Introduction to Communication, Control and Signal Processing

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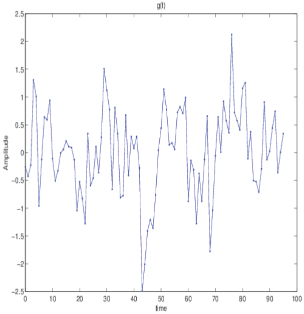
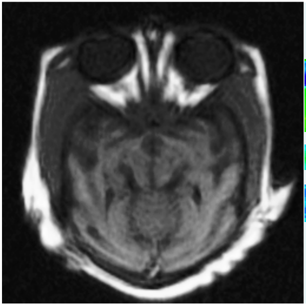
Signals and images

- Signal: $f(t), f(x), f(\nu)$
  - $f(t)$ Variation of temperature in a given position as a function of time t
  - $f(x)$ Variation of temperature as a function of the position x on a line
  - $f(\nu)$ Variation of temperature as a function of the frequency ν
- Image: $f(x, y), f(x, t), f(\nu, t), f(\nu_1, \nu_2)$
  - $f(x, y)$ Distribution of temperature as a function of the position (x, y)
  - $f(x, t)$ Variation of temperature as a function of x and t
  - ...
- 3D, 3D+t, 3D+ν, ... signals: $f(x, y, z), f(x, y, t), f(x, y, z, t)$
  - $f(x, y, z)$ Distribution of temperature as a function of the position (x, y, z)
  - $f(x, y, z, t)$ Variation of temperature as a function of (x, y, z) and t
  - ...

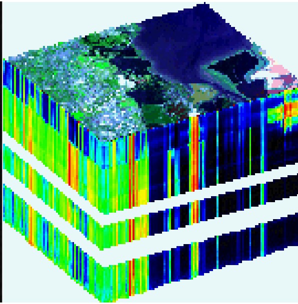
Representation of signals and images



1D signal
$f(t)$
variation of temperature



2D signal=image
$f(x, y)$
Medical image



3D signal
$f(x, y, \nu)$
hyperspectral image

Linear Transformations

$$g(\mathbf{s}) = \int_D f(\mathbf{r})\, h(\mathbf{r},\mathbf{s})\, \mathrm{d}\mathbf{r}$$

$$f(\mathbf{r}) \longrightarrow \boxed{h(\mathbf{r},\mathbf{s})} \longrightarrow g(\mathbf{s})$$

- 1–D :

$$g(t) = \int_D f(t')\, h(t,t')\, \mathrm{d}t'$$

$$g(x) = \int_D f(x')\, h(x,x')\, \mathrm{d}x'$$

- 2–D :

$$g(x,y) = \iint_D f(x',y')\, h(x,y;x',y')\, \mathrm{d}x'\, \mathrm{d}y'$$

$$g(r,\phi) = \iint_D f(x,y)\, h(x,y;r,\phi)\, \mathrm{d}x\, \mathrm{d}y$$

Linear and Invariant systems: convolution

$$h(\mathbf{r}, \mathbf{r}') = h(\mathbf{r} - \mathbf{r}')$$

$$f(\mathbf{r}) \longrightarrow \boxed{h(\mathbf{r})} \longrightarrow g(\mathbf{r}) = h(\mathbf{r}) * f(\mathbf{r})$$

- 1–D :

$$g(t) = \int_D f(t')\, h(t - t')\, \mathrm{d}t'$$

$$g(x) = \int_D f(x')\, h(x - x')\, \mathrm{d}x'$$

- 2–D :

$$g(x, y) = \iint_D f(x, y)\, h(x - x', y - y')\, \mathrm{d}x'\, \mathrm{d}y'$$

- $h(t)$ impulse response
- $h(x, y)$ Point Spread Function

Linear Transformations: Separable systems

$$g(\mathbf{s}) = \int_D f(\mathbf{r})\, h(\mathbf{r}, \mathbf{s})\, \mathrm{d}\mathbf{r}$$

$$h(\mathbf{r}, \mathbf{s}) = \prod_j h_j(r_j, s_j)$$

Examples:

- 2D Fourier Transform

$$g(\omega_x, \omega_y) = \iint f(x, y)\, \exp\left[-j(\omega_x x + \omega_y y)\right]\, \mathrm{d}x\, \mathrm{d}y$$

$$h(x, y, \omega_x, \omega_y) = h_1(\omega_x x)\, h_2(\omega_y y)$$

$$\exp\left[-j(\omega_x x + \omega_y y)\right] = \exp\left[-j(\omega_x x)\right]\, \exp\left[-j(\omega_y y)\right]$$

- nD Fourier Transform

$$g(\boldsymbol{\omega}) = \int f(\mathbf{x})\, \exp\left[-j\boldsymbol{\omega}'\mathbf{x}\right]\, \mathrm{d}\mathbf{x}$$

Fourier Transform

[Joseph Fourier, French Mathematicien (1768-1830)]

- 1D Fourier: $\mathcal{F}_1$

$$\begin{cases} g(\omega) & = \int f(t) \exp\left[-j\omega t\right] \, \mathrm{d}t \\ f(t) & = \frac{1}{2\pi} \int g(\omega) \exp\left[+j\omega t\right] \, \mathrm{d}\omega \end{cases}$$

- 2D Fourier: $\mathcal{F}_2$

$$\begin{cases} g(\omega_x, \omega_y) & = \iint f(x, y) \exp\left[-j(\omega_x x + \omega_y y\right] \, \mathrm{d}x \, \mathrm{d}y \\ f(x, y) & = (\frac{1}{2\pi})^2 \iint g(\omega_x, \omega_y) \exp\left[+j(\omega_x x + \omega_y y\right] \, \mathrm{d}\omega_x \, \mathrm{d}\omega_y \end{cases}$$

- nD Fourier: $\mathcal{F}_n$

$$\begin{cases} g(\boldsymbol{\omega}) & = \int f(\mathbf{x}) \exp\left[-j\boldsymbol{\omega}'\mathbf{x}\right] \, \mathrm{d}\mathbf{x} \\ f(\mathbf{x}) & = (\frac{1}{2\pi})^n \int g(\boldsymbol{\omega}) \exp\left[+j\boldsymbol{\omega}'\mathbf{x}\right] \, \mathrm{d}\boldsymbol{\omega} \end{cases}$$

1D Fourier Transform $\mathcal{F}_1$

$$\begin{cases} g(\omega) &= \int f(t) \exp\left[-j\omega t\right] \mathrm{d}t \\ f(t) &= \frac{1}{2\pi} \int g(\omega) \exp\left[+j\omega t\right] \mathrm{d}\omega \end{cases}$$

