Entropic Time

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Abstract. The formulation of quantum mechanics within the framework of entropic dynamics includes several new elements. In this paper we concentrate on one of them: the implications for the theory of time. Entropic time is introduced as a book-keeping device to keep track of the accumulation of changes. One new feature is that, unlike other concepts of time appearing in the so-called fundamental laws of physics, entropic time incorporates a natural distinction between past and future.

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INTRODUCTION

A general framework for dynamics based on the method of maximum entropy is applied to non-relativistic quantum mechanics. The basic assumption of entropic dynamics is that in addition to the particles of interest there exist other variables, which we may call "hidden", to which we can associate an entropy. The evolution of the particles is a diffusion driven by the entropy of the hidden variables. The second important assumption is that there is a conserved energy. In [1][2] it was shown that such a conservative diffusion is equivalent to the Schrödinger equation.

Entropic dynamics differs from other approaches to quantum mechanics in several important aspects. One is the explicitly epistemological emphasis: the laws of physics in this approach are rules for processing information. A second new element concerns the statistical interpretation of the wave function. The magnitude of the wave function is interpreted in the usual way. What is new is that the phase also receives a statistical interpretation: it is directly related to the entropy of the hidden variables.

Our goal in this paper is to focus on a third new aspect. We discuss how a dynamics driven by entropy naturally leads to an "entropic" notion of time. Entropic time is introduced as a convenient book-keeping device to keep track of the accumulation of change. Our task here is to develop a model that includes (a) something one might identify as an "instant", (b) a sense in which these instants can be "ordered", (c) a convenient concept of "duration" measuring the separation between instants. The welcome new feature is that entropic time is intrinsically directional. Thus, an arrow of time is generated automatically. We also discuss the relation between entropic time, which is a purely inferential device, and the presumably more objective notion of "physical" time. We argue that for the pragmatic purpose of predicting the empirically observable correlations among particles nothing more "physical" than entropic time is needed.

ENTROPIC DYNAMICS

The objective is to make inferences about the positions $x \in \mathcal{X}_N$ of N particles on the basis of information about some hidden variables $y \in \mathcal{Y}$ [2]. For a single particle the configuration space $\mathcal{X}$ is Euclidean and 3-dimensional with metric $\gamma_{ab} = \delta_{ab}/\sigma^2$. We will focus on a single particle because the generalization to N particles is straightforward – the anisotropy of $\mathcal{X}_N$ is easily represented by including different scale factors σ_i^2 for different particles.

The number, the nature, and the values y of the hidden variables need not be specified – they will remain "hidden" throughout the analysis. We only need to assume that the unknown y are described by a probability distribution $p(y|x)$ that depends on the position x of the particle. To each $x \in \mathcal{X}$ there corresponds a $p(y|x)$ and therefore the set $\mathcal{M} = \{p(y|x); x \in \mathcal{X}\}$ is a 3-dimensional statistical manifold. Most features of $p(y|x)$ turn out to be irrelevant to the dynamics of x; what turns out to be relevant is the entropy of the hidden variables,

$$S[p,q] = -\int dy\, p(y|x) \log \frac{p(y|x)}{q(y)} = S(x) \ , \tag{1}$$

where $q(y)$ is some underlying measure which need not be specified further. Note that x enters as a parameter in $p(y|x)$ so that the entropy is a function of x: $S[p,q] = S(x)$. As we shall see, $S(x)$ will later be determined from energy considerations so that it is not necessary to know $p(y|x)$ explicitly.

To find the probability $P(x'|x)$ of a short step from x to a nearby point x' we use the method of maximum entropy [3].

The relevant space is $\mathcal{X} \times \mathcal{Y}$. This is because we want to infer x' on the basis of information about the new y'. What we need is the joint distribution $P(x', y'|x)$ and the appropriate (relative) entropy is

$$\mathcal{S}[P,Q] = -\int dx' dy'\, P(x', y'|x) \log \frac{P(x', y'|x)}{Q(x', y'|x)} \ . \tag{2}$$

The relevant information is introduced through the prior $Q(x', y'|x)$ and through suitable constraints on the acceptable posteriors $P(x', y'|x)$.

The prior reflects a state of extreme ignorance: the new (x', y') could in principle be anywhere in $\mathcal{X} \times \mathcal{Y}$, and we have no idea how they are related to each other – knowing x' tells us nothing about y' and vice versa. This is described by a uniform distribution (probabilities are proportional to volumes). The chosen prior is $Q(x', y'|x) \propto q(y')$.

