Superresolution of a compact neutron spectrometer at energies relevant for fusion diagnostics

M. Reginatto and A. Zimbal

Physikalisch-Technische Bundesanstalt (PTB), Braunschweig, Germany

Abstract. The ability to achieve resolution that is better than the instrument resolution (i.e., superresolution) is well known in optics, where it has been extensively studied. Unfortunately, there are only a handful of theoretical studies concerning superresolution of particle spectrometers, even though experimentalists are familiar with the enhancement of resolution that is achievable when appropriate methods of data analysis are used, such as maximum entropy and Bayesian methods. Knowledge of the superresolution factor is in many cases important. For example, in applications of neutron spectrometry to fusion diagnostics, the temperature of a burning plasma is an important physical parameter which may be inferred from the width of the peak of the neutron energy spectrum, and the ability to determine this width depends on the superresolution factor.

Kosarev has derived an absolute limit for resolution enhancement using arguments based on a well known theorem of Shannon. Most calculations of superresolution factors in the literature, however, are based on the assumption of Gaussian, translationally invariant response functions and therefore not directly applicable to neutron spectrometers which typically have response functions not satisfying these requirements. In this work, we develop a procedure that allows us to overcome these difficulties and we derive estimates of superresolution for liquid scintillator spectrometers of a type commonly used for neutron measurements. Theoretical superresolution factors are compared to experimental results.

Keywords: Superresolution, maximum entropy, neutron spectrometry, fusion diagnostics
PACS: 06.20.Dk, 29.30.Hs, 29.40.Mc

INTRODUCTION

The concept of resolution was originally developed in optics, and it measures the ability of an instrument (e.g., a telescope) to distinguish between two point sources (e.g., two stars) separated by a small angular interval. One of the earliest measures of optical resolution identifies the resolution with the effective width of the point spread function of the apparatus. This approach, while often useful, is too simplistic because it does not take into consideration the algorithms that are commonly used to improve the quality of images and which have become an integral part of modern optical instruments.

For particle detectors, the energy resolution [1] is usually defined by analogy to optical resolution as described above. However, while such a definition is useful for some spectrometers, in the case of neutron spectrometers based on scintillators it can not be applied because the response functions are not localized around a given energy. This is illustrated in Fig. 1, which shows a spectrum of quasi-monoenergetic neutrons produced at the PTB accelerator facility together with the data, in the form of a pulse height spectrum (PHS), that was measured with an NE213 spectrometer [2]. Because

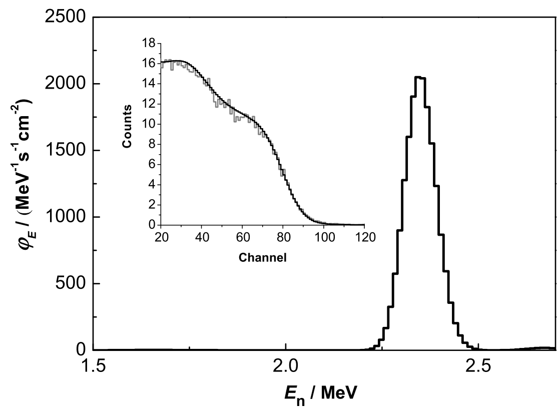

FIGURE 1. Neutron spectra measured at the PTB accelerator facility. The inset shows the measured PHS (histogram) as well as the PHS that results from folding the response functions with the unfolded neutron spectrum (line) [2].

the responses are not localized, the PHS does not resemble the neutron spectrum. PHS of measurements of quasi-monoenergetic neutrons at other energies are similar in shape to the one shown here [3].

Since the traditional definition of resolution can not be applied, we develop here a different approach which combines elements of the resolving power formalism of Backus and Gilbert [4, 5] with the superresolution formalism of Kozarev [6]. The paper is organized as follows. After a brief description of the NE213 spectrometer, we calculate its resolving power for energy regions that are relevant for neutron fusion diagnostics. We then do a linear transformation that maps the response functions to Gaussian-shaped response functions with a width determined by the resolving power. Next, the formalism of Kozarev is used to estimate the superresolution factor in these energy regions and theoretical predictions are compared to experimental results that have been previously published [2]. We end with some concluding remarks.