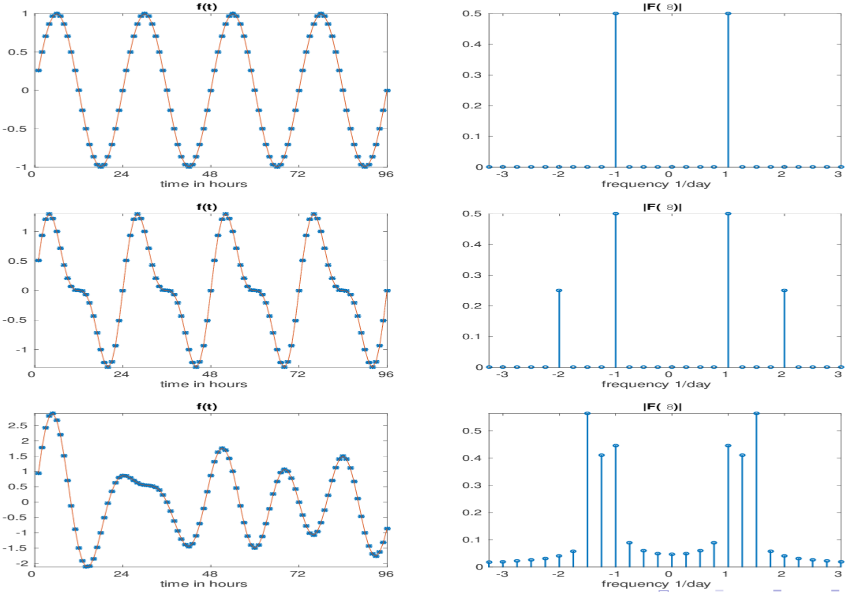
- $|g(\omega)|^2$ is called the spectrum of the signal $f(t)$
- For real valued signals $f(t)$, $|g(\omega)|$ is symmetric

Examples:

$f(t)$	$g(\omega)$
$\exp\left[-j\omega_0 t\right]$	$\delta(\omega - \omega_0)$
$\sin(\omega_0 t)$	?
$\cos(\omega_0 t)$	?
$\exp\left[-t^2\right]$	?
$\exp\left[-\frac{1}{2}(t-m)^2/\sigma^2\right]$	?
$\exp\left[-t/\tau\right], t > 0$	?
$1 \quad \text{if } \lvert t\rvert < T/2$	?

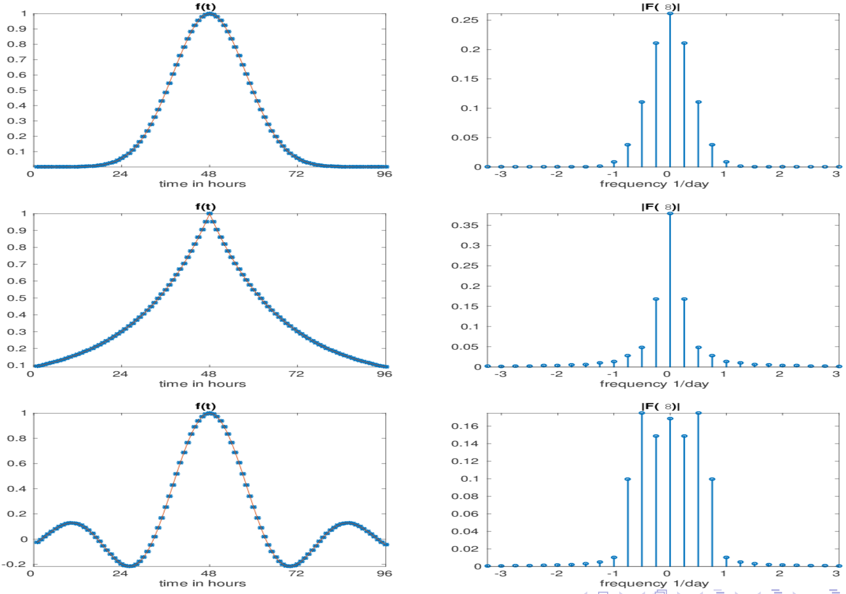
Time and Fourier representations

$f(t)$ $\quad\quad$ $|g(\omega)|$



Time and Fourier representations

$f(t)$ $|g(\omega)|$



2D Fourier Transform: $\mathcal{F}_2$

$$\left\{\begin{array}{lll} g(\omega_x,\omega_y) & = & \iint f(x,y) \exp\left[-j(\omega_x x+\omega_y y\right]\, \mathrm{d}x\, \mathrm{d}y \\ f(x,y) & = & (\frac{1}{2\pi})^2 \iint g(\omega_x,\omega_y) \exp\left[+j(\omega_x x+\omega_y y\right]\, \mathrm{d}\omega_x\, \mathrm{d}\omega_y \end{array}\right.$$

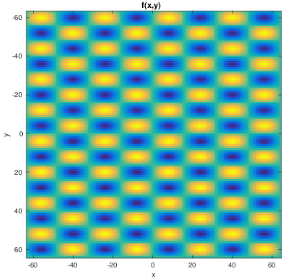
- $|g(\omega_x,\omega_y)|^2$ is called the spectrum of the image $f(x,y)$
- For real valued image $f(x,y)$, $|g(\omega_x,\omega_y)|$ is symetric with respect of the two axis ω_x and ω_y.

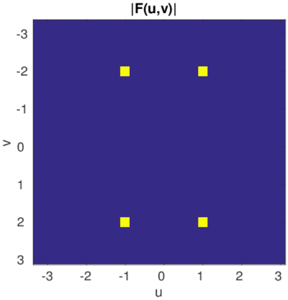
Examples:

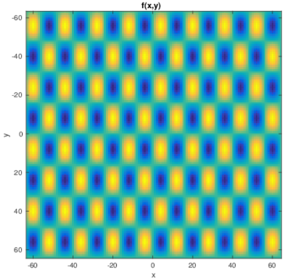
$f(x,y)$	$g(\omega_x,\omega_y)$
$\exp\left[-j(\omega_{x0}x+\omega_{y0}y)\right]$	$\delta(\omega_x-\omega_{x0})\delta(\omega_y-\omega_{y0})$
$\exp\left[-(x^2+y^2)\right]$	?
$\exp\left[-\frac{1}{2}[(x-m_x)^2/\sigma_x^2+(y-m_y)^2/\sigma_y^2]\right]$	?
$\exp\left[-(\lvert x\rvert+\lvert y\rvert)\right]$	?
$1 \quad \text{if } \lvert x\rvert < T_x/2\ \&\ \lvert y\rvert < T_y/2$	?
$1 \quad \text{if } (x^2+y^2) < a$	?