The first constraint on the posterior $P(x', y'|x) = P(x'|x)P(y'|x', x)$ establishes the known relation between y' and x'. It says that $P(y'|x', x) = p(y'|x') \in \mathcal{M}$. The uncertainty in y' depends only on the current position x', and not on any previous value x.

The second constraint reflects the physical fact that motion is continuous. Motion over large distances happens but only through the successive accumulation of many short steps. Let $x' = x + \Delta x$. We require that the expectation $\langle \Delta\ell^2 \rangle = \langle \gamma_{ab} \Delta x^a \Delta x^b \rangle = \kappa$ be some small but for now unspecified numerical value κ, which we take to be independent of x in order to reflect the translational symmetry of the space $\mathcal{X}$.

Varying $P(x'|x)$ to maximize $\mathcal{S}[P,Q]$ in (2) subject to the two constraints plus normalization gives

$$P(x'|x) = \frac{1}{\zeta} e^{S(x') - \frac{1}{2}\alpha\,\gamma_{ab}\,\Delta x^a \Delta x^b} \ , \tag{3}$$

where ζ is a normalization constant, and the Lagrange multiplier α is determined in the standard way, $\partial \log \zeta / \partial \alpha = -\kappa/2$.

$P(x'|x)$ shows that in the limit of large α we expect short steps in essentially random directions with a small anisotropic bias along the entropy gradient. Expanding the exponent in $P(x'|x)$ about its maximum gives a Gaussian distribution,

$$P(x'|x) \propto \exp\left[-\frac{\alpha}{2\sigma^2}\delta_{ab}\left(\Delta x^a - \Delta\bar{x}^a\right)\left(\Delta x^b - \Delta\bar{x}^b\right)\right], \tag{4}$$

The displacement $\Delta x^a = \Delta\bar{x}^a + \Delta w^a$ can be expressed as the expected drift plus a fluctuation

$$\Delta\bar{x}^a = \frac{\sigma^2}{\alpha}\delta^{ab}\partial_b S(x) \ , \tag{5}$$

$$\left\langle \Delta w^a \right\rangle = 0 \quad \text{and} \quad \left\langle \Delta w^a \Delta w^b \right\rangle = \frac{\sigma^2}{\alpha}\delta^{ab} \ . \tag{6}$$

As $\alpha \to \infty$ the fluctuations become dominant: the drift $\Delta\bar{x} \propto \alpha^{-1}$ while Δw is of order $\alpha^{-1/2}$. This implies that, as in Brownian motion, the trajectory is continuous but not differentiable.

ENTROPIC TIME

The necessity to keep track of the accumulation of small changes requires us to develop appropriate book-keeping tools. Here we show how a dynamics driven by entropy naturally leads to an "entropic" notion of time.

An ordered sequence of instants

The foundation of all notions of time is dynamics. In entropic dynamics, at least for infinitesimally short steps, change is given by the transition probability $P(x'|x)$ in eq.(4). The nth step takes us from $x = x_{n-1}$ to $x' = x_n$. Using the product rule for the joint probability, $P(x_n, x_{n-1}) = P(x_n|x_{n-1})P(x_{n-1})$, and integrating over x_{n-1}, we get

$$P(x_n) = \int d^3x_{n-1}\, P(x_n|x_{n-1})P(x_{n-1}) \ . \tag{7}$$

This equation is a direct consequence of the laws of probability. However, if $P(x_{n-1})$ happens to be the probability of different values of x_{n-1} *at a given instant of entropic time t*, then we will interpret $P(x_n)$ as the probability of values of x_n at the "later" instant of entropic time $t' = t + \Delta t$. Accordingly, we write $P(x_{n-1}) = \rho(x,t)$ and $P(x_n) = \rho(x',t')$ so that

$$\rho(x',t') = \int dx\, P(x'|x)\rho(x,t) \tag{8}$$

Nothing in the laws of probability that led to eq.(7) forces this interpretation on us—this is an independent assumption about what constitutes time in our model. We use eq.(8) to define what we mean by an instant: *if the distribution $\rho(x,t)$ refers to one instant, then the distribution $\rho(x',t')$ defines what we mean by the "next" instant* and eq.(8) allows entropic time to be constructed one instant after another.

