Harmonic maps relative to α-connections of statistical manifolds

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Abstract. In this paper, we study harmonic maps relative to α-connections, but not necessarily standard harmonic maps. A standard harmonic map is defined by the first variation of the energy functional of a map. A harmonic map relative to an α-connection is defined by an equation similar to a first variational equation, though it is not induced by the first variation of the standard energy functional. In this paper, we define energy functionals of maps relative to α-connections of statistical manifolds. Next, we show that, for harmonic maps relative to α-connections, the Euler-Lagrange equations are induced by first variations of energy functionals relative to α-connections.

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INTRODUCTION

Harmonic maps are important objects in certain branches of differential geometry and theoretical physics. Standard Harmonic maps are defined by the Levi-Civita connections of Riemannian manifolds. However, α-connections, but not necessarily Levi-Civita connections, of statistical manifolds are useful for information geometry. In our previous paper, we defined harmonic maps, the Euler-Lagrange equations, and tension fields relative to α-connections on Hessian domains and on their level sets. We also proposed several theorems in it. Though standard harmonic maps are induced by first variations of standard energy functionals, harmonic maps relative to α-connections are not necessarily induced by them.

In this paper, we define energy functionals of maps relative to α-connections of dually flat statistical manifolds. For the purpose, we explain dually flat statistical manifolds and Riemannian metrics preserved by α-connections. Next, we show that, for harmonic maps relative to α-connections, the Euler-Lagrange equations are induced by first variations of energy functionals relative to α-connections.

DUALLY FLAT STATISTICAL MANIFOLDS AND RIEMANNIAN METRICS PRESERVED BY α-CONNECTIONS

Given a torsion-free affine connection ∇ and a pseudo-Riemannian metric g on a manifold M, the triple (M,∇,g) is said to be a *statistical manifold* if ∇g is symmetric. If the curvature tensor R of ∇ vanishes, (M,∇,g) is said to be *flat*.

Let (M,∇,g) be a statistical manifold and let ∇' be an affine connection on M such that

$$Xg(Y,Z) = g(\nabla_X Y,Z) + g(Y,\nabla'_X Z) \text{ for } X,Y \text{ and } Z \in \Gamma(TM),$$

where $\Gamma(TM)$ is the set of smooth tangent vector fields on M. The affine connection ∇' is torsion free and $\nabla' g$ is symmetric. Then ∇' is called the *dual connection* of ∇. The triple (M,∇',g) is the *dual statistical manifold* of (M,∇,g), and (∇,∇',g) defines the *dualistic structure* on M. The curvature tensor of ∇' vanishes if and only if the curvature tensor of ∇ also vanishes. Under these conditions, (∇,∇',g) becomes a *dually flat structure*.

For $\alpha \in \mathbf{R}$, an affine connection defined by

$$\nabla^{(\alpha)} \equiv \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla' \tag{1}$$

is called an α-*connection* of (M,∇,g). The triple $(M,\nabla^{(\alpha)},g)$ is also a statistical manifold, and $\nabla^{(-\alpha)}$ is the dual connection of $\nabla^{(\alpha)}$. The 1-connection, the (-1)-connection, and the 0-connection correspond to the ∇, ∇', and the Levi-Civita connection of (M,g), respectively. An α-connection is torsion free, i. e.,

$$\nabla^{(\alpha)}_X Y - \nabla^{(\alpha)}_Y X - [X,Y] = 0$$

and, however, it does not need to be flat [1].

Let (M,∇,g) be a flat statistical manifold with a Riemannian metric g. Let $\{x^1,\ldots,x^m\}$ be a local affine coordinate system on M satisfying that $\nabla dx^i = 0$ for $i = 1,\cdots,m$. A local region of a flat statistical manifold is a Hessian domain, so the Riemannian metric g is described as $g = Dd\varphi = \sum_{i,j=1}^m (\partial^2\varphi/\partial x^i \partial x^j) dx^i dx^j$ for a function φ locally [2].