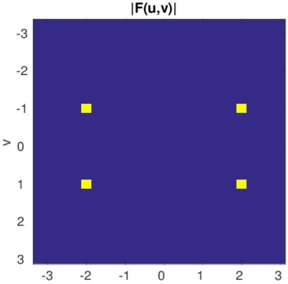
Time and Fourier representations

$f(x, y)$ $\qquad\qquad$ $|g(\omega_x, \omega_y)|$



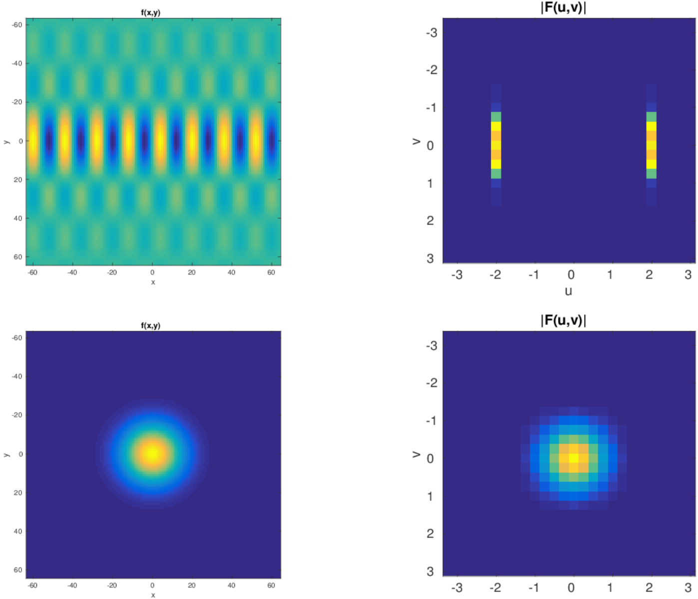






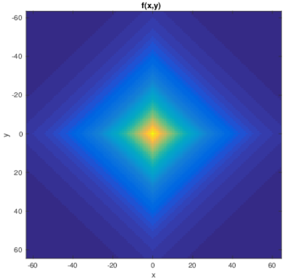
Time and Fourier representations

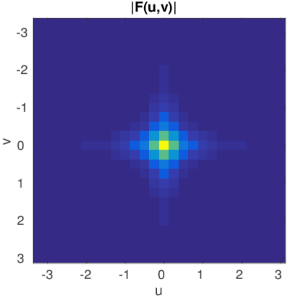
$f(x, y)$ $\qquad\qquad$ $|g(\omega_x, \omega_y)|$

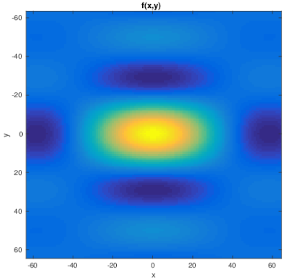


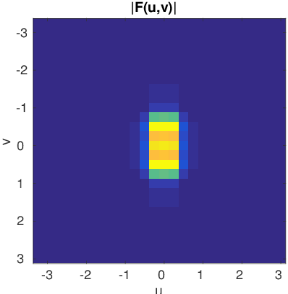
Time and Fourier representations

$f(x,y)$ $\qquad$ $|g(\omega_x,\omega_y)|$









nD Fourier Transform: $\mathcal{F}_n$

$$\begin{cases} g(\boldsymbol{\omega}) & = \int f(\mathbf{x}) \exp\left[-j\boldsymbol{\omega}'\mathbf{x}\right] \, \mathrm{d}\mathbf{x} \\ f(\mathbf{x}) & = (\frac{1}{2\pi})^n \int g(\boldsymbol{\omega}) \exp\left[+j\boldsymbol{\omega}'\mathbf{x}\right] \, \mathrm{d}\boldsymbol{\omega} \end{cases}$$

- $|g(\boldsymbol{\omega})|^2$ is called the spectrum of $f(\mathbf{x})$
- For real valued image $f(\mathbf{x})$, $|g(\boldsymbol{\omega})|$ is symetric with respect of all the axis ω_j.

Examples:

$f(\mathbf{x})$	$g(\boldsymbol{\omega})$
$\exp\left[-j(\boldsymbol{\omega}_0'\mathbf{x})\right]$	$\delta(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$
$\exp\left[-\mathbf{x}'\mathbf{x}\right] = \exp\left[-\|\mathbf{x}\|^2\right]$	$\exp\left[-\boldsymbol{\omega}'\boldsymbol{\omega}\right] = \exp\left[-\|\boldsymbol{\omega}\|^2\right]$
$\exp\left[-\|\mathbf{D}\mathbf{x}\|^2\right]$	?
$\exp\left[-\frac{1}{2}[(\mathbf{x}-\mathbf{m})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\mathbf{m})\right]$	?
$1 \quad \text{if } \|x\|^2 < R$	?
$1 \quad \text{if } |x_j| < R$	?

Laplace Transforms: $\mathcal{L}$

[Pierre-Simon Laplace, French Mathematicien (1749-1827)]
Let $f(t)$ be a signal with support in $[0,\infty)$ such that $\exp[-kt]\, f(t) \in L_1$ for some real number k.

$$F(s)=\int_0^\infty f(t)\,\exp[-s\,t]\,\mathrm{d}t$$

- $F(s)$ is defined at least in the right half of the complex plane defined by $\mathcal{R}\{s\}>k$.
- When the inversion conditions for the FT hold, we also have an inversion for the LT given by

$$f(t)=\frac{1}{j2\pi}\int_{a-j\infty}^{a+j\infty} F(s)\exp[+st]\,\mathrm{d}s, \quad \forall t>0$$

  where $a>k$ is a real number such that $\exp[-kt]\, f(t) \in L_1$
- Suppose $f(t)$ and $g(t)$ have support in $[0,\infty)$, $\exp[-k_1 t]\, f(t)$ and $\exp[-k_2 t]\, g(t)$ are in L_1, $f \leftrightarrow F$, and $g \leftrightarrow G$. Then:

$$h(t)=\int_0^t f(u)\,g(t-u)\,\mathrm{d}u=f(t)*g(t) \longleftrightarrow H(s)=F(s)G(s)$$

Laplace Transform: few examples

$f(t)$	$g(s)$
t	$1/s$
$\exp[at]$	$1/(s-a)$
$\sin \omega t$	$\omega/(s^2+\omega^2)$
...	

