The Order-Theoretic Origin of Special Relativity

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Abstract. In this paper, we present a novel derivation of special relativity and the information physics of events. We postulate that *events are fundamental*, and that some events have the potential to be influenced by other events. However, this potential is not reciprocal, nor are all pairs of events related in such a way. This leads to the concept of a *partially-ordered set of events*, which is often called a causal set. Quantification proceeds by distinguishing two chains of coordinated events, each of which represents an observer, and assigning a numerical valuation to each chain. By projecting events onto each chain, each event can be quantified by a pair of numbers, referred to as a *pair*. We show that each pair can be decomposed into a sum of symmetric and antisymmetric pairs, which correspond to time-like and space-like coordinates. We show that one can map a pair to a scalar and that this gives rise to the Minkowski metric. The result is an observer-based theory of special relativity that quantifies events with pairs of numbers. Events are fundamental and space-time is an artificial construct designed to make events look simple.

Keywords: causal, causal sets, information physics, lattice, measure, order, poset, relativity, valuation.
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INTRODUCTION

Information Physics views the laws of physics as originating from the laws that govern the ways in which we process information about the world around us. Such an approach inherently depends both on the description chosen to represent reality and the method used to add a layer of quantification to the chosen description. This paper introduces a novel derivation of the Minkowski metric of special relativity as a quantification of a partially ordered set of events. This derivation does not rely on any notion of space, time, motion, light, or even relativity. Instead, it arises naturally from a simple quantification scheme applied to a partially ordered set of events.

THE EVENT POSTULATE

We assert a single postulate, which embodies a simple description of physical reality:

> ***Event Postulate***: Events are fundamental. Some events have the potential to be influenced by, or informed about, other events. However, this potential is not reciprocal. That is, if an event A has the potential to be influenced by event B, then event B does not have the potential to be influenced by event A.

This potential to be influenced can be viewed as a binary ordering relation relating pairs of events. This results in a partial order where if event A has the potential to be influenced by event B, we say that A *includes* B and write $A \geq B$. This notion of inclusion is transitive, so that if $A \geq B$ and $B \geq C$, then it is also true that $A \geq C$. It is possible that there exist events that cannot possibly influence one another. In this case, we say that the events are *incomparable* and write $A||C$. The relationships $A \geq B$ and $B \geq A$ can only hold simultaneously if $A = B$. Together, a set of events and the described ordering relation results in a partially-ordered set, or poset, of events. Such a poset of events is called a *causal set* [1]. Causal sets have been employed in approaches to quantum gravity, although they are typically endowed with Lorentzian (Minkowski) geometry [2].

Our aim is to derive the relevant physical laws by introducing a layer of quantification to the poset. Though the chosen quantification scheme may be arbitrary, its implementation is subject to a set of constraints imposed by the ordering relation via the poset. These constraints give rise to the physical laws.

QUANTIFICATION

In this section we describe the quantification scheme. We assume that the poset of events is sufficiently dense that we can always find an event that meets our specifications. This is our only additional assumption and from it we recover special relativity rather than general relativity. This implies that interactions between masses serves to place constraints on the possible set of events.

We begin by identifying a distinguished set of events called a chain. A *chain* consists of a set of events that are totally ordered so that for a chain P consisting of N events each labeled by some index i, we have that $p_1 \leq p_2 \leq \ldots \leq p_i \leq \ldots p_N$. Here we consider finite chains and show that they are sufficient to recover special relativity. Countably infinite chains and uncountably infinite chains can be handled similarly.

An event x can be projected onto a chain P if there exists an event $p \in P$ such that $x \leq p$. Since any event $p_+ \geq p$ on the chain also includes x by transitivity, and the chain is finite, there must exist a least event $p_x \in P$ such that $p_x \geq x$. The *projection* of x onto the chain P is given by the least event p_x on the chain P such that $x \leq p_x$. If one considers the sub-poset consisting only of the element x and the elements comprising the chain P, then in this sub-poset p_x covers x, $p_x \succ x$ (Fig. 1A). In the event that the projection exists, the element x can then be "quantified" by assigning to the element x the numeric label assigned to the element $p_x \in P$.

We represent an *observer* as a chain of events and label particular events in the chain for the purpose of quantification. We can imagine that such *quantifying events* are the result of an event generator, or a clock. Note, however, that all observers are chains, but not all chains are observers. We introduce two observers and require that quantifying events in each observer chain are selected so that successive quantifying events in one chain project to successive quantifying events in the other chain (Fig. 1B). This method of distinguishing quantifying events is equivalent to synchronizing clocks.