The arrow of entropic time

Time constructed according to eq.(8) is remarkable in yet another respect: the inference implied by $P(x'|x)$ in eq.(4) incorporates an intrinsic directionality in entropic time: there is an absolute sense in which $\rho(x,t)$ is prior and $\rho(x',t')$ is posterior.

Suppose we wanted to find a time-reversed evolution. We would write

$$\rho(x,t)=\int dx'\, P(x|x')\rho(x',t')\,. \tag{9}$$

This is perfectly legitimate but in order to be correct $P(x|x')$ cannot be obtained from eq.(4) by merely exchanging x and x'. According to the rules of probability theory $P(x|x')$ is related to eq.(4) by Bayes' theorem,

$$P(x|x')=\frac{P(x)}{P(x')}P(x'|x)\ . \tag{10}$$

In other words, one of the two transition probabilities, either $P(x|x')$ or $P(x|x')$, *but not both*, can be given by the maximum entropy distribution eq.(4). The other is related to it by Bayes' theorem. I hesitate to say that this is what breaks the time-reversal symmetry because the symmetry was never there in the first place. There is no symmetry between prior and posterior; there is no symmetry between the inferential past and the inferential future.

The puzzle of the arrow of time has a long history (see *e.g.* [4]). The standard question is how can an arrow of time be derived from underlying laws of nature that are symmetric? Entropic dynamics offers a new perspective because it does not assume any underlying laws of nature – whether they be symmetric or not – and its goal is not to explain the asymmetry between past and future. The asymmetry is the inevitable consequence of entropic inference. From the point of view of entropic dynamics the challenge does not consist in explaining the arrow of time, but rather in explaining how it comes about that despite the arrow of time some laws of physics turn out to be reversible. Indeed, even when the derived laws of physics – in our case, the Schrödinger equation – turns out to be fully time-reversible, *entropic time itself only flows forward*.

A convenient scale of time

Duration is defined so that motion looks simple. Since longer steps presumably take a longer time, specifying the interval Δt between successive instants amounts to specifying the multiplier $\alpha(x,t)$ in terms of Δt. For large α the dynamics is dominated by the fluctuations Δw. In order that the fluctuations $\left\langle \Delta w^a \Delta w^b \right\rangle$ reflect the symmetry of translations in space and time we choose α independent of x and t, $\alpha(x,t)=\tau/\Delta t=$ constant, where τ is a constant that fixes the units of time. With this choice the dynamics is indeed simple: $P(x'|x)$ in (4) becomes a standard Wiener process. The displacement is $\Delta x=b(x)\Delta t+\Delta w$, where $b(x)$ is the drift velocity,

$$\langle \Delta x^a \rangle = b^a \Delta t \quad \text{with} \quad b^a(x)=\frac{\sigma^2}{\tau}\delta^{ab}\partial_b S(x)\ , \tag{11}$$

and Δw is the fluctuation,

$$\left\langle \Delta w^a \right\rangle = 0 \quad \text{and} \quad \left\langle \Delta w^a \Delta w^b \right\rangle = \frac{\sigma^2}{\tau} \Delta t \, \delta^{ab} \; . \tag{12}$$

The formal similarity to Nelson's stochastic mechanics [5]-[7] is evident. The new elements here are (1) that eq.(11) has been derived rather than postulated, and (2) that the previously uninterpreted scalar function $S(x)$ now receives a definite interpretation as the entropy of the hidden variables.

THE SCHRÖDINGER EQUATION

The evolution of $\rho(x,t)$ obtained by iterating (8) together with (11)-(12) is given by the Fokker-Planck equation (FP) which can be written as a continuity equation, [8]

$$\partial_t \rho = -\partial_a \left(v^a \rho \right) \; , \tag{13}$$

where $v^a = b^a + u^a$, the *current velocity*, is given in terms of the drift velocity b^a, eq.(11), and the *osmotic velocity*

$$u^a \stackrel{\text{def}}{=} -\frac{\sigma^2}{\tau} \partial^a \log \rho^{1/2} \; . \tag{14}$$

Thus the probability flow ρv^a has two components, one is the drift current ρb^a, and the other is the diffusion current, $\rho u^a = -\frac{\sigma^2}{2\tau} \partial^a \rho$. Using (11) and (14) the current velocity can be expressed as a gradient too,

$$v^a = \frac{\sigma^2}{\tau} \partial^a \phi \quad \text{where} \quad \phi = S - \log \rho^{1/2} \tag{15}$$

As long as the geometry of the statistical manifold $\mathcal{M}$ is rigidly fixed the density $\rho(x,t)$ is the only degree of freedom available and the dynamics described by the FP equation (13) is just standard diffusion – quantum mechanics requires a second degree of freedom.