Hilbert Transform: $\mathcal{H}$

[David Hilbert, German mathematicien (1862-1943)]

- Definition: If $f \in L_2$ on $(-\infty, \infty)$,

$$g(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{t-x} \, \mathrm{d}t$$

$$f(t) = \frac{-1}{\pi} \int_{-\infty}^{+\infty} \frac{g(x)}{x-t} \, \mathrm{d}x$$

  The integrals are interpreted in the Cauchy principal value(CPV) sense at $t = x$.

- Alternate expression useful in signal processing:

$$g(t) = \frac{1}{\pi} \lim_{\epsilon \mapsto 0} \int_{\epsilon}^{\infty} \frac{f(t+\tau) - f(t-\tau)}{\tau} \, \mathrm{d}\tau$$

Hilbert Transform: $\mathcal{H}$

If $f \in L_2$

- $\mathcal{H}(\mathcal{H}(f)) = f$
- f and $\mathcal{H}(f)$ are orthogonal, i.e.,

$$\lim_{r\to\infty} \int_{-r}^{r} [f\mathcal{H}(f)](u)\,\mathrm{d}u = 0$$

- The Hilbert transform of a constant is zero.
- Hilbert and Fourier Transforms

$$\mathcal{H}(f) = f * \frac{-1}{\pi t} \quad \longrightarrow \mathcal{F}\{\mathcal{H}(f)\} = -j\mathrm{sgn}(\omega)F(\omega)$$

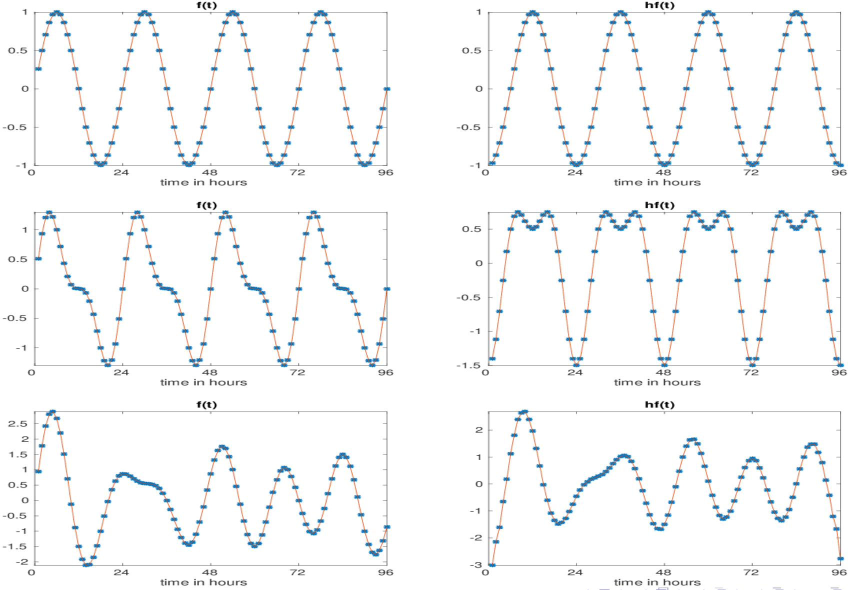
Hilbert Transform: $\mathcal{H}$

$$[\mathcal{H}f](t) = f(t) * \frac{-1}{\pi t} \quad \longrightarrow [\mathcal{H}f](t) = \mathcal{F}^{-1}\left\{-j\text{sgn}(\omega)[\mathcal{F}f](\omega)\right\}$$

$f(t)$	$g(t) = \mathcal{H}[f](t)$
$\sin(t)$	$-\cos(t)$
$\cos(t)$	$\sin(t)$
$\exp(jt)$	$j\exp(-jt)$
$\frac{1}{t^2+1}$	$\frac{t}{t^2+1}$
$\delta(t)$	$\frac{1}{\pi t}$
$\frac{\sin(t)}{t}$	$\frac{1-\cos(t)}{t}$

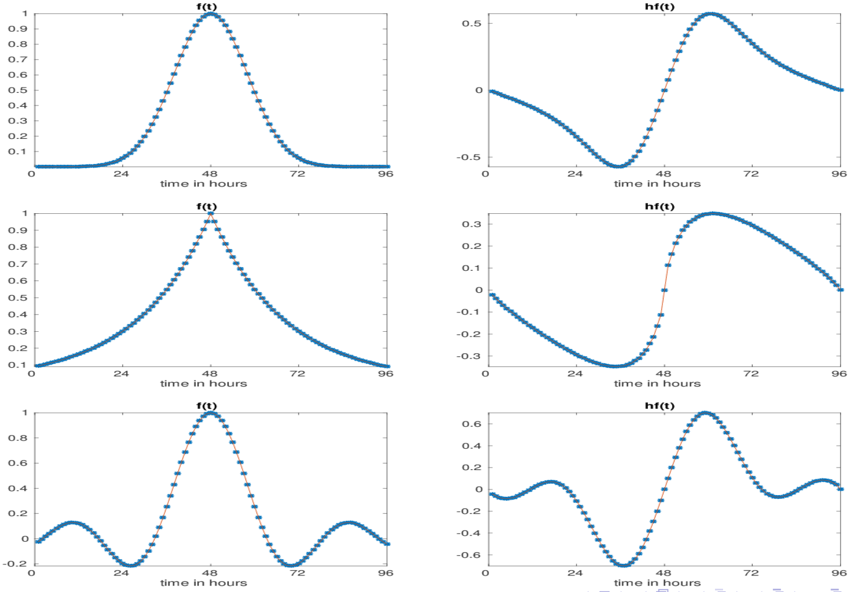
Hilbert Transform examples

$f(t)$ $\mathcal{H}[f](t)$



Time and Fourier representations

$f(t)$ $\qquad\qquad$ $\mathcal{H}[f](t)$



Background on Probability theory

- Why we need Probability theory ?
- What is the significance of probability ?
- What means a random variable ?