We set $g_{ij} = \partial^2\varphi/\partial x^i \partial x^j$ and $[g^{ij}] = [g_{ij}]^{-1}$. For $\alpha \in \mathbf{R}$, we define a Riemannian metric $g^{(\alpha)}$ on M by

$$[{g^{(\alpha)}}_{ij}] = [g_{ij}]^{1-\alpha} = S\, diag[\lambda_1^{1-\alpha},\lambda_2^{1-\alpha},\cdots,\lambda_m^{1-\alpha}]\, S^T,$$

where $[g_{ij}] = S\, diag[\lambda_1,\lambda_2,\cdots,\lambda_m]\, S^T$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$) is the Schur decomposition of a positive definite matrix $[g_{ij}]$, S is an orthogonal matrix, and S^T is the transporsed matrix. Note that

$$[{g^{(1)}}_{ij}] = [g_{ij}]^0 = e; \quad \text{the identity matrix},$$

$$[{g^{(0)}}_{ij}] = [g_{ij}], \quad [{g^{(-1)}}_{ij}] = [g_{ij}]^2.$$

Then the next lemma holds.

Lemma 1 For any $\alpha \in \mathbf{R}$, an α-connection $\nabla^{(\alpha)}$ preserves a Riemannian metric $g^{(\alpha)}$, i. e.,

$$Xg^{(\alpha)}(Y,Z) = g^{(\alpha)}(\nabla^{(\alpha)}_X Y,Z) + g^{(\alpha)}(Y,\nabla^{(\alpha)}_X Z) \text{ for } X,Y \text{ and } Z \in \Gamma(TM).$$

Proof. A component of $Xg^{(\alpha)}(Y,Z)$ is described by

$$\frac{\partial g^{(\alpha)}{}_{ij}}{\partial x^k} = \frac{\partial([g_{ij}]^{1-\alpha})_{ij}}{\partial x^k}$$

$$= \sum_{l=1}^{m} (\frac{\partial S_{il}}{\partial x^k}\lambda_l{}^{1-\alpha}S_{jl} + (1-\alpha)S_{il}\lambda_l{}^{-\alpha}\frac{\partial \lambda_l}{\partial x^k}S_{jl} + S_{il}\lambda_l{}^{1-\alpha}\frac{\partial S_{jl}}{\partial x^k})$$

$$= \sum_{l=1}^{m} \lambda_l{}^{-\alpha}(\frac{\partial S_{il}}{\partial x^k}\lambda_l S_{jl} + (1-\alpha)S_{il}\frac{\partial \lambda_l}{\partial x^k}S_{jl} + S_{il}\lambda_l\frac{\partial S_{jl}}{\partial x^k})$$

$$= (1-\alpha)\sum_{l=1}^{m} \lambda_l{}^{-\alpha}(\frac{\partial S_{il}}{\partial x^k}\lambda_l S_{jl} + S_{il}\frac{\partial \lambda_l}{\partial x^k}S_{jl} + S_{il}\lambda_l\frac{\partial S_{jl}}{\partial x^k}) + \alpha\sum_{l=1}^{m} \lambda_l{}^{1-\alpha}(\frac{\partial S_{il}}{\partial x^k}S_{jl} + S_{il}\frac{\partial S_{jl}}{\partial x^k}).$$

For $\alpha = 0$, we have

$$\frac{\partial g_{ij}}{\partial x^k} = \sum_{l=1}^{m} (\frac{\partial S_{il}}{\partial x^k}\lambda_l S_{jl} + S_{il}\frac{\partial \lambda_l}{\partial x^k}S_{jl} + S_{il}\lambda_l\frac{\partial S_{jl}}{\partial x^k}).$$

For $\alpha = 1$, we have

$$\frac{\partial g^{(1)}{}_{ij}}{\partial x^k} = \sum_{l=1}^{m} (\frac{\partial S_{il}}{\partial x^k}S_{jl} + S_{il}\frac{\partial S_{jl}}{\partial x^k}) = 0$$

by $[g^{(1)}{}_{ij}] = SS^T = e$. Then we obtain

$$\frac{\partial g^{(\alpha)}{}_{ij}}{\partial x^k} = (1-\alpha)[P\, diag[\lambda_1{}^{-\alpha}, \lambda_2{}^{-\alpha}, \cdots, \lambda_m{}^{-\alpha}]\, P^T]_{ij}$$

$$+\alpha[Q\, diag[\lambda_1{}^{1-\alpha}, \lambda_2{}^{1-\alpha}, \cdots, \lambda_m{}^{1-\alpha}]\, Q^T]_{ij},$$

where $P = [p_{ij}]$ and $Q = [q_{ij}]$ satisfy that

$$PP^T = [\frac{\partial g_{ij}}{\partial x^k}] = [\frac{\partial g_{ji}}{\partial x^k}], \quad QQ^T = [\frac{\partial g^{(1)}_{ij}}{\partial x^k}] = [\frac{\partial g^{(1)}_{ji}}{\partial x^k}] = 0; \text{ the zero matrix.}$$

