{P}_{k+1|k} &= \mathbf{F}_k\mathbf{P}_{k|k}\,\mathbf{F}^t_k + \mathbf{G}_k\,\mathbf{Q}_k\,\mathbf{G}^t_k
\end{aligned}$$

- **Very compact form for prediction :**

$$\begin{aligned}
\widehat{\mathbf{x}}_{k+1|k} &= \mathbf{F}_k\widehat{\mathbf{x}}_{k|k-1} + \mathbf{K}^g_k[\mathbf{y}_k - \mathbf{H}_k\,\widehat{\mathbf{x}}_{k|k-1}] \\
\mathbf{K}^g_k &= \mathbf{P}_{k|k-1}\,\mathbf{H}^t_k[\mathbf{R}_k + \mathbf{H}_k\,\mathbf{P}_{k|k-1}\,\mathbf{H}^t_k]^{-1} \\
\mathbf{P}_{k+1|k} &= \mathbf{F}_k\mathbf{P}_{k|k-1}\,\mathbf{F}^t_k - \mathbf{F}_k\mathbf{K}^g_k\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{F}^t_k + \mathbf{G}_k\,\mathbf{Q}_k\,\mathbf{G}^t_k
\end{aligned}$$

where $\mathbf{K}_k$ is called the *Kalman filter gain* and $\mathbf{K}^g_k = \mathbf{K}_k(\mathbf{R}^e_k)^{-1}$ the generalized *Kalman filter gain*.

Kalman Filtering: Time invariant system

For Time invariant system:

$$\begin{aligned}
\widehat{\mathbf{x}}_{n+1|n} &= \mathbf{F}\widehat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n[\mathbf{y}_n - \mathbf{H}\widehat{\mathbf{x}}_{n|n-1}] \\
\mathbf{K}_n &= \mathbf{P}_{n|n-1}\,\mathbf{H}'[\mathbf{R} + \mathbf{H}\mathbf{P}_{n|n-1}\,\mathbf{H}']^{-1} \\
\mathbf{P}_{n+1|n} &= \mathbf{F}\mathbf{P}_{n|n-1}\mathbf{F}' - \mathbf{F}\mathbf{K}_n\mathbf{H}\mathbf{P}_{n|n-1}\mathbf{F}' + \mathbf{G}\,\mathbf{Q}\,\mathbf{G}'
\end{aligned}$$

For scalar data $y_n = \mathbf{h}'\mathbf{x}_n + v_n$:

$$\begin{aligned}
\widehat{\mathbf{x}}_{n+1|n} &= \mathbf{F}\widehat{\mathbf{x}}_{n|n-1} + \mathbf{k}_n[y_n - \mathbf{h}'\widehat{\mathbf{x}}_{n|n-1}] \\
\mathbf{k}_n &= \mathbf{P}_{n|n-1}\,\mathbf{h}[r + \mathbf{h}'\mathbf{P}_{n|n-1}\mathbf{h}]^{-1} \\
\mathbf{P}_{n+1|n} &= \mathbf{F}\mathbf{P}_{n|n-1}\mathbf{F}' - \mathbf{F}\mathbf{k}_n\mathbf{h}'\mathbf{P}_{n|n-1}\mathbf{F}' + \mathbf{G}\,\mathbf{Q}\,\mathbf{G}'
\end{aligned}$$

For scalar data and scalar state vector:

$$\begin{aligned}
\widehat{x}_{n+1|n} &= f\widehat{x}_{n|n-1} + k_n[y_n - h\widehat{x}_{n|n-1}] \\
k_n &= p_{n|n-1}\,h[r + h^2 p_{n|n-1}]^{-1} \\
p_{n+1|n} &= f^2 p_{n|n-1} - f^2 k_n h\, p_{n|n-1} + q
\end{aligned}$$