NE213 SPECTROMETER

Scintillation spectrometers based on organic liquids (commercially available under the names NE213 and BC501A) are routinely used for neutron and photon measurements [7, 8]. The physical principle used to measure the particles is the conversion of incoming radiation (neutrons or γ particles) into charged particles (recoil protons and Compton electrons) which produce scintillation light in the organic liquid related to the kinetic energy of the particles. The energy transfer to the secondary charged particles is in general incomplete, however, which results in broad distributions of the intensity of light pulses reflecting the variety of possible scattering angles even for incoming particles with the same or nearly the same energy.

Spectrometry in mixed neutron/photon fields is possible using pulse shape analysis to separate neutron induced events from photon induced events, but in this paper we restrict to neutron measurements. The ranges of neutron energies E_{n} that can be measured with this specific detector is 1.5 MeV < E_{n} < 20 MeV. The spectrometer is well suited for fusion diagnostics [9].

A measurement carried out with a scintillation detector provides an indirect rather than a direct measurement of the particle differential energy spectrum $\Phi_E(E)$. The PHS measured by the detector is related to $\Phi_E(E)$ by the linear equations

$$N_k + e_k = \int dE\ R_k(E)\Phi_E(E), \tag{1}$$

where N_k is the number of counts in channel k ($k = 1, \ldots, n$, with n the number of channels in the PHS), $R_k(E)$ is the detector response of channel k to particles of energy E, and e_k is a term which accounts for effects that are not described by the model of the measurement; e.g., statistical fluctuations in the number of counts, discrepancies between N_k and $\int dE\ R_k(E)\Phi_E(E)$ due to deviations of $R_k(E)$ from the true value of the response, etc. The value of e_k is not known *a priori*, but it is expected to be of the same order of magnitude as the estimated uncertainty σ_k that is assigned to the value N_k measured in channel k.

Estimation of $\Phi_E(E)$ in general requires a deconvolution, but in this paper we do not address this issue; see Ref. [2] for the use of maximum entropy and Bayesian methods to analyze NE213 data.

RESOLVING POWER

The initial step of our calculation of the superresolution factor consists in estimating the resolving power of the spectrometer. Following Backus and Gilbert [4, 5], we consider linear averages of $\Phi_E(E)$ about a given energy E_i. These can always be written in the form

$$< \Phi_E >_{E_i} = \int dE\ A(E_i, E)\ \Phi_E(E), \tag{2}$$

where $A(E_i, E)$ is the averaging kernel and $\int dE\ A(E_i, E) = 1$. To construct the averaging kernel, Backus and Gilbert consider functions that are linear in the response functions,

$$A(E_i, E) = \sum_{k=1}^{n} a_k(E_i) R_k(E). \tag{3}$$

The $a_k(E_i)$ are constants with respect to the energy E, but their values will depend on the choice of E_i. Notice that the $A(E_i, E)$ can be used to provide an estimate of $< \Phi_E >_{E_i}$ directly in terms of the measured data, since

$$< \Phi_E >_{E_i} \ \sim \ \sum_{k=1}^{n} a_k(E_i) N_k \tag{4}$$

where we have used Eqs. (1-3) and neglected the e_k. This estimate will be accurate whenever $e_k << N_k$, the averaging kernel $A(E_i, E)$ is peaked at E_i, and the spectrum $\Phi_E(E)$ does not vary much over the regions in which $A(E_i, E)$ is non-zero.

Ideally, one would like to have $A(E_i,E) = \delta(E-E_i)$, where δ is the Dirac delta function. While this is not possible if we define $A(E_i,E)$ according to equation (3), it is possible to choose the constants $a_k(E_i)$ so that that $A(E_i,E)$ will resemble a delta function as closely as possible.

To do this, Backus and Gilbert introduce the following scheme. Pick a function $J(E_i,E)$ which vanishes at $E = E_i$ and increases monotonically as E increases or decreases away from E_i; for example, $J(E_i,E) \sim (E-E_i)^2$ is one possible choice. Then, introduce the functional

$$I[A] = \int dE\, J(E_i,E)\,[A(E_i,E)]^2 \,. \tag{5}$$

I is always non-negative and vanishes if $A(E_i,E)$ is a delta function. If one now chooses the $a_k(E_i)$ in Eq. (5) that minimize I subject to the normalization constraint, one obtains the linear combination of $R_k(E)$ that "best approximates" a delta function. While many "δ-ness criteria" are possible, the choice of Backus and Gilbert, Eq. (5), is convenient in that it leads to linear equations for the $a_k(E_i)$ when I is minimized. These equations can be solved using matrix inversion. The $a_k(E_i)$ must satisfy the following set of $n+1$ equations,

$$\sum_{j=1}^{n} \int dE \left[J(E_i,E)R_k(E)R_j(E)\right] a_j(E_i) + \lambda \int dE\, R_k(E) = 0 \tag{6}$$
$$\sum_{j=1}^{n} \int dE\, R_j(E) a_j(E_i) = 1$$

where λ is a Lagrange multiplier.