- Discrete variables $\{x_1, \cdots, x_n\}$
  Probability distribution: $\{p_1, \cdots, p_n\}$ with $\sum p_n = 1$
- Continuous variables $x \in \mathcal{R}$ or $x \in \mathcal{R}_+$ or $x \in [a, b]$
  Probability density function $p(x)$ with $\int_{-\infty}^{+\infty} p(x) \, \mathrm{d}x = 1$,
  Partition function: $F(x) = P(X \leq x) = \int_{\infty}^{x} p(x) \, \mathrm{d}x$
- Expected value: $\mathrm{E}\{X\} = \int x \, p(x) \, \mathrm{d}x$
- Variance value: $\mathrm{Var}\{X\} = \int (x - \mathrm{E}\{X\})^2 \, p(x) \, \mathrm{d}x$
- Mode value $\mathrm{Mode} = \arg\max_x \{p(x)\}$

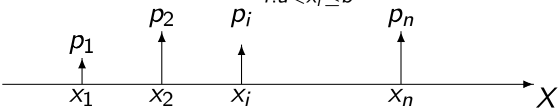
- Normal distribution $\mathcal{N}(x|m, v)$
- Gamma distribution $\mathcal{G}(x|\alpha, \beta)$

Discrete random variables

- X takes values x_i with probabilities $p_i, \; i = 1, \cdots, n$.
- $P(X = x_i) = p_i, \; i = 1, \cdots, n$ is probability distribution (pd).
- If we sort x_i in such a way that $x1 \leq x_2 \leq \cdots \leq x_n$, then we can define the "probability cumulative distribution (pcd)":

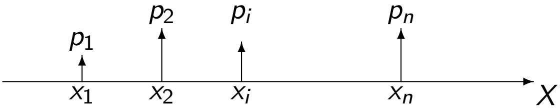
$$F(x) = P(X \leq x) = \sum_{i : x_i \leq x} P(X = x_i) \tag{1}$$

$$P(a < X \leq b) = \sum_{i : a < x_i \leq b} P(X = x_i) \tag{2}$$

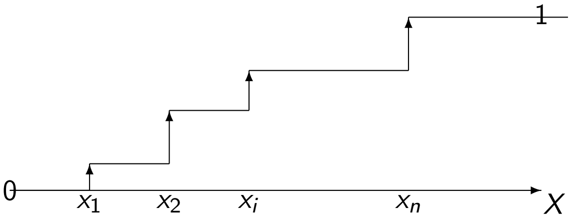


Discrete events

- Probabilities Distribution



- Cumulative Probablity Distribution



Discrete events

- Expected value

$$\mathrm{E}\{X\} = <X> = \sum_i p_i\, x_i \tag{3}$$

- Variance

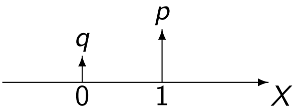
$$\mathrm{Var}\{X\} = \sum_i p_i\, (x_i - \mathrm{E}\{X\})^2 = \sum_i p_i\, (x_i - <X>)^2 \tag{4}$$

- Entropy

$$\mathrm{H}(X) = -\sum_i p_i \ln p_i \tag{5}$$

Discrete variables probability distributions

- Bernouilli distribution: A variable with two outcomes only
  $X = \{0, 1\}, \quad P(X = 1) = p, \quad P(X = 0) = q = 1 - p$



- Bernoulli trial $B(n, p)$: n independent trials of an experiment with two outcomes only 0010001100000010
  - p probability of success
  - $q = 1 - p$ probability of failure
- Binomial distribution $\mathcal{B}in(.|n, p)$:
  The probability of k successes in n trials:

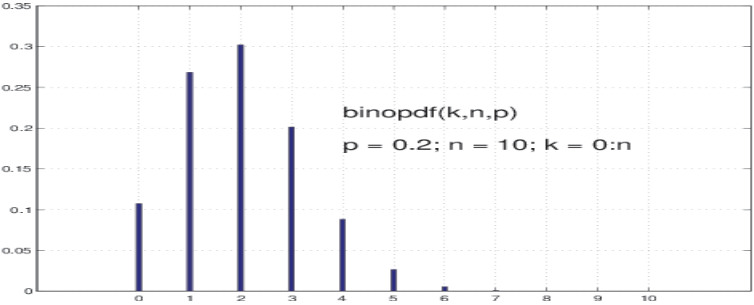
$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \tag{6}$$

Binomial distribution $\mathcal{B}in(.|n, p)$

The probability of k successes in n trials:

$$P(X = k) = \binom{n}{k} p^k \, (1-p)^{n-k}, k = 0, 1, \cdots, n \tag{7}$$

$$\mathrm{E}\{X\} = n\,p, \quad \mathrm{Var}\{X\} = n\,p\,q = n\,p\,(1-p) \tag{8}$$



Poisson distribution

- The Poisson distribution can be derived as a limiting case to the binomial distribution as the number of trials goes to infinity and the expected number of successes remains fixed

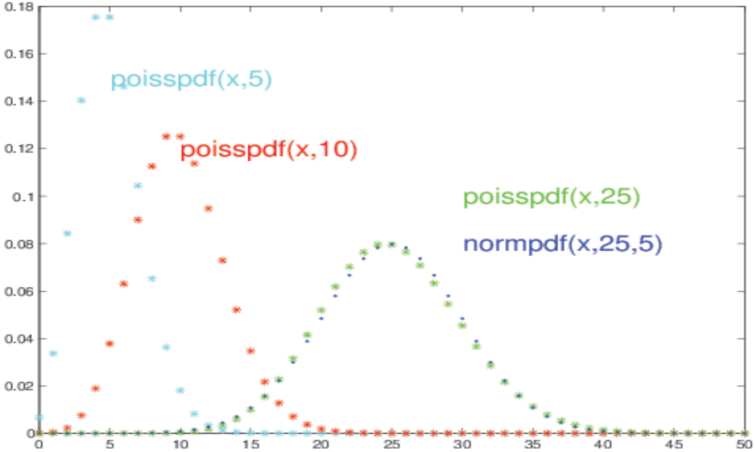
$$X \sim \mathcal{B}in(n, p) \lim_{n \mapsto \infty, np \mapsto \lambda} X \sim \mathcal{P}(\lambda) \tag{9}$$

$$P(X = k|\lambda) = \frac{\lambda^k \exp[-\lambda]}{k!} \tag{10}$$

$$\mathrm{E}\{X\} = \lambda, \quad \mathrm{Var}\{X\} = \lambda \tag{11}$$

- If $X_n \sim \mathcal{B}in(n, \lambda/n)$ and $Y \sim \mathcal{P}(\lambda)$ then for each fixed k, $\lim_{n \to \infty} P(X_n = k) = P(Y = k)$.

