Sensors, Measurement systems
Signal processing and Inverse problems
Exercises

Ali Mohammad-Djafari
Laboratoire des Signaux et Systèmes,
UMR8506 CNRS-SUPELEC-UNIV PARIS SUD 11
SUPELEC, 91192 Gif-sur-Yvette, France
http://lss.supelec.free.fr
Email: djafari@lss.supelec.fr
http://djafari.free.fr

Files: http://djafari.free.fr/Cours/Master_MNE/Cours/Cours_MNE_2014_01.pdf

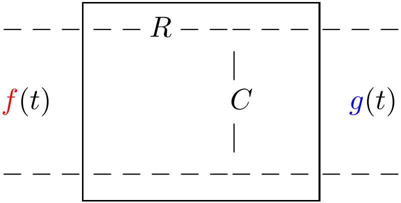
Exercise 1: Fourier and Laplace transforms

- Consider the following signals:
  1. $f(t) = a\ \sin(\omega t)$
  2. $f(t) = a\ \cos(\omega t)$
  3. $f(t) = \sum_{k=1}^{K} [a_k\ \sin(\omega_k t) + b_k \sin(\omega_k t)]$
  4. $f(t) = \sum_{k=1}^{K} a_k\ \exp\left[-j(\omega_k t)\right]$
  5. $f(t) = a\ \exp\left[-t^2\right]$
  6. $f(t) = \sum_{k=1}^{K} a_k\ \exp\left[-\frac{1}{2}(t - m_k)^2 / v_k\right]$
  7. $f(t) = a\ \sin(\omega t)/(\omega t)$
  8. $f(t) = 1,$ if $|t| < a,\quad 0$ elsewhere
- For each of these signals, first compute their Fourier Transform $F(\omega)$, then write a Matlab program to plot these signals and their corresponding $|F(\omega)|$.

- Consider the following signals:
  1. $f(t) = a\ \exp\left[-t/\tau\right], \quad t > 0$
  2. $f(t) = 0, \quad t \leq 0, \quad 1$
- For each of these signals, compute their Laplace Transform $g(s)$.

Exercise 2: Input-Output modeling, Transfert function

Consider the following system:



with $RC = 1$.

- Write the expression of the transfer function $H(\omega) = \frac{G(\omega)}{F(\omega)}$
- Write the expression of the impulse response $h(t)$
- Write the expression of the relation linking the output $g(t)$ to the input $f(t)$ and the impulse response $h(t)$
- Write the expression of the relation linking the Fourier transforms $G(\omega)$, $F(\omega)$ and $H(\omega)$
- Write the expression of the relation linking the Laplace transforms $G(s)$, $F(s)$ and $H(s)$

Exercise 2 (continued)

- Give the expression of the output when the input is $f(t) = \delta(t)$
- Give the expression of the output when the input is a step function $f(t) = u(t) = \begin{cases} 0 & \forall t < 0, \\ 1 & \forall t \geq 0 \end{cases}$
- Give the expression of the output when the input is $f(t) = a \sin(\omega_0 t)$
- Give the expression of the output when the input is $f(t) = \sum_k f_k \sin(\omega_k t)$
- Give the expression of the output when the input ist $f(t) = \sum_j f_j \, \delta(t - t_j)$

Exercise 3: Averaging to increase accuracy

Let note
$$\bar{x}_N = \frac{1}{N}\sum_{n=1}^{N} x(n), \qquad v_N = \frac{1}{N}\sum_{n=1}^{N}(x(n)-\bar{x}_N)^2$$
$$\bar{x}_{N-1} = \frac{1}{N-1}\sum_{n=1}^{N-1} x(n), \qquad v_{N-1} = \frac{1}{N-1}\sum_{n=1}^{N-1}(x(n)-\bar{x}_N)^2$$

Show that

- Updating mean and variance:

$$\bar{x}_N = \frac{N-1}{N}\bar{x}_{N-1} + \frac{1}{N}x(n) = \bar{x}_{N-1} + \frac{1}{N}(x(n)-\bar{x}_{N-1})$$
$$v_N = \frac{N-1}{N}v_{N-1} + \frac{N-1}{N^2}(x(n)-\bar{x}_N)^2$$

- Updating inverse of the variance:

$$v_N^{-1} = \frac{N}{N-1}v_{N-1}^{-1} + \frac{N}{(N-1)(N+\rho_N)}(x(n)-\bar{x}_N)^2 v_{N-1}^{-2}$$
with $\rho_N = (x(n)-\bar{x}_N)^2 v_{N-1}^{-1}$