Kalman Filtering equations: 1D case

An AR1 signal is observed through a perturbed chanel:

$$\left\{\begin{array}{lcl} x_{n+1} & = & f\, x_n + u_n \\ y_n & = & h\, x_n + v_n \end{array}\right. \qquad \left\{\begin{array}{lcl} u_n & \sim & \mathcal{N}(0, q) \\ v_n & \sim & \mathcal{N}(0, r) \\ x_0 & \sim & \mathcal{N}(m_0, p_0) \end{array}\right.$$

$$\left\{\begin{array}{lcl} \widehat{x}_{n+1|n} & = & f\, \widehat{x}_{n|n} \\ \widehat{x}_{n|n} & = & \widehat{x}_{n|n-1} + k_n\,(y_n - h\, \widehat{x}_{n|n-1}) \\ k_n & = & (h^2 p_{n|n-1} + r)^{-1} p_{n|n-1} h = \frac{p_{n|n-1} h}{h^2 p_{n|n-1} + r} = \frac{1}{h}\, \frac{p_{n|n-1}}{p_{n|n-1} + r/h^2} \end{array}\right.$$

$$\left\{\begin{array}{lcl} p_{n+1|n} & = & f^2\, p_{n|n} + q \\ p_{n|n} & = & \frac{1}{h}\, \frac{p_{n|n-1}}{\frac{r}{h^2}\, p_{n|n-1+1}} \end{array}\right.$$

We can eliminate the coupling between these equations and obtain

$$p_{n+1|n} = \frac{f^2\, p_{n|n-1}}{\frac{h^2}{r}\, p_{n|n-1} + 1} + q$$

As n increases, $p_{n+1|n}$ and so the gain k_n approaches a constant.

Kalman Filtering equations: 1D case

If $p_{n+1|n}$ does approach a constant, say p_∞, then p_∞ must satisfy

$$p_\infty = \frac{f^2\, p_\infty}{\frac{h^2}{r}\, p_\infty + 1} + q$$

This equation is quadratic and has a unique positive solution

$$p_\infty = \frac{1}{2}\left\{\left[\frac{r}{h^2}(1-f^2) - q\right]^2 + \frac{4rq}{h^2}\right\}^{1/2} - \frac{r}{2h^2}(1-f^2) + q$$

$$\begin{aligned} \left|p_{n+1|n} - p_\infty\right| &= f^2 \left|\frac{p_{n|n-1}}{\frac{h^2}{r}\, p_{n|n-1} + 1} - \frac{p_\infty}{\frac{h^2}{r}\, p_\infty + 1}\right| \\ &\leq f^2 \left|p_{n|n-1} - p_\infty\right| \end{aligned}$$

$$\left|p_{n+1|n} - p_\infty\right| \leq f^{2(n+1)} \left|p_0 - p_\infty\right| \leq f^2 \left|p_{n|n-1} - p_\infty\right|$$

- If $|f| < 1$ then $p_{n+1|n}$ converges to p_∞.
- $|f| < 1$ is a sufficient condition for Kalman-Bucy filter to approach a steady state.

Kalman Filtering: Applications

Track-While-Scan (TWS) Radar

$$\begin{aligned}\begin{bmatrix} X_{n+1} \\ V_{n+1} \end{bmatrix} &= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_n \\ V_n \end{bmatrix} + \begin{bmatrix} 0 \\ T \end{bmatrix} A_n \\ Y_n &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_n \\ V_n \end{bmatrix} + e_n \end{aligned}$$

- For a more general case in 3D we have a state vector with 6 components (3 positions and 3 velocities).
- But, if we assume that the measurement noise in 3 dimensions are independent of one another and independent to the components of the acceleration, the problem can be treated as 3 independent one-dimensional moving target.

Kalman Filtering: Applications

Track-While-Scan (TWS) Radar

$$\begin{aligned}
\begin{bmatrix} \widehat{X}_{n+1|n} \\ \widehat{V}_{n+1|n} \end{bmatrix} &= \begin{bmatrix} \widehat{X}_{n|n} + T\widehat{V}_{n|n} \\ \widehat{V}_{n|n} \end{bmatrix} \\
\begin{bmatrix} \widehat{X}_{n|n} \\ \widehat{V}_{n|n} \end{bmatrix} &= \begin{bmatrix} \widehat{X}_{n|n-1} \\ \widehat{V}_{n|n-1} \end{bmatrix} + \begin{bmatrix} K_{n,1} \\ K_{n,2} \end{bmatrix} \begin{bmatrix} Y_n - \widehat{X}_{n|n-1} \end{bmatrix} \\
\begin{bmatrix} K_{n,1} \\ K_{n,2} \end{bmatrix} &= \begin{bmatrix} \frac{P(1,1)}{P(1,1)+r} \\ \frac{P(2,1)}{P(1,1)+r} \end{bmatrix}
\end{aligned}$$

where $P(k,l)$ is the $(k-l)$th component of the matrix $P_{n|n-1}$. To reduce the computation, the time varying elements of the Kalman gain vector can be replaced with some constants (the steady states values) to obtain

$$\begin{bmatrix} \widehat{X}_{n|n} \\ \widehat{V}_{n|n} \end{bmatrix} = \begin{bmatrix} \widehat{X}_{n|n-1} \\ \widehat{V}_{n|n-1} \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix} \begin{bmatrix} Y_n - \widehat{X}_{n|n-1} \end{bmatrix}$$

with constatnt values for α and β.