The natural solution is to allow the manifold $\mathcal{M}$ to participate in the dynamics. Then the entropy of the hidden variables becomes a second dynamical field, $S = S(x,t)$. Quantum dynamics consists of the coupled evolution of $\rho(x,t)$ and $S(x,t)$. An equivalent but more convenient choice of variables is the scalar function $\phi = S - \log \rho^{1/2}$ in (15).

To specify the dynamics of the manifold $\mathcal{M}$ we follow Nelson [6] and impose that the dynamics be "conservative," that is, one requires the conservation of a certain functional $E[\rho,\phi]$ of $\rho(x,t)$ and $\phi(x,t)$ that we will call the "energy". In the non-relativistic regime the "energy" functional is [2]

$$E[\rho,\phi] = \int d^3x \, \rho(x,t) \left(\frac{1}{2} m v^2 + \frac{1}{2} m u^2 + V(x) \right) \; , \tag{16}$$

which includes terms that are at time reversal invariant and at most quadratic in the velocities v^2 and u^2, and where $V(x)$ represents a "potential" energy. The coefficient m is related to other constants in the theory by $m = \eta\tau/\sigma^2$, where σ^2 is the length scale in the metric of $\mathcal{X}$ (required for the squared velocities), and the new constant η is introduced to take care of units – if E is given in units of energy, then m has units of mass. It is possible to assign different coefficients to the v^2 and u^2 terms but this does not change the final conclusions. (For further details see [2].) In these units the current and osmotic velocities, eqs.(15) and (14), are

$$mv_a = \eta\,\partial_a\phi \quad \text{and} \quad mu_a = -\eta\partial_a \log\rho^{1/2} \,. \tag{17}$$

The FP equation and the energy E become

$$\dot\rho = -\partial_a\left(\rho v^a\right) = -\frac{\eta}{m}\partial^a\left(\rho\partial_a\phi\right) \tag{18}$$

and

$$E = \int dx\,\rho\left(\frac{\eta^2}{2m}(\partial_a\phi)^2 + \frac{\mu\eta^2}{2m^2}(\partial_a\log\rho^{1/2})^2 + V\right) \,. \tag{19}$$

Imposing that $\dot E = 0$ for arbitrary choices of $\dot\rho$ leads after some algebra [2] to

$$\eta\dot\phi + \frac{\eta^2}{2m}(\partial_a\phi)^2 + V - \frac{\eta^2}{2m}\frac{\nabla^2\rho^{1/2}}{\rho^{1/2}} = 0 \,. \tag{20}$$

Equations (18) and (20) are the coupled dynamical equations we seek. The evolution of $\rho(x,t)$, eq.(18), is guided by $\phi(x,t)$; and the evolution of $\phi(x,t)$, eq.(20), is determined by $\rho(x,t)$.

The two real equations can be condensed into a single complex equation by combining ρ and ϕ into a complex function $\Psi = \rho^{1/2}\exp(i\phi)$. Computing the time derivative $\dot\Psi$ and using eqs. (18) and (20) leads to the Schrödinger equation,

$$i\eta\frac{\partial\Psi}{\partial t} = -\frac{\eta^2}{2m}\nabla^2\Psi + V\Psi \,, \tag{21}$$

which allows us to identify m with the particle mass and η with Planck's constant, $\eta = \hbar$. Therefore the Schrödinger equation describes a conservative diffusion driven by the entropy of the hidden variables.