Since g is a Hessian metric and $\nabla^{(0)}$ is the Levi-Civita connection ∇^{LC}, the next follows

$$g(\nabla^{(0)}_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = g(\frac{\partial}{\partial x^i}, \nabla^{(0)}_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^j}) = \frac{1}{2}\frac{\partial g_{kj}}{\partial x^i} = \frac{1}{2}\frac{\partial g_{ki}}{\partial x^j} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}.$$

By (1) and $\nabla dx^i = 0$, α-connections are described by

$$\Gamma^{(\alpha)}{}^k_{ij} = (1-\alpha)\Gamma^{LC}{}^k_{ij}, \quad \nabla^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \sum_{k=1}^{m} \Gamma^{(\alpha)}{}^k_{ij}\frac{\partial}{\partial x^k},$$

$$\Gamma^{(1)}{}^{k}_{ij} = 0, \quad \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{m} \Gamma^{(1)}{}^{k}_{ij} \frac{\partial}{\partial x^k}.$$

Then it holds that

$$\begin{aligned}
\frac{(1-\alpha)}{2}[P\, diag[\lambda_1{}^{-\alpha}, \lambda_2{}^{-\alpha}, \cdots, \lambda_m{}^{-\alpha}]\, P^T]_{ij} &= (1-\alpha) g^{(\alpha)}(\nabla^{(0)}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\
&= g^{(\alpha)}(\nabla^{(\alpha)}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), \\
\frac{(1-\alpha)}{2}[P\, diag[\lambda_1{}^{-\alpha}, \lambda_2{}^{-\alpha}, \cdots, \lambda_m{}^{-\alpha}]\, P^T]_{ij} &= (1-\alpha) g^{(\alpha)}(\frac{\partial}{\partial x^i}, \nabla^{(0)}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}) \\
&= g^{(\alpha)}(\frac{\partial}{\partial x^i}, \nabla^{(\alpha)}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}).
\end{aligned}$$

Hence we obtain

$$(1-\alpha)[P\, diag[\lambda_1{}^{-\alpha}, \lambda_2{}^{-\alpha}, \cdots, \lambda_m{}^{-\alpha}]\, P^T]_{ij} = g^{(\alpha)}(\nabla^{(\alpha)}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) + g^{(\alpha)}(\frac{\partial}{\partial x^i}, \nabla^{(\alpha)}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}).$$

We also have

$$g^{(\alpha)}(\nabla^{(1)}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = 0 \quad \text{for any } \alpha.$$

Thus it holds that

$$[Q\, diag[\lambda_1{}^{1-\alpha}, \lambda_2{}^{1-\alpha}, \cdots, \lambda_m{}^{1-\alpha}]\, Q^T]_{ij} = g^{(\alpha)}(\nabla^{(1)}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = 0.$$

Hence the next holds.

$$\frac{\partial g^{(\alpha)}{}_{ij}}{\partial x^k} = g^{(\alpha)}(\nabla^{(\alpha)}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) + g^{(\alpha)}(\frac{\partial}{\partial x^i}, \nabla^{(\alpha)}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}).$$

Thus $\nabla^{(\alpha)}$ preserves a Riemannian metric $g^{(\alpha)}$. $\diamondsuit$

HARMONIC MAPS AND FIRST VARIATIONS OF ENERGY FUNCTIONALS RELATIVE TO α-CONNECTIONS

Let (M, ∇, g) be a flat statistical manifold with a Riemannian metric g and $(N, \hat{\nabla}, h)$ be a flat statistical manifold with a Riemannian metric h. For $\alpha, \hat{\alpha} \in \mathbf{R}$, let $\nabla^{(\alpha)}$ and $\hat{\nabla}^{(\hat{\alpha})}$ be an α-connection of ∇ and an $\hat{\alpha}$-connection of $\hat{\nabla}$, respectively. Let $\{x^1, \ldots, x^m\}$ be a local affine coordinate system on M satisfying that $\nabla dx^i = 0$.