This quantification scheme distinguishes two observers, represents them as chains, and quantifies them in such a way to make observers look simple.

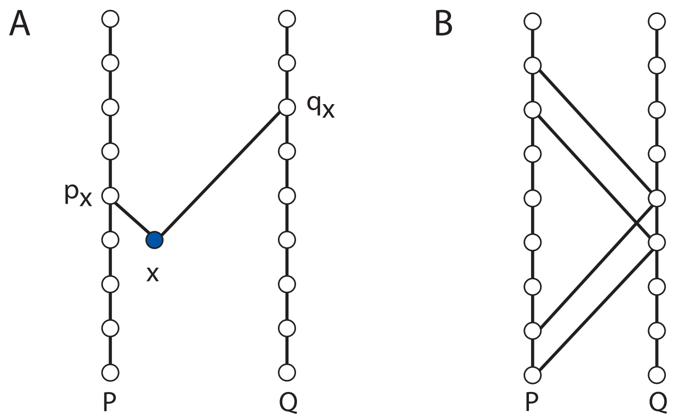

FIGURE 1. A) The projection of an event x onto a chain is the least event on the chain that can be informed about x. (B) Chains can be synchronized by identifying events on the chains such that successive events on one chain project to successive events on the other.

Interval Pair (Pair)

We can quantify an event x by the pair of numbers (p_x, q_x) obtained by projecting it onto the two observer chains **P** and **Q**. Similarly, we can consider the *interval* between two events by considering the difference in the way that the two events project onto the observer chains. For events labeled 0 and 1, we can quantify the interval by computing the difference.

$$(\Delta p, \Delta q) = (p_1, q_1) - (p_0, q_0) = (p_1 - p_0, q_1 - q_0). \tag{1}$$

Since we will be focusing on intervals, we will suppress the deltas in the notation, and refer to such a pair of differences as a *pair*.

Note that some pairs of events project so that both chains agree as to the order in which they are observed (Figure 2A); whereas other pairs of events project so that the order in which they are observed by one chain is reverse that of the other chain (Figure 2B). This suggests that is may be convenient to decompose a pair (p,q) it into the sum of a symmetric pair and an antisymmetric pair

$$(p,q) = \left(\frac{p+q}{2}, \frac{p+q}{2}\right) + \left(\frac{p-q}{2}, \frac{-(p-q)}{2}\right). \tag{2}$$

We call this the *symmetric/antisymmetric decomposition* (Figure 2C).

Scalar Measures

Here we aim to identify a scalar measure that is a non-trivial function of the pair. We define the function f as an unknown map from a pair to a real scalar, and insist that the scalar obeys the symmetric/antisymmetric decomposition (2)

$$f(a,b) = f\left(\frac{(a+b)}{2}, \frac{(a+b)}{2}\right) + f\left(\frac{(a-b)}{2}, \frac{-(a-b)}{2}\right). \tag{3}$$

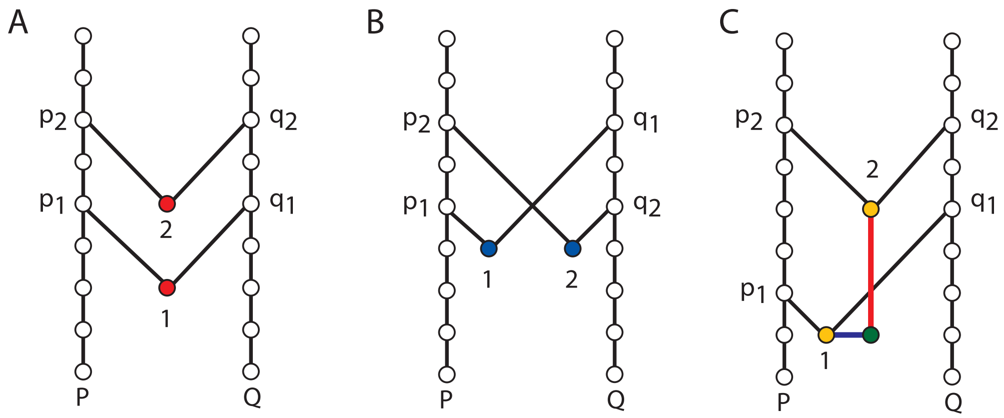

FIGURE 2. (A) Some pairs of events project so that both chains agree as to the order in which they are observed. This is the symmetric, or chain-like, configuration, which corresponds to a time-like separation. (B) Other pairs of events project so that the order in which they are observed by one chain is reverse that of the other chain. This is the anti-symmetric, or antichain-like, configuration, which corresponds to a space-like separation. (C) Any interval can be decomposed into the sum of a symmetric interval and an antisymmetric interval.

