Detecting and estimating the shape of a periodic component in short duration signals

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Description of the problem

- Detecting a periodic component in a short duration signal, estimating its period and its shape.
- Low noise example

![Graph showing a periodic component with low noise.](image1)

- High noise example

![Graph showing a periodic component with high noise.](image2)
Description of the problem

- Detecting a periodic component in a short duration signal, estimating its period and its shape.
  - Low noise example

- High noise example
Description of the problem

\[ \mathbf{g} = [g_1, \ldots, g_N, g_{N+1}, \ldots, \ldots, g_{KN+1}, \ldots, g_M]' \]
\[ = [f_1, \ldots, f_N, f_1 \ldots, f_N, \ldots, f_1, \ldots] ' \]

where \( M = KN + r \) where \( K \) is the number of complete repetition of the periodic shape and \( r \) is the rest.

This relation can be written as a linear relation:

\[ \mathbf{g} = \mathbf{H}_N \mathbf{f} \]

where \( \mathbf{H} \) has the following structure

\[ \mathbf{H}_N = [\mathbf{I}_N|\mathbf{I}_N|\ldots|\mathbf{I}_N|\mathbf{I}(\cdot,1:r)] \]

where \( \mathbf{I}_N \) is the unitary matrix of size \( N \times N \) and \( \mathbf{I}(\cdot,1:r) \) is its first \( r \) columns.

\[ \mathbf{g} = \mathbf{H}_N \mathbf{f} + \epsilon \]

the vector \( \epsilon \) represents those errors.
Also, we require some regularity in the shape of \( f \). This regularity can be modelled as

\[
f = Df + \xi \rightarrow (I - D)f = \xi \rightarrow Cf = \xi
\]

where \( C \) can be of the form

\[
C_N = \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
0 & 1 & -1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1 & -1 \\
-1 & 0 & \cdots & 0 & 1
\end{bmatrix}
\]

A criterion which measures the regularity can be \( \|Cf\|^2 \)
Deterministic regularization method

\[ g = H_N f + \epsilon, \quad C f = \xi \]

With these two equations, we have at least two possibilities:

- **Deterministic regularization:**
  \[
  \hat{f} = \arg \min_f \{ J(f) \} \quad \text{with} \quad J(f) = \| g = H_N f \|_2^2 + \lambda \| C_N f \|_2^2
  \]
  The solution is given by:
  \[
  \hat{f} = \left( H_N' H_N + \lambda C_N' C_N \right)^{-1} H_N' g
  \]

- The above criterion was given for a given value of \( N \). We can now try to define a criterion which depends explicitly on \( N \):
  \[
  J(N, f) = \| g = H_N f \|_2^2 + \lambda \| C_N f \|_2^2
  \]
  and try to optimize it to find both the sought period \( N \) and the shape \( f \):
  \[
  (\hat{N}, \hat{f}) = \arg \min_{(N, f)} \{ J(N, f) \}
  \]
Bayesian approach

\[ g = H_N f + \epsilon, \quad Cf = \xi \]

- **Likelihood**
  \[ p(g|f) = \mathcal{N}(g|H_N f, \nu_\epsilon I), \]

- **Prior**
  \[ p(f) = \mathcal{N}(f|0, \nu_\xi (C'_N C_N)^{-1}) \]

- **Posterior**
  \[ p(f|g) \propto p(g|f)p(f) = \mathcal{N}(f|\hat{f}, \hat{\Sigma}) \]
  with
  \[ \hat{f} = \arg \max_f \{ p(f|g) \} = \arg \min_f \{ J(f) \} \]

which is equivalent to the quadratic regularization as before

\[ \hat{f} = (H'_N H_N + \lambda C'_N C_N)^{-1} H'_N g \]

with \( \lambda = \nu_\epsilon / \nu_\xi \).
Bayesian approach

- **Poserion**

\[ p(f|g) \propto p(g|f)p(f) = \mathcal{N}(f|\hat{f}, \hat{\Sigma}) \]

with

\[ \hat{f} = \arg \max_f \{ p(f|g) \} = \arg \min_f \{ J(f) \} \]

and

\[ \hat{\Sigma} = [H_N H_N + \lambda C_N C_N]^{-1} \]

which can be used to put error bars on the solution.