Poisson distribution



Continuous case

- Cumulative Distribution Function (cdf): $F(x) = P(X < x)$
- Measure theory

$$\begin{aligned} P(a \leq X < b) &= F(b) - F(a) \\ P(x \leq X < x + dx) &= F(x + dx) - F(x) = \mathrm{d}F(x) \end{aligned}$$

- If $F(x)$ is a continuous function

$$p(x) = \frac{\partial F(x)}{\partial x} \tag{12}$$

- $p(x)$ probability density function (pdf)

$$P(a < X \leq b) = \int_a^b p(x)\,\mathrm{d}x \tag{13}$$

- Cumulative distribution function (cdf)

$$F(x) = \int_{-\infty}^{x} p(x)\,\mathrm{d}x \tag{14}$$

Continuous case

- Expected value

$$\mathrm{E}\left\{X\right\} = \int x\, p(x)\, \mathrm{d}x =< X > \tag{15}$$

- Variance

$$\mathrm{Var}\left\{X\right\} = \int (x - \mathrm{E}\left\{X\right\})^2\, p(x)\, \mathrm{d}x = \left\langle (x - \mathrm{E}\left\{X\right\})^2 \right\rangle \tag{16}$$

- Entropy

$$\mathrm{H}(X) = \int -\ln p(x)\, p(x)\, \mathrm{d}x = \langle -\ln p(X) \rangle \tag{17}$$

- Mode: $\mathrm{Mode}(X) = \arg\max_x \left\{p(x)\right\}$
- Median $\mathrm{Med}(X)$:

$$\int_{-\infty}^{\mathrm{Med}(X)} p(x)\, \mathrm{d}x = \int_{\mathrm{Med}(X)}^{+\infty} p(x)\, \mathrm{d}x$$

Uniform and Beta distributions

- Uniform:

$$X \sim \mathrm{U}(.|a,b) \longrightarrow p(x) = \frac{1}{b-a}, \quad x \in [a,b] \qquad (18)$$

$$\mathrm{E}\{X\} = \frac{a+b}{2}, \quad \mathrm{Var}\{X\} = \frac{(b-a)^2}{12} \qquad (19)$$

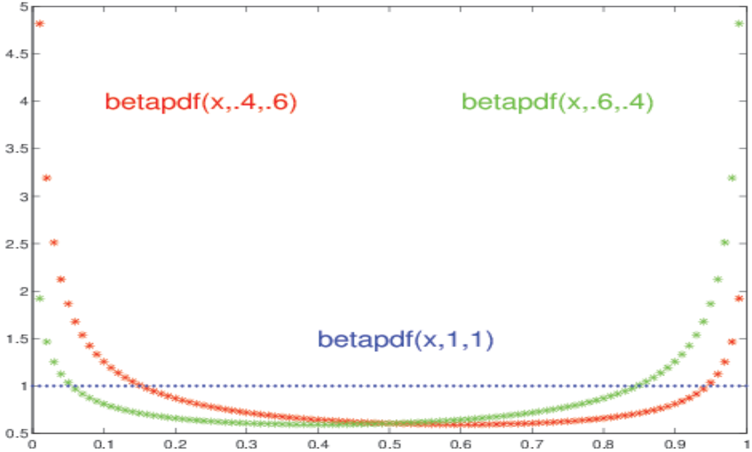
- Beta:

$$X \sim \mathrm{Beta}(.|\alpha,\beta) \longrightarrow p(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, x \in [0,1] \qquad (20)$$

$$\mathrm{E}\{X\} = \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}\{X\} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \qquad (21)$$

- $\mathrm{Beta}(.|1,1) = \mathrm{U}(.|0,1)$

Uniform and Beta distributions



Gaussian distributions

Different notations:

- classical one with mean and variance:

$$X \sim \mathcal{N}(.|\mu, \sigma^2) \longrightarrow p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] \tag{22}$$

$$\mathrm{E}\{X\} = \mu, \quad \mathrm{Var}\{X\} = \sigma^2 \tag{23}$$

- mean and precision parameters:

$$X \sim \mathcal{N}(.|\mu, \lambda) \longrightarrow p(x) = \frac{\lambda}{\sqrt{2\pi}} \exp\left[-\frac{\lambda}{2}(x-\mu)^2\right] \tag{24}$$

$$\mathrm{E}\{X\} = \mu, \quad \mathrm{Var}\{X\} = \sigma^2 = \frac{1}{\lambda} \tag{25}$$

Generalized Gaussian distributions

- Gaussian:

$$X \sim \mathcal{N}(.|\mu, \sigma^2) \longrightarrow p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2\right] \quad (26)$$

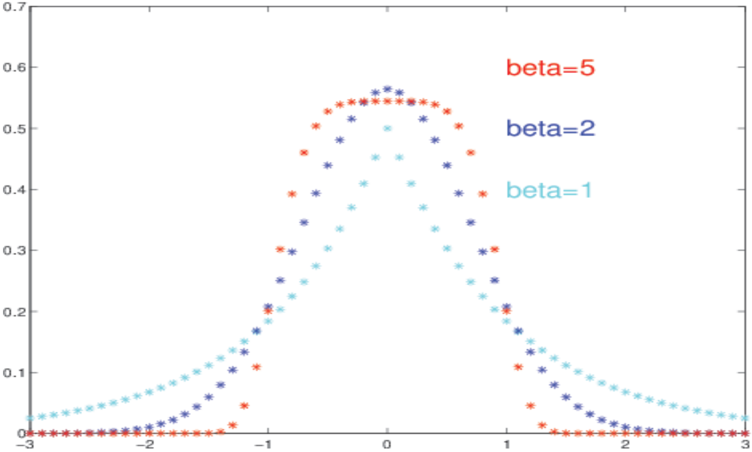
- Generalized Gaussian:

$$X \sim \mathcal{GG}(.|\alpha, \beta) \longrightarrow p(x) = \frac{\beta}{2\alpha\Gamma(1/\beta)} \exp\left[-\left(\frac{|x-\mu|}{\alpha}\right)^\beta\right] \quad (27)$$

$$\mathrm{E}\{X\} = \mu, \quad \mathrm{Var}\{X\} = \frac{\alpha^2\Gamma(3/\beta)}{\gamma(1/\beta)} \quad (28)$$

- $\beta > 0$, $\beta = 1$ Laplace, $\beta = 2$: Gaussian, $\beta \mapsto \infty$: Uniform

Gaussian and Generalized Gaussian distributions



Gamma distributions

- Forme 1:

$$p(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \text{ for } x \geq 0 \tag{29}$$