The Backus-Gilbert procedure consists of solving the set of Eqs. (6), substituting the $a_j(E_i)$ into Eq. (3) and inspecting the resulting averaging kernels. If $A(E_i,E)$ resembles a blurred delta function, then the integral in equation Eq. (4) can be thought of as a good approximation to the value of $\Phi_E(E)$ about E_i. The width of $A(E_i,E)$ provides then a measure of the *resolving power* at E_i. If the solution $A(E_i,E)$ does not resemble a blurred delta function, then it is not possible to obtain a good approximation to the value of $\Phi_E(E)$ about E_i and the resolving power of the spectrometer is not well defined at that energy. For the NE213 spectrometer, the resolving power is well defined over the entire neutron energy range.

We now introduce a new representation which we call the *Backus-Gilbert representation*. It is defined by a new set of response functions $R_j^{BG}(E)$ and corresponding PHS data N_j^{BG} given by

$$R_j^{BG}(E) = \sum_{k=1}^{n} a_{jk} R_k(E)\,, \qquad N_j^{BG} = \sum_{k=1}^{n} a_{jk} N_k \tag{7}$$

where $a_{ij} = a_j(E_i)$ for an appropriate set of n energies E_i (in our case, we have chosen a set of equidistant E_i defined over an appropriate energy range).

We now consider measurements of quasi-monoenergetic neutrons at $E_\mathrm{n} \sim 2.5$ MeV and $E_\mathrm{n} \sim 14$ MeV and examine the response functions and PHS in the Backus-Gilbert

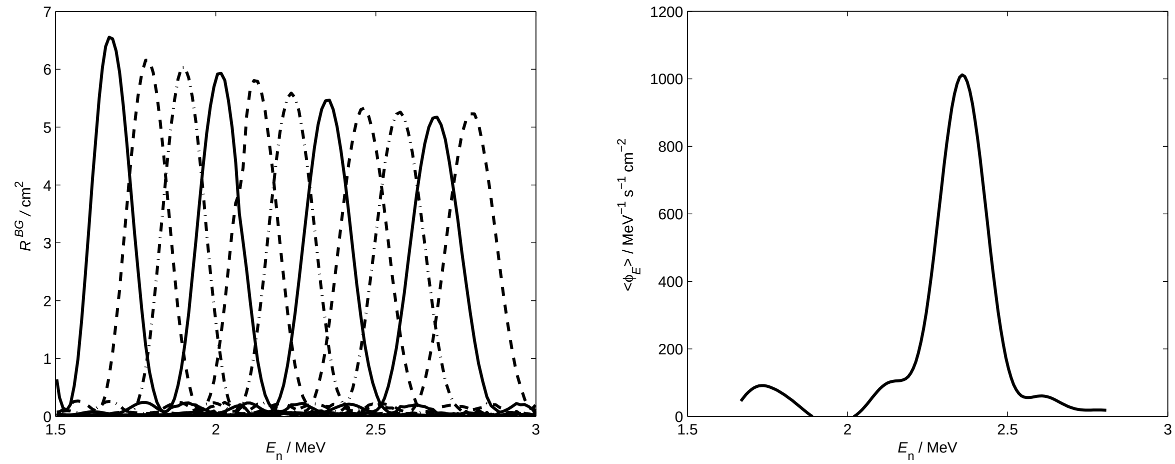

FIGURE 2. Response functions $R_j^{BG}(E)$ for energies $E_\mathrm{n} \sim 2.5$ MeV (left) and PHS (right), both in the Backus-Gilbert representation. For clarity, only a few of the 143 $R_j^{BG}(E)$ that were calculated are shown.