- **Joint posterior** \( p(N, f|g) \) and and try to optimize it to find the seeked solution

\[ (\hat{N}, \hat{f}) = \arg \max_{(N, f)} \{ p(N, f|g) \} \]

- **We can do better.**
Bayesian approach with more appropriate priors

- Forward and prior model equations:

\[ g = H_N f + \epsilon, \quad C f = \xi \]

- \( \epsilon_i \) and \( \xi_j \) are Gaussian but with unknown variances that we want to estimate to.

\[
p(\epsilon_i | v_{\epsilon_i}) = \mathcal{N}(\epsilon_i | 0, v_{\epsilon_i}), \quad p(v_{\epsilon_i} | \alpha_{\epsilon_0}, \beta_{\epsilon_0}) = \mathcal{IG}(v_{\epsilon_i} | \alpha_{\epsilon_0}, \beta_{\epsilon_0})
\]

\[
p(\xi_i | v_{\xi_i}) = \mathcal{N}(\xi_i | 0, v_{\xi_i}), \quad p(v_{\xi_i} | \alpha_{\xi_0}, \beta_{\xi_0}) = \mathcal{IG}(v_{\xi_i} | \alpha_{\xi_0}, \beta_{\xi_0})
\]

- This can also be interpreted as a wish to model them by a heavier tailed probability laws such as Student-t:

\[
S_t(\epsilon_i | \alpha_{\epsilon_i}, \beta_{\epsilon_i}) = \int_0^\infty \mathcal{N}(\epsilon_i | 0, v_{\epsilon_i}) \mathcal{IG}(v_{\epsilon_i} | \alpha_{\epsilon_0}, \beta_{\epsilon_0}) \, dv_{\epsilon_i}
\]

\[
S_t(\xi_j | \alpha, \beta) = \int_0^\infty \mathcal{N}(\xi_j | 0, v_{\xi_j}) \mathcal{IG}(v_{\xi_j} | \alpha_{\xi_0}, \beta_{\xi_0}) \, dv_{\xi_j}
\]
Non stationary noise and sparsity enforcing model

- Forward model:
  \( g = Hf + \epsilon, \ \epsilon_i \sim N(\epsilon_i|0, \nu_{\epsilon_i}) \rightarrow \epsilon \sim N(\epsilon|0, \nu_{\epsilon}), \nu_{\epsilon} = \text{diag} [\nu_{\epsilon 1}, \cdots, \nu_{\epsilon M}] \)

- Prior model:
  \( C_N f = \xi, \ \xi_j \sim N(\xi_j|0, \xi_j) \rightarrow \xi \sim N(\xi|0, \nu_{\xi}), \nu_{\xi} = \text{diag} [\nu_{\xi 1}, \cdots, \nu_{\xi N}] \)

\[
\begin{align*}
  p(g|f, \nu_{\epsilon}) &= N(g|Hf, \nu_{\epsilon}), \quad \nu_{\epsilon} = \text{diag} [\nu_{\epsilon}] \\
p(f|\nu_f) &= N(f|0, \nu_{\xi}CC'), \quad \nu_{\xi} = \text{diag} [\nu_{\xi}] \\
p(\nu_{\epsilon}) &= \prod_i IG(\nu_{\epsilon_i}|\alpha_{\epsilon 0}, \beta_{\epsilon 0}) \\
p(\nu_{\xi}) &= \prod_j IG(\nu_{\xi_j}|\alpha_{\xi 0}, \beta_{\xi 0}) \\
p(f, \nu_{\epsilon}, \nu_{\xi}|g) &\propto p(g|f, \nu_{\epsilon}) p(f|\nu_{\xi}) p(\nu_{\epsilon}) p(\nu_{\xi})
\end{align*}
\]

Objective: Infer \((f, \nu_{\epsilon}, \nu_{\xi})\)

- VBA: Approximate \(p(f, \nu_{\epsilon}, \nu_{\xi}|g)\)
  by \(q_1(f) q_2(\nu_{\epsilon}) q_3(\nu_{\xi})\)
Results

N=96, period=29,

- By using Fourier Transform technic, no way to find the right value of period.
- It is also difficult to estimate the shape of this repeating scheme.

Low noise case:

High noise case:
Results

\[ N=96, \text{ period}=29, \]

Low noise case:

High noise case:
Results
Results
Conclusions

- the first step in any inference is to write down the relation between what you observe (data $g$) and the unknowns $f$.
- The second step is to model and assign priors to account for all uncertainties.
- The third step is to use the Bayes rule to find the expression of the joint probability law of all the unknowns given the data and all the hyper parameters.
- **Do the Bayesian computation**, show the results.
- Interpret your results.
- Enjoy.