- Vectorial data $\boldsymbol{x}_n$

$$\bar{\boldsymbol{x}}_N = \frac{N-1}{N}\bar{\boldsymbol{x}}_{N-1} + \frac{1}{N}\boldsymbol{x}(n) = \bar{\boldsymbol{x}}_{N-1} + \frac{1}{N}(\boldsymbol{x}(n)-\bar{\boldsymbol{x}}_{N-1})$$
$$\boldsymbol{V}_N = \frac{N-1}{N}\boldsymbol{V}_{N-1} + \frac{N-1}{N^2}(\boldsymbol{x}(n)-\bar{\boldsymbol{x}}_N)(\boldsymbol{x}(n)-\bar{\boldsymbol{x}}_N)'$$
$$\boldsymbol{V}_N^{-1} = \frac{N}{N-1}\boldsymbol{V}_{N-1}^{-1} + \frac{N}{(N-1)(N+\rho_N)}\boldsymbol{V}_{N-1}^{-1}(x(n)-\bar{x}_N)(x(n)-\bar{x}_N)'\boldsymbol{V}_{N-1}^{-1}$$
with $\rho_N = (\boldsymbol{x}(n)-\bar{\boldsymbol{x}}_N)'\boldsymbol{V}_{N-1}^{-1}(\boldsymbol{x}(n)-\bar{\boldsymbol{x}}_N)'$

Exercise 4: Forward modeling

Consider the following system:

$$f(t) \longrightarrow \boxed{H(\omega)} \longrightarrow g(t)$$

with $H(\omega) = \frac{1}{1+j\omega}$.

- Find $h(t)$.
- For a given input $f(t)$ give the general expression of the output $g(t)$.
- $f(t)$ give the general expression of the output $g(t)$.
- Give the expression of the output when the input is a step function $f(t) = u(t) = \begin{cases} 0 & \forall t < 0, \\ 1 & \forall t \geq 0 \end{cases}$
- Give the expression of the output when the input is $f(t) = a \sin(\omega_0 t)$

Exercise 5: Discretization and forward computation

Consider the following general system:

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow g(t)$$

- For a given input $f(t)$ give the general expression of the output $g(t)$.
- Give the expression of the output when the input is $f(t) = \sum_{n=0}^{N} f_n \, \delta(t - jn\Delta)$ with $\Delta = 1$.
- Suppose that $h(t) = \displaystyle\sum_{k=0}^{K} h_k \delta(t - k\Delta)$ with $\Delta = 1$, Compute the output $g(t)$ for $t = 0, \cdots, M\Delta$ with $\Delta = 1$ and $M > N$.
- Show that if $g(t)$ is sampled at the same sampling period $\delta = 1$, we have $g(t) = \sum_{m=0}^{M} g_m \delta(t - m\Delta)$. Then show that

$$g_m = \sum_{k=0}^{K} h_k f_{n_k}$$

Exercise 5 (continued)

- Show that the relation between $\boldsymbol{f} = [f_0, \cdots, f_N]'$, $\boldsymbol{h} = [h_0, \cdots, h_K]'$ and $\boldsymbol{g} = [g_0, \cdots, g_M]'$ can be written as $\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f}$ or as $\boldsymbol{g} = \boldsymbol{F}\boldsymbol{h}$. give the expressions and the structures of the matrices $\boldsymbol{H}$ and $\boldsymbol{F}$.
- What do you remark on the structure of these two matrices?
- Write a Matlab programs which compute $\boldsymbol{g}$ when $\boldsymbol{f}$ and $\boldsymbol{h}$ are given.
- Let name this program `g=direct(h,f,method)` where method will indicate different methods to use to do the computation. Test it with creating different inputs and different impulse responses and compute the outputs.

Exercise 6: Least Squares and Regularisation

In a measurement system, we have established the following relation: $\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$ where
$\boldsymbol{g}$ is a vector containig the measured data $\{g_m, m = 1 \cdots, M\}$,
$\boldsymbol{\epsilon}$ is a vector representing the errors $\{\epsilon_m, m = 1 \cdots, M\}$,
$\boldsymbol{f}$ is a vector representing the unknowns $\{f_n, n = 1 \cdots, N\}$, and
$\boldsymbol{H}$ is a matrix with the elements $\{a_{mn}\}$ depending on the geometry of the measurement system and assumed to be known.