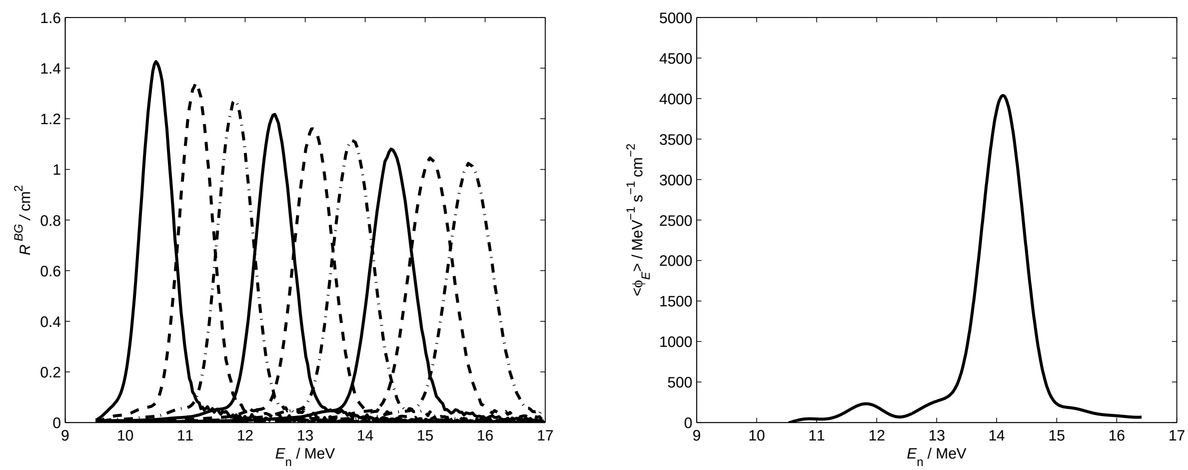

FIGURE 3. Response functions $R_j^{BG}(E)$ for energies $E_\mathrm{n} \sim 14$ MeV (left) and PHS (right), both in the Backus-Gilbert representation. For clarity, only a few of the 1022 $R_j^{BG}(E)$ that were calculated are shown.

representation. For details of the measurements, the neutron fields, and the spectrometer used, see Ref. [2]. Figs. 2 and 3 show a few selected response functions $R_j^{BG}(E)$ and the corresponding PHS, both in the Backus-Gilbert representation. In both cases, the width of the averaging kernels increases with increasing energy, indicating that the resolving power is energy dependent. The PHS are now similar in shape to the neutron spectrum that is being measured, as expected given that the $R_j^{BG}(E)$ are well localized.

Finally, we consider another application of this formalism, this time to measurements of an ^{241}AmBe(α,n) neutron source which has a spectrum that extends over a wide range of energies. Fig. 4 shows the PHS in the Backus-Gilbert representation together with a

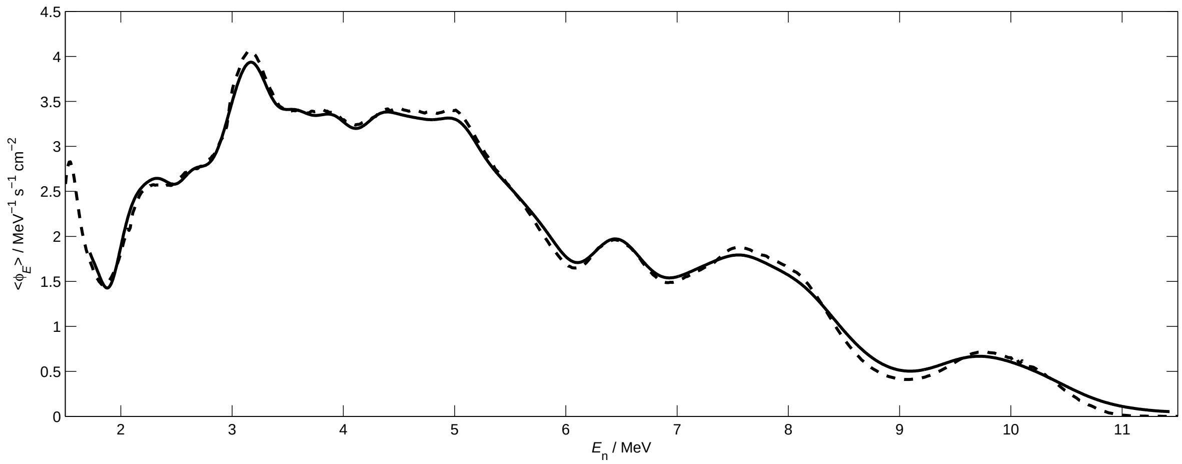

FIGURE 4. PHS in the Backus-Gilbert representation (solid line) compared to the neutron spectrum derived using maximum entropy deconvolution [10] (dashed line) for an ^{241}AmBe(α,n) source.

deconvolution using maximum entropy [10], for comparison purposes. It is remarkable that the agreement is already very good.