Kalman Filtering: Applications

Track-While-Scan (TWS) Radar with dependent acceleration sequences

- heavy target:

$$A_{n+1} = \rho A_n + W_n, \quad n = 0, 1, \ldots$$

- ρ near to 0: low inertia target and ρ near to 1: high inertia target.

$$\begin{aligned}
\begin{bmatrix} X_{n+1} \\ V_{n+1} \\ A_{n+1} \end{bmatrix} &= \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} X_n \\ V_n \\ A_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} W_n \\
Y_n &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_n \\ V_n \\ A_n \end{bmatrix} + e_n
\end{aligned}$$

Kalman Filtering: Applications

- **Track-While-Scan (TWS) Radar**

$$\begin{bmatrix} \widehat{X}_{n+1|n} \\ \widehat{V}_{n+1|n} \\ \widehat{A}_{n+1|n} \end{bmatrix} = \begin{bmatrix} \widehat{X}_{n|n} + T\widehat{V}_{n|n} \\ \widehat{V}_{n|n} + T\widehat{A}_{n|n} \\ \rho\widehat{A}_{n|n} \end{bmatrix}$$

$$\begin{bmatrix} \widehat{X}_{n|n} \\ \widehat{V}_{n|n} \\ \widehat{A}_{n|n} \end{bmatrix} = \begin{bmatrix} \widehat{X}_{n|n-1} \\ \widehat{V}_{n|n-1} \\ \widehat{A}_{n|n-1} \end{bmatrix} + \begin{bmatrix} K_{n,1} \\ K_{n,2} \\ K_{n,3} \end{bmatrix} \begin{bmatrix} Y_n - \widehat{X}_{n|n-1} \end{bmatrix}$$

$$\begin{bmatrix} K_{n,1} \\ K_{n,2} \\ K_{n,3} \end{bmatrix} = \begin{bmatrix} \frac{P(1,1)}{P(1,1)+r} \\ \frac{P(2,1)}{P(1,1)+r} & \frac{P(3,1)}{P(1,1)+r} \end{bmatrix}$$

where $P(k,l)$ is the $(k-l)$th component of the matrix $P_{n|n-1}$.

- To reduce the computation:

$$\begin{bmatrix} \widehat{X}_{n|n} \\ \widehat{V}_{n|n} \\ \widehat{A}_{n|n} \end{bmatrix} \begin{bmatrix} \widehat{X}_{n|n-1} \\ \widehat{V}_{n|n-1} \\ \widehat{A}_{n|n-1} \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta/T \\ \gamma/T^2 \end{bmatrix} \begin{bmatrix} Y_n - \widehat{X}_{n|n-1} \end{bmatrix}$$

Kalman Filtering: Fast Kalman Algorithms

$$\begin{aligned}
\widehat{\mathbf{x}}_{k+1|k} &= \mathbf{F}\widehat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{R}^e_k)^{-1}[\mathbf{z}_k - \mathbf{H}\widehat{\mathbf{x}}_{k|k-1}] \\
\mathbf{R}^e_k &= \mathbf{R} + \mathbf{H}\mathbf{P}_{k|k-1}\,\mathbf{H}^t \\
\mathbf{K}_k &= \mathbf{F}\mathbf{P}_{k|k-1}\,\mathbf{H}^t \\
\mathbf{P}_{k+1|k} &= \mathbf{F}\mathbf{P}_{k|k-1}\,\mathbf{F}^t + \mathbf{G}\mathbf{Q}\mathbf{G}^t - \mathbf{K}_k(\mathbf{R}^e_k)^{-1}\,\mathbf{K}^t_k
\end{aligned}$$

$$\begin{aligned}
\delta\mathbf{P}_k &= \mathbf{P}_{k|k-1} - \mathbf{P}_{k-1|k-2} \\
\delta\mathbf{K}^g_k &= \mathbf{K}^g_k - \mathbf{K}^g_{k-1} \\
\delta\mathbf{R}^e_k &= \mathbf{R}^e_k - \mathbf{R}^e_{k-1}
\end{aligned}$$