This functional equation has several potential solutions:

$$F1.\ f(a,b) = a \tag{4}$$
$$F2.\ f(a,b) = b \tag{5}$$
$$F3.\ f(a,b) = ab \tag{6}$$
$$F4.\ f(a,b) = (a+b)^n \quad n \in \text{odd} \tag{7}$$
$$F5.\ f(a,b) = a^2+b^2 \tag{8}$$

Instead of mapping the pair to a scalar, we could also take the lattice product of the two chains and identify a scalar valuation on the product lattice from the valuations assigned to each chain. Consistency requires that these two approaches should agree with one another. Since the lattice product is associative, the scalar measure also must obey the associativity equation [3, 4]

$$g(f(a,b)) = g(a)+g(b), \tag{9}$$

where g is an arbitrary function.

This results in two solutions. The first solution, $f(a,b) = a+b$, is given by $F4$ with $n = 1$ and $g(\cdot)$ equal to the identity. Since this scalar is proportional to the symmetric component of the decomposition, we referred to it as the *symmetric scalar*. The symmetric scalar trivially satisfies additivity under the symmetric/antisymmetric decomposition. Note that, while the antisymmetric component does satisfy additivity, it does not satisfy associativity and therefore it is not a consistent measure for the interval. The second solution is given by $F3$, where we have $f(a,b) = ab$ with $g(\cdot) = log(\cdot)$, so that the scalar associated with the pair (a,b) is the product of its components ab. We refer to this as the *interval scalar* and denote it with the *symbol* Δs^2

$$\Delta s^2 = (p_b - p_a)(q_b - q_a).$$

Note that nothing is really being squared—it is simply the product of two numbers. The interval scalar obeys additivity under this decomposition, since

$$pq = \left(\frac{p+q}{2}\right)\left(\frac{p+q}{2}\right) + \left(\frac{p-q}{2}\right)\left(\frac{q-p}{2}\right), \tag{10}$$

which can be rewritten as

$$pq = \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2. \tag{11}$$

Coordinates

We now introduce a change of variables to emphasize the symmetric and antisymmetric nature of the pair. Given events a and b, we have the interval pair given by

$$(\Delta p, \Delta q) = (p_a - p_b, q_a - q_b). \tag{12}$$

We define coordinates

$$\begin{aligned} \Delta t &= \frac{\Delta p + \Delta q}{2} \\ \Delta x &= \frac{\Delta p - \Delta q}{2} \end{aligned}$$

which results in

$$(\Delta p, \Delta q) = (\Delta t + \Delta x, \Delta t - \Delta x) = (\Delta t, \Delta t) + (\Delta x, -\Delta x). \tag{13}$$

The interval scalar is then simply

$$\Delta s^2 = \Delta p \Delta q = \Delta t^2 - \Delta x^2, \tag{14}$$

which we recognize immediately as the Minkowski metric.

SPECIAL RELATIVITY

The pair of synchronized chains selected for quantification is arbitrary. Instead of choosing synchronized chains **P** and **Q**, we could have chosen two other synchronized chains **P**$'$ and **Q**$'$, (Figure 3) such that they are coordinated with **P** and **Q** so that successive events in **P**$'$ and **Q**$'$ project to intervals where $\Delta p = m$ and $\Delta q = n$. We say that chains **P** and **P**$'$ are *coordinated*, and refer to each pair of chains as an *inertial frame of reference*, or a *frame* for short.