- Suppose first $M = N$ and that the matrix $\boldsymbol{H}$ be invertible. Why the solution $\widehat{\boldsymbol{f}}_0 = \boldsymbol{H}^{-1}\boldsymbol{g}$ is not, in general, a satisfactory solution?
  What relation exists between $\frac{\|\delta\widehat{\boldsymbol{f}}_0\|}{\|\widehat{\boldsymbol{f}}_0\|}$ and $\frac{\|\delta\boldsymbol{g}\|}{\|\boldsymbol{g}\|}$?

Exercise 6 (continued)

- Let come back to the general case $M \neq N$. Show then that the Least Squares (LS) solution, *i.e.* $\widehat{\boldsymbol{f}}_1$ which minimises

$$J_1(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2$$

  is also a solution of equation $\boldsymbol{H}'\boldsymbol{H}\boldsymbol{f} = \boldsymbol{H}'\boldsymbol{g}$ and if $\boldsymbol{H}'\boldsymbol{H}$ is invertible, then we have

$$\widehat{\boldsymbol{f}}_1 = [\boldsymbol{H}'\boldsymbol{H}]^{-1}\boldsymbol{H}'\boldsymbol{g}$$

  What is the relation between $\frac{\|\delta\widehat{\boldsymbol{f}}_1\|}{\|\widehat{\boldsymbol{f}}_1\|}$ and $\frac{\|\delta\boldsymbol{g}\|}{\|\boldsymbol{g}\|}$?

- What is the relation between the covarience of $\widehat{\boldsymbol{f}}_1$ and covarience of $\boldsymbol{g}$?

Exercise 6 (continued)

- Consider now the case $M < N$. Evidently, $\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f}$ has infinite number of solutions. The minimum norm solution is:

$$\widehat{\boldsymbol{f}} = \arg \min_{\boldsymbol{H}\boldsymbol{f}=\boldsymbol{g}} \left\{ \|\boldsymbol{f}\|^2 \right\}$$

  Show that this solution is obtained via:

$$\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{H}^t \\ \boldsymbol{H} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{g} \end{bmatrix}$$

  which gives:

$$\widehat{\boldsymbol{f}}_2 = \boldsymbol{H}^t (\boldsymbol{H}\boldsymbol{H}^t)^{-1} \boldsymbol{g}$$

  if $\boldsymbol{H}\boldsymbol{H}^t$ is invertible.

- Show that with this solution we have: $\widehat{\boldsymbol{g}} = \boldsymbol{H}\widehat{\boldsymbol{f}}_2 = \boldsymbol{g}$.
- What is the relation between the covarience of $\widehat{\boldsymbol{f}}_2$ and covarience of $\boldsymbol{g}$?

Exercise 6 (continued)

- Let come back to the general case $M \neq N$ and define

  $$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{with} \quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{f}\|^2$$

  Show that for any $\lambda > 0$, this solution exists and is unique and is obtained by:

  $$\widehat{\boldsymbol{f}} = [\boldsymbol{H}'\boldsymbol{H} + \lambda\boldsymbol{I}]^{-1}\boldsymbol{H}'\boldsymbol{g}$$

- What relation exists between $\widehat{\boldsymbol{g}} = \boldsymbol{H}\widehat{\boldsymbol{f}}$ and $\boldsymbol{g}$?
- What is the relation between the covarience of $\widehat{\boldsymbol{f}}$ and covarience of $\boldsymbol{g}$?
- Another regularized solution $\widehat{\boldsymbol{f}}_2$ to this problem is to minimize a criterion such as:

  $$J_2(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2,$$

  where $\boldsymbol{D}$ is a matrix approximating the operator of derivation. Show that this solution is given by:

  $$\widehat{\boldsymbol{f}}_2 = \arg\min_{\boldsymbol{f}} \{J_2(\boldsymbol{f})\} = \left[\boldsymbol{H}'\boldsymbol{H} + \lambda\boldsymbol{D}'\boldsymbol{D}\right]^{-1}\boldsymbol{H}'\boldsymbol{g}$$

Exercise 6 (continued)

- Suppose that $\boldsymbol{H}$ and $\boldsymbol{D}$ be circulent matrices and symmetric. Then, show that the regularised solution $\widehat{\boldsymbol{f}}_2$ can be written using the DFT by:

$$F(\omega) = \frac{1}{H(\omega)} \frac{|H(\omega)|^2}{|H(\omega)|^2 + \lambda |D(\omega)|^2} G(\omega)$$

  where

  - $H(\omega)$ is the DFT of the first ligne of the matrix $\boldsymbol{H}$,
  - $D(\omega)$ is the DFT of the first ligne of the matrix $\boldsymbol{D}$
  - $F(\omega)$ is the DFT of the solution vector $\widehat{\boldsymbol{f}}_2$, et
  - $G(\omega)$is the DFT of the data measurement vector $\boldsymbol{g}$.