$$\begin{aligned}
\delta\mathbf{P}_{k+1} &= [\mathbf{F} - \mathbf{K}^g_{k-1}\mathbf{H}][\delta\mathbf{P}_k - \delta\mathbf{P}_k\,\mathbf{H}^t(\mathbf{R}^e_k)^{-1}\,\mathbf{H}\mathbf{P}_k][\mathbf{F} - \mathbf{K}^g_{k-1}\mathbf{H}]^t \\
&= [\mathbf{F} - \mathbf{K}^g_k\mathbf{H}][\delta\mathbf{P}_k + \delta\mathbf{P}_k\,\mathbf{H}^t(\mathbf{R}^e_{k-1})^{-1}\,\mathbf{H}\mathbf{P}_k][\mathbf{F} - \mathbf{K}^g_k\mathbf{H}]^t
\end{aligned}$$

Kalman Filtering: Fast Kalman Algorithms

- If $\delta\mathbf{P}_k$ can be factorized as:

$$\delta\mathbf{P}_1 = \mathbf{z}_0\mathbf{M}_0\mathbf{z}_0^t, \longrightarrow \delta\mathbf{P}_{k+1} = \mathbf{z}_k\,\mathbf{M}_k\,\mathbf{z}_k^t$$

$$\begin{array}{rcl}
\delta\mathbf{P}_{k+1} & = & \mathbf{z}_k\,\mathbf{M}_k\,\mathbf{z}_k^t \\
\mathbf{z}_k & = & [\mathbf{F} - \mathbf{K}_k^g\mathbf{H}]\mathbf{z}_{k-1} \\
\mathbf{M}_k & = & \mathbf{M}_{k-1} + \mathbf{M}_{k-1}\,\mathbf{z}_{k-1}^t\mathbf{H}^t(\mathbf{R}_{k-1}^e)^{-1}\,\mathbf{H}\mathbf{z}_{k-1}\,\mathbf{M}_{k-1} \\
\mathbf{R}_{k-1}^e & = & \mathbf{R}_k^e + \mathbf{H}\mathbf{z}_k\,\mathbf{M}_k\,\mathbf{z}_k^t\mathbf{H}^t \\
\mathbf{K}_{k+1} & = & \mathbf{K}_{k+1}^g\,\mathbf{R}_{k+1}^e = \mathbf{K}_k + \mathbf{F}\mathbf{z}_k\,\mathbf{M}_k\,\mathbf{z}_k^t\mathbf{H}^t \\
\mathbf{P}_{k|k-1} & = & \mathbf{P}_0 + \displaystyle\sum_{j=0}^{k-1}\mathbf{z}_j\mathbf{M}_j\mathbf{z}_j^t
\end{array}$$

which are called *Chandrasekhar* equations.

Kalman Filtering: Fast Kalman Algorithms

- Note that if $\alpha = \text{rang}\,\{\delta\mathbf{P}_1\}$ where

$$\delta\mathbf{P}_1 = \mathbf{F}\mathbf{P}_0\mathbf{F}^t + \mathbf{G}\mathbf{Q}\mathbf{G}^t - \mathbf{K}_0(\mathbf{R}_0^e)^{-1}\mathbf{K}_0^t - \mathbf{P}_0$$

  then $\mathbf{z}_k$ has dimensions (N, α), $\mathbf{M}_k$ has dimensions (α, α).
- So, in place of updating $\mathbf{P}_k$ with dimensions (N, N) we only have to update the matrixes $\mathbf{z}_k$ and $\mathbf{M}_k$ with dimensions (N, α) and (α, α).
- Note also that $\mathbf{M}_0$ is the signature matrix of $\delta\mathbf{P}_1$ and the value of α depends on the choice of the initial covariance matrix $\mathbf{P}_0$.
- It is not unusual to have $\alpha = 1$ which greately reduces the computation cost.