SUPERRESOLUTION

The proof of the existence of a well defined superresolution limit by Kozarev [6] is modeled after the theorem of Shannon which concerns the maximum information transmission speed through a noisy information channel [11]. Because it requires the use of techniques from information theory which are beyond the scope of this paper, we will not reproduce the proof here but instead we summarize the main result.

We follow the presentation in [6]. In place of Eq. (1), we introduce the equation

$$g(x)+e(x)=\int_{y_{min}}^{y_{max}} dy\, K(x,y)\ f(y) \tag{8}$$

where $K(x,y)=K(x-y)$. We refer to $e(x)$ as the “noise” in the data. We normalize K by the condition $K(0)=1$ and define the effective width Δ of $K(x-y)\equiv K(z)$ by

$$\Delta=\int_{-\infty}^{\infty} dz\, K^2. \tag{9}$$

To help interpret the construction, consider a Gaussian response $K(z)=\exp(-z^2/2\sigma^2)$. In this case, $\Delta=\sqrt{2\pi}\sigma=2.51\sigma$ which is close to the $FWHM=\sqrt{2\ln 2}\sigma=2.35\sigma$.

The improvement in resolution that can be achieved with respect to Δ is always limited by noise. The limiting superresolution factor is given by [6]

$$SR=\frac{\Delta}{\delta}=\frac{1}{3}log_2\left(1+\frac{P_s}{P_n}\right) \tag{10}$$

where $P_s = \int_{-\infty}^{\infty} dx\, g^2$ is the signal energy and $P_n = \int_{-\infty}^{\infty} dx\, e^2$ is the noise energy. As is apparent from Eq. (10), the superresolution depends logarithmically on the signal-to-noise ratio. Eq. (10) is the fundamental result that is needed to calculate the superresolution of the spectrometer. Some further work, however, is needed before this can be done.

Neither the original response functions $R_j(E)$ nor the $R_j^{BG}(E)$ of the Backus-Gilbert representation are translationally invariant; therefore, they can not be used to estimate the superresolution factor. One may define a new representation however which has responses that *are* translationally invariant and which satisfy all the conditions needed for the proof of Kozarev to be applicable. Since the $R_j^{BG}(E)$ are approximately Gaussian in shape (see Figs. 2 and 3), it is reasonable to introduce a Gaussian representation with

$$R_j^G(E) = G(E_j - E) = \sum_{k=1}^{n} g_{jk} R_k(E) \tag{11}$$

where the g_{jk} are chosen so that $G(E_j - E)$ is a good approximation to a Gaussian centered at E_j with a FWHM that matches the *largest* FWHM of the $R_j^{BG}(E)$ for the energy region of interest. In this way, we use the information about resolving power to determine an appropriate representation for the superresolution analysis.

We now calculate superresolution factors at $E_\mathrm{n} \sim 2.5$ MeV and $E_\mathrm{n} \sim 14$ MeV. First, introduce data N_k^G and uncertainties σ_k^G in the Gaussian representation, derived by doing the corresponding transformation based on Eq. (11). These quantities are then used to estimate the signal energy and the noise energy and to calculate the superresolution factor following the analysis in Ref. [6].

The FWHM of the Gaussian responses $R_j^G(E)$ was set to 0.18 MeV for $E_\mathrm{n} \sim 2.5$ MeV and to 0.93 MeV for $E_\mathrm{n} \sim 14$ MeV. For the data shown in Figs. 2 and 3, the superresolution factor obtained was $SR = 5$ in both cases, which implies that it should be possible to resolve structure with a width of 0.04 MeV at $E_\mathrm{n} \sim 2.5$ MeV and 0.19 MeV at $E_\mathrm{n} \sim 14$ MeV.

The results that we obtained are consistent with experiment: measurements made under controlled conditions indicate that the NE213 spectrometer can resolve peaks with a FWHM of 0.08 MeV at $E_\mathrm{n} \sim 2.5$ MeV and 0.2 MeV at $E_\mathrm{n} \sim 14$ MeV. In both cases, the FWHM of the measured spectra were determined using maximum entropy deconvolution; furthermore, these FWHM are in good agreement with numerical simulations [2]. Unfortunately, quasi-monoenergetic neutron calibration sources with smaller FWHM are not available at these energies, so it is not possible to check experimentally if the superresolution limits calculated here are indeed absolute limits.