- Comment the expressions of $\widehat{\boldsymbol{f}}_2$ in the question 3. and $F(\omega)$ in the question 4. when $\lambda = 0$ and when $\lambda \longrightarrow \infty$.

Exercise 7: Sensor output noise filtering: Bayesian approach

We have a sensor output signal $g(t)$ which is very noisy. Let note the non-noisy signal $f(t)$, then we have: $g(t) = f(t) + \epsilon(t)$. Let, first assume the noise to be modelled by a centered and Gaussian probability law with known variance σ_ϵ^2. We have observed this signal at times: $t = 1, 2, \cdots, n$ and let note by $\boldsymbol{g} = [g_1, \cdots, g_n]'$, $\boldsymbol{f} = [f_1, \cdots, f_n]'$ and $\boldsymbol{\epsilon} = [\epsilon_1, \cdots, \epsilon_n]'$, the vectors containing, respectively, the samples of $g(t)$, $f(t)$ and $\epsilon(t)$.

Part 1: First we model the signal by a separable Gaussian model: $p(f_j) = \mathcal{N}(f_0, \sigma_f^2), \forall j$.

- Give the expressions of $p(\epsilon_j)$ et $p(g_j|f_j)$ and then $p(f_j|g_j)$ using the Bayes rule.
- Show that $p(f_j|g_j) = \mathcal{N}(\widehat{\mu}_j, \widehat{v}_j)$ and give the expressions of $\widehat{\mu}_j$ et $\widehat{v}_j$.
- Give the expression of the Maximum A posteriori estimate $\widehat{f_j}$ of f_j.

Part 2:

Now, let model the input signal by a first order AR model:
$f_j = a\, f_{j-1} + \xi_j$ where we assume $\xi_j \simeq \mathcal{N}(0, \sigma_f^2)$.

- write the expression of $p(f_j|f_{j-1})$ and then $p(\boldsymbol{f})$ and show that it can be written as:
  $p(\boldsymbol{f}) \propto \exp\left[-\frac{1}{\sigma_f^2}\|\boldsymbol{D}\boldsymbol{f}\|^2\right]$ where $\boldsymbol{D}$ is a matrix that you give the expression.
- Write the expressions of $p(\boldsymbol{g}|\boldsymbol{f})$ and then using the a priori $p(\boldsymbol{f})$ give the expression of the a posteriori $p(\boldsymbol{f}|\boldsymbol{g})$.
- Show that the a posteriori law $p(\boldsymbol{f}|\boldsymbol{g})$ is given by:
  $p(\boldsymbol{f}|\boldsymbol{g}) \propto \exp\left[-\frac{1}{2}J(\boldsymbol{f})\right]$ where you give the expression of $J(\boldsymbol{f})$and the expression of the Maximum A posteriori (MAP estimate: $\widehat{\boldsymbol{f}}_{MAP} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\}$.

Exercise 8: Identification and Deconvolution

Consider the problem of deconvolution where the measured signal $g(t)$ is related to the input signal $f(t)$ and the impulse response $h(t)$ by $g(t) = h(t) * f(t) + \epsilon(t)$ and where we are looking to estimate $h(t)$ from the knowledge of the input $f(t)$ and output $g(t)$ and to estimate $f(t)$ from the knowledge of the impulse response $h(t)$ and output $g(t)$.

- Given $f(t)$ and $g(t)$, describe different methods for estimating $h(t)$.
- Write a Matlab program which can compute $\boldsymbol{h}$ given $\boldsymbol{f}$ and $\boldsymbol{g}$. Let name it: `h=identification(g,f,method)`. Test it by creating different inputs $\boldsymbol{f}$ and outputs $\boldsymbol{g}$. Think also about the noise. Once test your programs without noise, then add some noise on the output $\boldsymbol{g}$ and test them again.

Exercise 8: (continued)

- Given $g(t)$ and $h(t)$, describe different methods for estimating $f(t)$.
- Write a Matlab program which can compute $\boldsymbol{f}$ given $\boldsymbol{g}$ and $\boldsymbol{h}$. Let name it: `f=inversion(g,h,method)`. Test it by creating different inputs $\boldsymbol{f}$ and outputs $\boldsymbol{g}$. Think also about the noise. Once test your programs without noise, then add some noise on the output $\boldsymbol{g}$ and test them again.
- Bring back your experiences and comments.