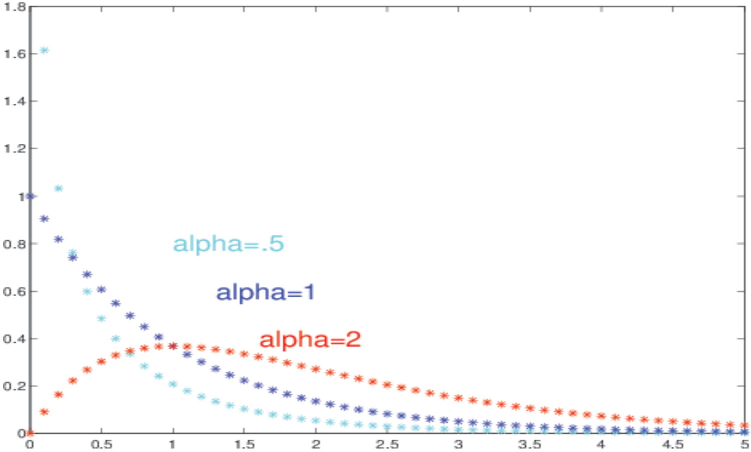
$$\mathrm{E}\{X\} = \frac{\alpha}{\beta}, \quad \mathrm{Var}\{X\} = \frac{\alpha}{\beta^2}, \quad \mathrm{Mod}(X) = \frac{\alpha-1}{\alpha+\beta-2} \tag{30}$$

- Forme 2: $\theta = 1/\beta$

$$p(x|\alpha,\theta) = \frac{\theta^{-\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \text{ for } x \geq 0 \tag{31}$$

- $\alpha = 1$: Exponential,
- $0 < \alpha < 1$ decreasing,
- $\alpha > 1$ Mode$=\frac{\alpha-1}{\beta}$

Gamma distributions



Student-t and Cauchy distributions

- Student's t-distribution has the probability density function:

$$p(x|\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} = \frac{1}{\sqrt{\nu}\,B\left(\frac{1}{2},\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} \quad (32)$$

  where
  - ν is the number of degrees of freedom,
  - Γ is the Gamma function and
  - B is the Beta function.
- $\nu = 1$ gives Cauchy distribution.

$$p(x) = \frac{\pi}{1+x^2} \quad (33)$$

- Cauchy distribution:

$$p(t|\mu) = \frac{\pi}{1+(x-\mu)^2} \quad (34)$$

Student-t and Cauchy distributions

- Three parameters location (μ) / scale (λ) / degree of freedom (ν) version

$$p(x|\mu,\lambda,\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \left(\frac{\lambda}{\pi\nu}\right)^{\frac{1}{2}} \left[1 + \frac{\lambda(x-\mu)^2}{\nu}\right]^{-\frac{\nu+1}{2}} \qquad (35)$$

$$\begin{array}{lll} \mathrm{E}\{X\} & = \mu & \text{for } \nu > 1, \\ \mathrm{Var}\{X\} & = \frac{1}{\lambda}\frac{\nu}{\nu-2} & \text{for } \nu > 2, \\ \mathrm{mode}(X) & = \mu. & \end{array} \qquad (36)$$

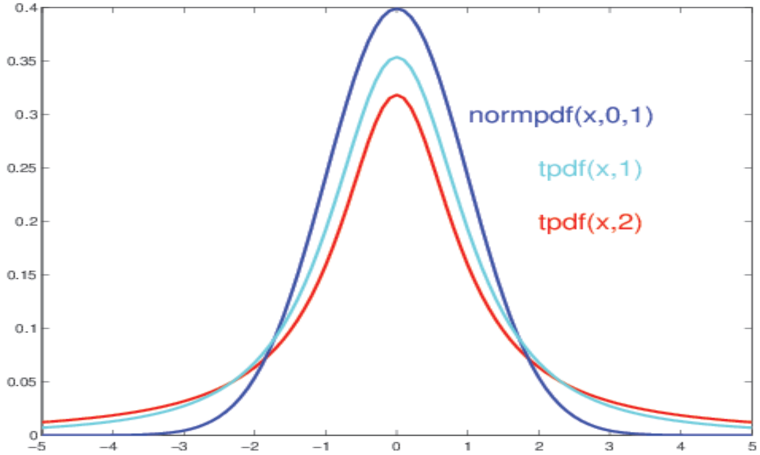
- Interesting relation between Student-t, Normal and Gamma distributions:

$$\mathcal{S}(x|\mu,1,\nu) = \int \mathcal{N}(x|\mu,1/\lambda)\,\mathcal{G}(\lambda|\nu/2,\nu/2)\,\mathrm{d}\lambda \qquad (37)$$

$$\mathcal{S}(x|0,1,\nu) = \int \mathcal{N}(x|0,1/\lambda)\,\mathcal{G}(\lambda|\nu/2,\nu/2)\,\mathrm{d}\lambda \qquad (38)$$

Student and Cauchy

$$p(x|\nu) \propto \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} \tag{39}$$



Vector variables

- Vector variables: $\mathbf{X} = [X_1, X_2, \cdots, X_n]'$
- $p(\mathbf{x})$ probability density function (pdf)
- Expected value

$$\mathrm{E}\{\mathbf{X}\} = \int \mathbf{x}\, p(\mathbf{x})\, \mathrm{d}\mathbf{x} =< \mathbf{X} > \tag{40}$$

- Covariance

$$\begin{aligned} \operatorname{cov}[\mathbf{X}] &= \int (\mathbf{X} - \mathrm{E}\{\mathbf{X}\})(\mathbf{X} - \mathrm{E}\{\mathbf{X}\})'\, p(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= \left\langle (\mathbf{X} - \mathrm{E}\{\mathbf{X}\})(\mathbf{X} - \mathrm{E}\{\mathbf{X}\})' \right\rangle \end{aligned}$$

- Entropy

$$\mathrm{E}(\mathbf{X}) = \int -\ln p(\mathbf{x})\, p(\mathbf{x})\, \mathrm{d}\mathbf{x} = \langle \ln p(\mathbf{X}) \rangle \tag{41}$$

- Mode: $\mathrm{Mode}(p(\mathbf{x})) = \arg\max_{\mathbf{x}} \{p(\mathbf{x})\}$

Vector variables

- Case of a vector with 2 variables: $\mathbf{X} = [X_1, X_2]'$
- $p(\mathbf{x}) = p(x_1, x_2)$ joint probability density function (pdf)
- Marginals

$$\begin{aligned} p(x_1) &= \int p(x_1, x_2)\, \mathrm{d}x_2 \\ p(x_2) &= \int p(x_1, x_2)\, \mathrm{d}x_1 \end{aligned}$$

- Conditionals

$$\begin{aligned} p(x_1|x_2) &= \frac{p(x_1, x_2)}{p(x_2)} \\ p(x_2|x_1) &= \frac{p(x_1, x_2)}{p(x_1)} \end{aligned}$$

Multivariate Gaussian

Different notations:

- mean and covariance matrix (classical): $\mathbf{X} \sim \mathcal{N}(.|\boldsymbol{\mu}, \boldsymbol{\sigma})$

$$p(\mathbf{x}) = (2\pi)^{-n/2}|\boldsymbol{\Sigma}|^{-1/2} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right] \quad (42)$$