Kalman Filter equations for signal deconvolution

$$y(k)=\sum_{i=0}^{p-1} h(i)x(k-i)+\epsilon(k)$$

$$\begin{bmatrix} y(1) \\ \vdots \\ y(k) \\ \vdots \\ y(M) \end{bmatrix} = \begin{bmatrix} h_{(p-1)} & \cdots & h_{(0)} & \cdots & \cdots \\ \vdots & & & & \vdots \\ 0 & h_{(p-1)} & \cdots & h_{(0)} & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & h_{(p-1)} & \cdots & h_{(0)} \end{bmatrix} \begin{bmatrix} x(-p) \\ \vdots \\ x(0) \\ \vdots \\ x(M) \end{bmatrix} + \begin{bmatrix} \epsilon(1) \\ \vdots \\ \epsilon(k) \\ \vdots \\ \epsilon(M) \end{bmatrix}$$

- **Constant state vector model**

$$\left\{ \begin{array}{lll} \mathbf{x}_{k+1} & = & \mathbf{x}_k = \mathbf{x} = [x_{-p}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n]^t \\ y_k & = & \mathbf{h}_k^t \cdot \mathbf{x}_k + v_k \end{array} \right.$$

$$\mathbf{h}_k = \begin{bmatrix} 0 & 0 & 0 & h_{p-1} & \ldots & h_0 & 0 & 0 \end{bmatrix}^t$$

where coefficient h_0 is in the k-th position.

Kalman Filter equations for signal deconvolution

- **Constant state vector model**

$$\left\{\begin{array}{lcl} \mathbf{u}_k & = & \mathbf{0} \\ \mathbf{F}_k & = & \mathbf{G}_k = \mathbf{I} \\ \mathbf{h}_{k+1}^t & = & \mathbf{D}\mathbf{h}_k^t \end{array}\right. \quad \text{with } \mathbf{D} = \begin{bmatrix} 0 & \dots & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \vdots & \vdots \\ \vdots & & & \vdots & \vdots \\ 0 & & \dots & 1 & 0 \end{bmatrix}$$

$$\mathrm{E}\left\{\mathbf{x}\right\} = \mathbf{x}_0 \qquad \mathrm{E}\left\{[\mathbf{x}-\mathbf{x}_0][\mathbf{x}-\mathbf{x}_0]^t\right\} = \mathbf{P}_0$$

$$\mathrm{E}\left\{v_k\right\} = 0 \qquad \mathrm{E}\left\{v_k v_j\right\} = r\,\delta_{kj}$$

$$\begin{array}{rcl} \widehat{\mathbf{x}}_{k|k} & = & \widehat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k (r_k^e)^{-1} [y_k - \mathbf{h}_k^t \cdot \widehat{\mathbf{x}}_{k|k-1}] \\ r_k^e & = & r + \mathbf{h}_k^t \mathbf{P}_{k|k-1}\,\mathbf{h}_k \\ \mathbf{K}_k & = & \mathbf{P}_{k|k-1}\,\mathbf{h}_k^t \\ \mathbf{P}_{k+1|k} & = & \mathbf{P}_{k|k-1} - \mathbf{K}_k (r_k^e)^{-1}\,\mathbf{K}_k^t \end{array}$$

- y_k and r_k^e are scalar.
- $\mathbf{x}$ is a N-dimensional.
- The covariance matrix $\mathbf{P}$ has the dimensions $(N \times N)$.

Kalman Filter equations for signal deconvolution

- **Non constant state space model**

$$\mathbf{h} = [h_0, \ldots, h_{p-1}]^t, \quad \mathbf{x}_k = [x_k, x_{k-1}, \ldots, x_{k-p+1}]^t$$

$$y(k) = \mathbf{h}^t \mathbf{x}_k + \epsilon(k), \quad \text{dimension of } \mathbf{x}_k = p \leq N$$

$$x(n+1) = \sum_{i=1}^{q} a(i)\, x(n-i+1) + u(n+1)$$

$$\mathrm{E}\left\{u_n\right\} = 0, \quad \mathrm{E}\left\{|u_n|^2\right\} = \beta^2, \quad \mathrm{E}\left\{u_m u_n\right\} = 0, \quad m \neq n$$

$$\begin{bmatrix} x(n+1) \\ x(n) \\ \vdots \\ \vdots \\ x(n-q+2) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \ldots & \ldots & a_q \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ \vdots \\ x(n-q+1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} u(n{+}1)$$

$$\left\{ \begin{array}{lcl} \mathbf{x}_{k+1} & = & \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{u}_{k+1} \\ y_k & = & \mathbf{h}^t \cdot \mathbf{x}_k + \epsilon_k \end{array} \right.$$

Kalman Filter equations for signal deconvolution

- **Non constant state space model**

$$\left\{\begin{array}{lcl} \mathbf{x}_{k+1} & = & \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{u}_{k+1} \\ y_k & = & \mathbf{h}^t \cdot \mathbf{x}_k + \epsilon_k \end{array}\right.$$

$$\mathbf{F} = \begin{bmatrix} a_1 & a_2 & \ldots & \ldots & a_q \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & 1 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Advantages: **F**, **G** and **H** are constant and we can use fast algorithms.
- Drawbacks: Determination of q and a_k, $k = 1, \ldots, q$.