It is the ratio of intervals that is relevant when comparing frames, so we define $\rho_{21} = m/n$, which describes Frame 2 with respect to Frame 1. One can show that the pair (p_1, q_1) transforms as

$$(p_2, q_2) = (p_1 \rho_{21}{}^{-1}, q_1 \rho_{21}), \tag{15}$$

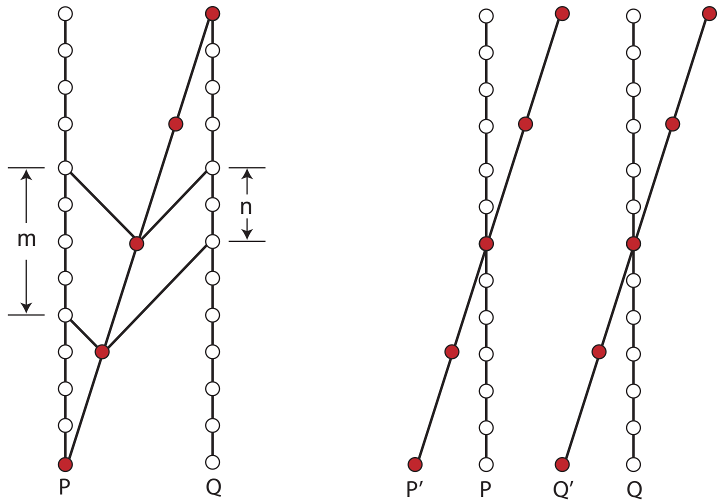

FIGURE 3. On the left we illustrate another possible relationship among chains. The new chain is coordinated to the original pair such that successive events result in projections $\Delta p = m$ and $\Delta q = n$. This new chain can be used to construct a second pair of observers and thus define another frame of reference.

and that the scalar interval remains invariant. Rewriting the pair in terms of coordinates, we have

$$(\Delta t_2 + \Delta x_2, \Delta t_2 - \Delta x_2) = ((\Delta t_1 + \Delta x_1)\rho_{21}{}^{-1}, (\Delta t_1 - \Delta x_1)\rho_{21}). \quad (16)$$

Solving for Δt_2 and Δx_2, we find that the space and time components mix

$$\Delta t_2 = \frac{\rho_{21} + \rho_{21}{}^{-1}}{2}\Delta t_1 + \frac{\rho_{21} - \rho_{21}{}^{-1}}{2}\Delta x_1 \quad (17)$$

$$\Delta x_2 = \frac{\rho_{21} - \rho_{21}{}^{-1}}{2}\Delta t_1 + \frac{\rho_{21} + \rho_{21}{}^{-1}}{2}\Delta x_1. \quad (18)$$

By defining

$$\beta_{21} = \frac{\rho_{21}{}^2 - 1}{\rho_{21}{}^2 + 1} \quad (19)$$

we find the Lorentz transformation in coordinate form

$$\Delta t_2 = \frac{1}{\sqrt{1 - \beta_{21}{}^2}}\Delta t_1 + \frac{-\beta_{21}}{\sqrt{1 - \beta_{21}{}^2}}\Delta x_1 \quad (20)$$

$$\Delta x_2 = \frac{-\beta_{21}}{\sqrt{1 - \beta_{21}{}^2}}\Delta t_1 + \frac{1}{\sqrt{1 - \beta_{21}{}^2}}\Delta x_1. \quad (21)$$

The result is that we have derived that the speed β is the relevant quantity relating inertial frames. Invariance of the speed $\beta = \pm 1$ follows immediately.

CONCLUSION

The theory of special relativity is derived as a quantification of a partially ordered set of events. This is performed without assuming anything about space, time, motion, light or the principle of relativity. Instead we assume that events are fundamental, and that some events have the potential to be influenced by other events, and that this potential is not reciprocal. We also assume that the poset of events is dense such that we can identify events to meet our specifications. This leads to special relativity, and suggests that general relativity arises when there are specific constraints on the events themselves.

It is important to keep in mind that this method of quantification is not unique, and that other methods are possible. Furthermore, the distinction between chain-like and antichain-like intervals leads to the distinction between time-like and space-like relationships with time being inherently one-dimensional and space being multi-dimensional through further decomposition of the spatial pair [5]. While convenient, the proposed symmetric/antisymmetric decomposition is arbitrary, and this suggests that the concepts of space and time, while convenient, are not fundamental. This is further supported by the fact that space and time coordinates are not preserved by Lorentz transformations. What is fundamental is the concept that events can be ordered.

Ordered of events was found to be both necessary and sufficient in our recent derivation of Feynman's rules for quantum mechanics [6, 7]. Furthermore, in that derivation, we assumed that pairs of numbers are required to quantify a quantum state, and here we show that pairs are fundamental as well. This not only suggests a deeper connection between the two theories, but also that pairs play a fundamental role in quantifying partially ordered sets and that the resulting constraints give rise to physical laws.

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