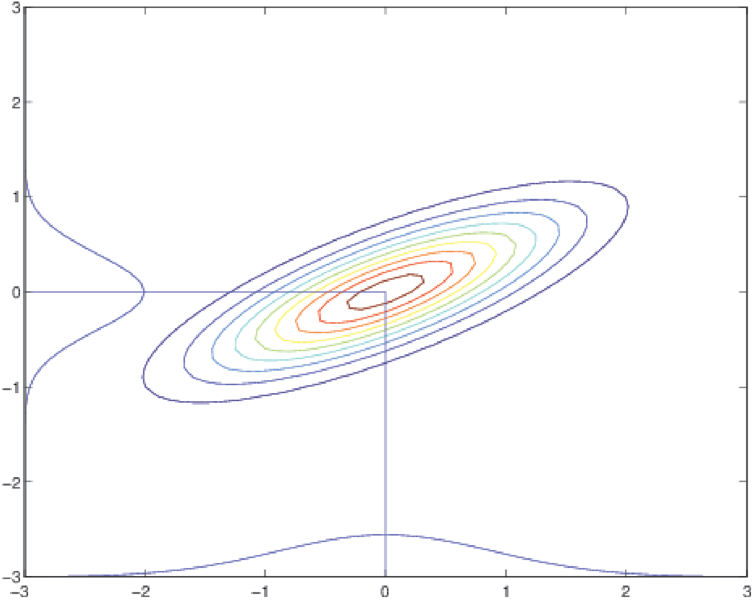
$$\mathrm{E}\{\mathbf{X}\} = \boldsymbol{\mu}, \quad \mathrm{cov}[\mathbf{X}] = \boldsymbol{\Sigma} \quad (43)$$

- mean and precision matrix: $\mathbf{X} \sim \mathcal{N}(.|\boldsymbol{\mu}, \boldsymbol{\Lambda})$

$$p(\mathbf{x}) = (2\pi)^{-n/2}|\boldsymbol{\Lambda}|^{1/2} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Lambda}(\mathbf{x}-\boldsymbol{\mu})\right] \quad (44)$$

$$\mathrm{E}\{\mathbf{X}\} = \boldsymbol{\mu}, \quad \mathrm{cov}[\mathbf{X}] = \boldsymbol{\Lambda}^{-1} \quad (45)$$

Multivariate normal distributions



Multivariate Student-t

$$p(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu) \propto |\boldsymbol{\Sigma}|^{-1/2} \left[1 + \frac{1}{\nu}(\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right]^{(\nu+p)/2} \quad (46)$$

- $p = 1$

$$f(t) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\nu\pi}} (1 + t^2/\nu)^{\frac{-(\nu+1)}{2}} \quad (47)$$

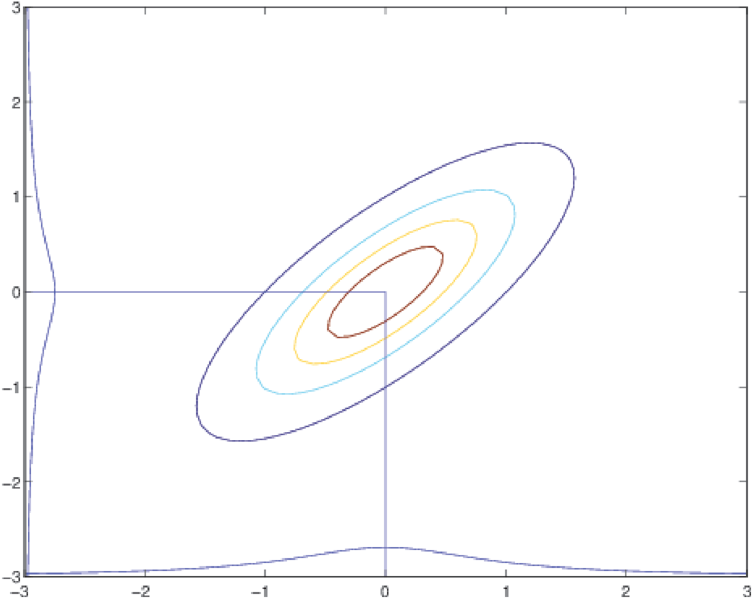
- $p = 2$, $\boldsymbol{\Sigma}^{-1} = \mathbf{A}$

$$f(t_1, t_2) = \frac{\Gamma((\nu+p)/2)}{\Gamma(\nu/2)\sqrt{\nu^p \pi^p}} \frac{|\mathbf{A}|^{1/2}}{2\pi} \left(1 + \sum_{i=1}^{p} \sum_{j=1}^{p} A_{ij} t_i \, t_j / \nu \right)^{\frac{-(\nu+2)}{2}} \quad (48)$$

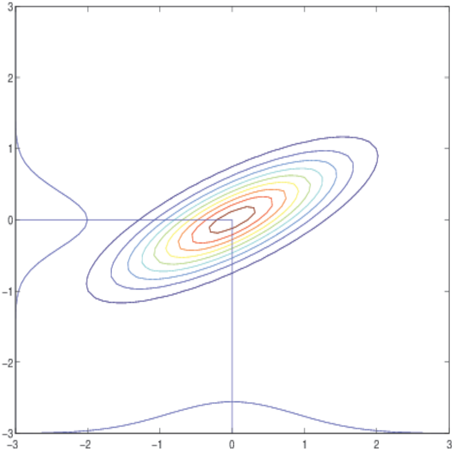
- $p = 2$, $\boldsymbol{\Sigma} = \mathbf{A} = \mathbf{I}$

$$f(t_1, t_2) = \frac{1}{2\pi} (1 + (t_1^2 + t_1^2)/\nu)^{\frac{-(\nu+2)}{2}} \quad (49)$$

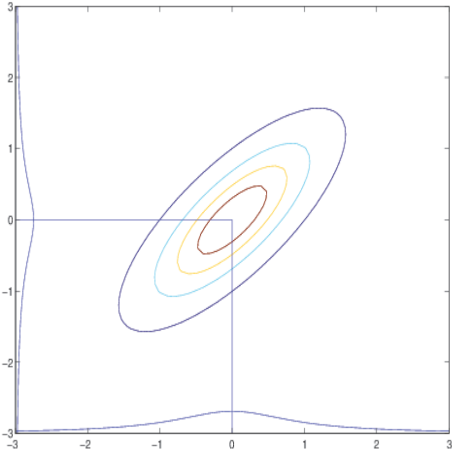
Multivariate Student-t distributions



Multivariate normal distributions



Normal



Student-t