Kalman Filter equations for signal deconvolution

$$\left\{\begin{array}{lll} \mathbf{x}_{k+1} & = \mathbf{F}_k\,\mathbf{x}_k + \mathbf{G}_k\,\mathbf{u}_k & \text{state equation} \\ \mathbf{y}_k & = \mathbf{H}_k\,\mathbf{x}_k + \mathbf{v}_k & \text{observation equation} \end{array}\right.$$

$$\begin{array}{rcl} \widehat{\mathbf{x}}_{k+1|k} & = & \mathbf{F}_k\,\widehat{\mathbf{x}}_{k|k} \\ \mathbf{P}_{k+1|k} & = & \mathbf{F}_k\,\mathbf{P}_{k|k}\,\mathbf{F}_k^t + \mathbf{G}_k\,\mathbf{Q}_k\,\mathbf{G}_k^t \\ \widehat{\mathbf{x}}_{k+1|k+1} & = & \widehat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}^f[\mathbf{y}_{k+1} - \mathbf{H}_{k+1}\,\widehat{\mathbf{x}}_{k+1|k}] \\ \mathbf{K}_{k+1}^f & = & \mathbf{P}_{k+1|k}\,\mathbf{H}_{k+1}^t(\mathbf{R}_{k+1}^e)^{-1} \\ \mathbf{R}_{k+1}^e & = & \mathbf{R}_{k+1} + \mathbf{H}_{k+1}\,\mathbf{P}_{k+1|k}\,\mathbf{H}_{k+1}^t \\ \mathbf{P}_{k+1|k+1} & = & [\mathbf{I} - \mathbf{K}_{k+1}^f\mathbf{H}_{k+1}]\mathbf{P}_{k+1|k} \end{array}$$

Case study: AR1 + FIR 3

$$\left\{\begin{array}{l} x(n) = a x(n-1) + u(n) \\ y(n) = h_0 x(n) + h_1 x(n-1) + h_2 x(n-2) + \epsilon(n) \end{array}\right.$$

$$\mathbf{x}_n = [x(n), x(n-1), x(n-2)]', \quad \mathbf{x}_{n+1} = [x(n+1), x(n), x(n-1)]'$$

$$\begin{bmatrix} x(n+1) \\ x(n) \\ x(n-1) \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \\ x(n-2) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(n)$$

$$y(n) = \begin{bmatrix} h_0 & h_1 & h_2 \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \\ x(n-2) \end{bmatrix} + v(n)$$

$$\mathbf{F} = \begin{bmatrix} a & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{g}, \quad \mathbf{H} = [h_0, h_1, h_2] = \mathbf{h}'$$

$$\left\{\begin{array}{lcl} \mathbf{x}_{n+1} & = & \mathbf{F}\,\mathbf{x}_n + \mathbf{G} u_n = \mathbf{F}\,\mathbf{x}_n + \mathbf{g} u_n \\ y_n & = & \mathbf{H}\,\mathbf{x}_n + v_n = \mathbf{h}'\,\mathbf{x}_n + v_n \end{array}\right.$$

Case study: AR1 + FIR 3

$$\left\{\begin{array}{ll} \widehat{\mathbf{x}}_{n+1|n} & = \mathbf{F}\widehat{\mathbf{x}}_{n|n} + \mathbf{P}_{n+1|n}\mathbf{H}' \left[\mathbf{R} + \mathbf{H}\mathbf{P}_{n+1|n}\mathbf{H}'\right]^{-1} \left[\mathbf{y}_{n+1} - \mathbf{H}\mathbf{F}\widehat{\mathbf{x}}_{n|n}\right] \\ \mathbf{P}_{n+1|n} & = \mathbf{F}\mathbf{P}_{n|n}\mathbf{F}' + \mathbf{G}\mathbf{Q}\mathbf{G}' \\ \mathbf{P}^{-1}_{n+1|n+1} & = \mathbf{P}^{-1}_{n+1|n} + \mathbf{H}'\mathbf{R}^{-1}\mathbf{H} \end{array}\right.$$