Other attempts to derive quantum theory start from an underlying, perhaps stochastic, classical mechanics. The entropic dynamics approach is different in that it does not assume an underlying classical substrate; entropic dynamics provides a derivation of *both Schrödinger's equation and also Newton's* $F = ma$. Classical mechanics is recovered in the usual limits of $\hbar \to 0$ or $m \to \infty$. Indeed, writing $S_{HJ} = \eta\phi$ in eq.(20) and letting $m \to \infty$ with S_{HJ}/m fixed leads to the classical Hamilton-Jacobi equation

$$\dot S_{HJ} + \frac{1}{2m}(\partial_a S_{HJ})^2 + V = 0 \,, \tag{22}$$

while eq.(17) gives $mv = \partial S_{HJ}$ and $u = 0$. Since $mv \approx mb = \eta \partial S$ we see that *classical particles move along the gradient of the entropy of the hidden variables*, with vanishing fluctuations $\langle \Delta w^a \Delta w^b \rangle = \frac{\eta}{m} \Delta t \, \delta^{ab} \to 0$. The classical Hamilton-Jacobi action is interpreted as the entropy of the hidden variables, $S_{HJ} = \eta S$.

ENTROPIC TIME *VS.* "PHYSICAL" TIME

We can now return to the question of the actual connection between entropic and physical time. If the order of the sequence of inferential steps generated by the Fokker-Planck equation is not the same as the order relative to a presumably more fundamental "physical" time, why should 'entropic time' deserve to be called 'time' at all?

The answer is that we are typically concerned with systems that include, in addition to the particles of interest, also at least one other system that one might call a "clock". The goal is to make inferences about correlations between the particles and the various states of the clock. Whether the inferred sequence of states of the particle-clock composite agrees or not with an order in "physical" time turns out to be quite irrelevant. It is only the correlations among the particles and the clock that are observable and not their "absolute" order.

Earlier we introduced a concept of simultaneity through the density $\rho(x,t)$ which gives the probability of different values of x at the same instant. We can now tackle the same issue from a different angle. Since the particle can follow different paths we need a criterion to decide whether an event x' reached along one path is earlier or later than another event x'' reached along a different path. This is where a clock comes in handy. The clock could be, for example, just a sufficiently massive particle that follows a deterministic trajectory, $x_C = \bar{x}_C(t)$, and remains largely unaffected by the motion of other particles.

The idea is that when we compute the probability that, say, after n steps the particle is found at the point x_n we implicitly *assume* that its three coordinates x_n^1, x_n^2, and x_n^3 are attained *simultaneously*. We make the same *assumption* in the larger configuration space of the composite particle-clock system, $x_n^A = (x_n^a, x_{Cn}^\alpha)$. The particle coordinates x_n^a $(a = 1,2,3)$ are assumed to be simultaneous with the clock coordinates x_{Cn}^α $(\alpha = 4,5,\ldots)$. Thus, when we say that at the nth step the particle is at x_n^a while the clock is at x_{Cn}^α it is implicit that these positions are attained *at the same time*.

By "the time is t" we just mean that "the clock is in its state $x_C = \bar{x}_C(t)$." We say that the possible event that the particle reached x' along one path is simultaneous with another possible event x'' reached along a different path when both are simultaneous with the same state $\bar{x}_C(t)$ of the clock. We usually omit referring to the clock and just say that x' and x'' happen "at the same time t." This justifies using the distribution $\rho(x,t)$ as the definition of an instant of time.

In the end the justification for the assumptions underlying entropic dynamics lies in their empirical success. The ordering scheme provided by entropic time allows one to predict correlations. Since these predictions, which are given by the Schrödinger equation, turn out to be tremendously successful one concludes that nothing deeper or more "physical" than entropic time is needed.

SOME REMARKS

Since it is the correlations between particles and clocks that are empirically accessible and not their absolute order we argued that entropic time is all we need. The time t that appears in the laws of physics, be it the Schrödinger equation or Newton's $F = ma$, is entropic time.

Julian Barbour in his relational approach to time in the context of classical dynamics [9] has made the strong claim that time is not real, that time is a mere illusion. Barbour's claim is much stronger than ours. The reason is that from Barbour's perspective the laws of physics directly reflect the laws of nature. Then the absence of physical time in the laws of physics is strong evidence for its non-existence. In this view, just as the ether of an earlier generation, time is an obsolete feature of outdated physics.

From the perspective of entropic dynamics, however, the laws of physics are not laws of nature but mere rules of inference. The fact that it is entropic time that appears in the laws of physics does not entitle us to conclude that physical time does not exist. We can only conclude that physical time is not the carrier of any piece of information that happens to be relevant for our current inferences.

One possibility is that physical time is an illusion, which explains why it would be irrelevant in the first place. Another possibility is that physical time actually exists and is identical with entropic time – which explains why it shows up in the equations of physics. Whether physical time is real or a mere illusion remains a question for the future.

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