CONCLUSIONS

A measure of resolution that is general enough to be applicable to most neutron spectrometers must allow for response functions that are not localized and must take into consideration the enhancement in resolution that is achievable with deconvolution methods. Standard measures of resolution do not pass this test.

In this paper, we present a new approach based on the formalisms of Backus and Gilbert and of Kozarev which is *generally* applicable. Our method follows a two-step procedure. In the first step, we carry out an analysis of the resolving power and produce the Backus-Gilbert representation. For the cases that we considered, the transformed response functions are localized and the transformed PHS resemble the measured neutron spectra. The FWHM of these responses provide an estimate of the resolution of the spectrometer *before* deconvolution of the data. We point out that the PHS in the Backus-Gilbert representation may be useful when preparing initial estimates that are needed for deconvolution methods. In the second step, we introduce a linear transformation that maps the original response functions to Gaussians of a FWHM which are determined by the resolving power in the energy region of interest. In this representation, the response functions satisfy the requirements of the theorem of Kozarev and the superresolution factor can be calculated.

The superresolution factors that we obtained are consistent with measurements performed at the PTB accelerator of quasi-monoeneretic neutrons with energies $E_{\mathrm{n}} \sim 2.5$ MeV and $E_{\mathrm{n}} \sim 14$ MeV. In particular, the result for $E_{\mathrm{n}} \sim 14$ MeV indicates that maximum entropy deconvolution succeeds in reaching the theoretical limit. The number of counts in the PHS were in both cases about 10^7, and in both cases a superrresolution factor of 5 was obtained. Our calculations indicate that it should be possible to resolve structures as small as 0.04 MeV at $E_{\mathrm{n}} \sim 2.5$ MeV and 0.2 MeV at $E_{\mathrm{n}} \sim 14$ MeV. These results are of relevance for fusion diagnostics, metrology, and other applications. For example, in fusion diagnostics, the temperature of a burning plasma is an important physical parameter which may be inferred from the width of the neutron emission peak, and the ability to determine this width depends on the superresolution factor.

There are a number of interesting questions that require further investigation. In order to calculate the superresolution we have mapped to Gaussian response functions but other representations should also be considered. The superresolution factor depends on the signal-to-noise ratio, and this dependence should be investigated both theoretically and experimentally. Finally, the Backus-Gilbert representation seems very promising, and it should be further developed. These issues are currently being examined and they will be the subject of a future publication.

REFERENCES

1. G.F. Knoll, *Radiation Detection and Measurement*, John Wiley, New York, 2000.
2. M. Reginatto and A. Zimbal, *Rev. Sci. Instrum.* **79**, 023505 (2008).
3. A. Zimbal, H. Klein, M. Reginatto, H. Schuhmacher, L. Bertalot, A. Murari and the JET-EFDA Contributors, "High resolution neutron spectrometry with liquid scintillation detectors for fusion applications" in *Proceedings of the International Workshop on Fast Neutron Detectors and Applications*, PoS(FNDA2006)035 (2006).
4. G. Backus and J. F. Gilbert, *Geophys. J. R. astr. Soc.* **13**, 247-276 (1967).
5. G. Backus and J. F. Gilbert, *Geophys. J. R. astr. Soc.* **16**, 169-205 (1968).
6. E. L. Kosarev, *Inverse Problems* **6**, 55-76 (1990).
7. H. Klein, and S. Neumann, *Nucl. Instrum. and Meth.* **A 476**, 132-142 (2002).
8. H. Klein, *Radiat. Prot. Dosimetry* **107**, 95-109 (2003).
9. A. Zimbal, M. Reginatto, H. Schuhmacher, L. Bertalot, B. Esposito, F. Poli, J.M. Adams, S. Popovichev, V. Kiptily and A. Murari, *Rev. Sci. Instrum.* **75**, 3553 (2004).
10. A. Zimbal, *Radiat. Prot. Dosimetry* **126**, 413-417 (2007).
11. C.E. Shannon and W. Weaver, *The Mathematical Theory of Communication*, University of Illinois Press, Urbana, Illinois, 1949.