As $u(n)$ and $v(n)$ are scalars, then their covariances $\mathbf{Q}$ and $\mathbf{R}$ become scalar $q = \sigma_u = 1$ and $r = \sigma_\epsilon^2 = .01$. Also, As $\mathbf{y}_{n+1}$ is scalar, $\mathbf{G}$ is a vector $\mathbf{G} = \mathbf{g} = [1, 0, 0]'$ and $\mathbf{H} = \mathbf{h}' = [h_0, h_1, h_2]$.

$$\left\{\begin{array}{ll} \widehat{\mathbf{x}}_{n+1|n} & = \mathbf{F}\widehat{\mathbf{x}}_{n|n} + \mathbf{P}_{n+1|n}\mathbf{h} \left[r + \mathbf{h}'\mathbf{P}_{n+1|n}\mathbf{h}\right]^{-1} \left[y_{n+1} - \mathbf{h}'\mathbf{F}\widehat{\mathbf{x}}_{n|n}\right] \\ \mathbf{P}_{n+1|n} & = \mathbf{F}\mathbf{P}_{n|n}\mathbf{F}' + q\mathbf{g}\mathbf{g}' \\ \mathbf{P}^{-1}_{n+1|n+1} & = \mathbf{P}^{-1}_{n+1|n} + \frac{1}{r}\mathbf{h}\mathbf{h}' \end{array}\right.$$

We can also assume $x(-2) = x(-1) = x(0) = 0$ and so starting by $\widehat{\mathbf{x}}_{0|0} = [0, 0, 0]'$ and $\mathbf{P}^{-1}_{1|0} = \mathbf{P}_{1|0} = \mathbf{I}$, the first iteration for $n = 0$ becomes: $\widehat{\mathbf{x}}_{1|0} = \mathbf{h}\left[r + \mathbf{h}'\mathbf{h}\right]^{-1} y(1) = \frac{1}{r+\mathbf{h}'\mathbf{h}}\mathbf{h}y(1)$.

Then, from the third equation: $\mathbf{P}^{-1}_{1|1} = \mathbf{I} + \frac{1}{r}\mathbf{h}\mathbf{h}'$ and finally

$\mathbf{P}_{2|1} = \mathbf{F}\mathbf{P}_{1|1}\mathbf{F}' + q\mathbf{g}\mathbf{g}'$

We can then continue for $n = 1, 2, 3, ...$

Case study: AR1 + FIR 3

We could also use:

$$
\begin{array}{rcl}
\widehat{\mathbf{x}}_{n+1|n} & = & \mathbf{F}\widehat{\mathbf{x}}_{n|n} \\
\mathbf{P}_{n+1|n} & = & \mathbf{F}\mathbf{P}_{n|n}\mathbf{F}^t + q\mathbf{g}\mathbf{g}^t \\
\widehat{\mathbf{x}}_{n+1|n+1} & = & \widehat{\mathbf{x}}_{n+1|n} + \mathbf{k}_{n+1}[y(n+1) - \mathbf{h}'\widehat{\mathbf{x}}_{n+1|n}] \\
\mathbf{k}_{n+1} & = & \mathbf{P}_{n+1|n}\,\mathbf{h}(r_{n+1})^{-1} \\
r_{n+1} & = & r + \mathbf{h}'\,\mathbf{P}_{n+1|n}\,\mathbf{h} \\
\mathbf{P}_{n+1|n+1} & = & [\mathbf{I} - \mathbf{k}_{n+1}\mathbf{h}']\mathbf{P}_{n+1|n}
\end{array}
$$

Starting by $P_{1|0} = \mathbf{I}$ we get:
$r_1 = r + \mathbf{h}'\mathbf{h}$, $\mathbf{k}_1 = \mathbf{h}/r_1$, $\mathbf{P}_{1|1} = (\mathbf{I} - \mathbf{k}_1\mathbf{h}') = (\mathbf{I} - \frac{1}{r_1}\mathbf{h}\mathbf{h}')$
Now, we can set $\mathbf{x}_{1|0} = [0, 0, 0]'$ and so:
$\widehat{\mathbf{x}}_{1|1} = \widehat{\mathbf{x}}_{1|0} + \mathbf{k}_1[y(1) - \mathbf{h}'\widehat{\mathbf{x}}_{1|0}] = \frac{1}{r_1}\mathbf{h}'y(1)$
and then, we can go through the iterations for $n = 1, 2, \ldots$