For a smooth map $\phi : M \to N$, we define a *density function* $e_{(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})}(\phi)$ *relative to* $(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})$ on M by

$$e_{(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})}(\phi)(x) = \frac{1}{2} \sum_{i,j=1}^{m} g^{(\alpha)ij}(\phi^* h^{(\hat{\alpha})})((\frac{\partial}{\partial x^i})_x, (\frac{\partial}{\partial x^j})_x) \tag{2}$$

$$= \frac{1}{2}\sum_{i,j=1}^{m} g^{(\alpha)ij} h^{(\hat{\alpha})}(\phi_*(\frac{\partial}{\partial x^i})_x, \phi_*(\frac{\partial}{\partial x^j})_x), \quad x \in M,$$

where $\phi^* h^{(\hat{\alpha})}$ is the pull-back Riemannian metric of $h^{(\hat{\alpha})}$ and ϕ_* is the differential of ϕ. If $\alpha = \hat{\alpha} = 0$, the definition (2) coincides with the standard density function of ϕ [3], [4].

We define a *energy functional of* ϕ *relative to* $(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})$ by

$$E(\phi) = \int_M e_{(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})}(\phi)\, v_{g^{(\alpha)}}, \tag{3}$$

where $v_{g^{(\alpha)}} = \sqrt{\det[g^{(\alpha)}{}_{ij}]}\, dx_1 \cdots dx_m = \sqrt{\det([g_{ij}]^{1-\alpha})}\, dx_1 \cdots dx_m$ is the volume form of $g^{(\alpha)}$. If $\alpha = \hat{\alpha} = 0$, the definition (3) coincides with the standard energy functional of ϕ [3], [4].

For a variation $\phi_t \in C^\infty(M,N)$, where $t \in (-\varepsilon, \varepsilon)$ is sufficiently small and $\phi_0 = \phi$, we call ϕ a *harmonic map relative to* $(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})$ if ϕ is a critical point of E, i. e.,

$$\frac{d}{dt}|_{t=0} E(\phi_t) = 0. \tag{4}$$

We assume that the closure of $\{x \in M | \phi_t(x) \neq 0\}$ has a compact support. For the purpose of calculating $(d/dt)E(\phi_t)$, we consider $E(\phi_t)$ a map $\Phi : (-\varepsilon, \varepsilon) \times M \to N$. Then the induced connection $\hat{\nabla}^{(\hat{\alpha})}$ on $\Phi_*^{-1}TN$ preserves the induced Riemannian metric $h^{(\hat{\alpha})}$ on $(-\varepsilon, \varepsilon) \times M$. Hence we obtain

$$\frac{d}{dt}E(\phi_t) = \frac{1}{2}\int_M \sum_{i,j=1}^{m} \frac{d}{dt} g^{(\alpha)ij} h^{(\hat{\alpha})}(\phi_{t*}(\frac{\partial}{\partial x^i}), \phi_{t*}(\frac{\partial}{\partial x^i}))\, v_{g^{(\alpha)}}$$

and

$$\frac{d}{dt} h^{(\hat{\alpha})}(\phi_{t*}(\frac{\partial}{\partial x^i}), \phi_{t*}(\frac{\partial}{\partial x^i})) = \frac{d}{dt} h^{(\hat{\alpha})}(\Phi_*(\frac{\partial}{\partial x^i}), \Phi_*(\frac{\partial}{\partial x^i}))$$

$$= 2h^{(\hat{\alpha})}(\hat{\nabla}^{(\hat{\alpha})}_{\frac{\partial}{\partial x^i}} \Phi_*(\frac{\partial}{\partial t}), \Phi_*(\frac{\partial}{\partial x^i}))$$

$$= 2\{\frac{\partial}{\partial x^i} h^{(\hat{\alpha})}(\Phi_*(\frac{\partial}{\partial t}), \Phi_*(\frac{\partial}{\partial x^j})) - h^{(\hat{\alpha})}(\Phi_*(\frac{\partial}{\partial t}), \hat{\nabla}^{(\hat{\alpha})}_{\frac{\partial}{\partial x^i}} \Phi_*(\frac{\partial}{\partial x^j}))\},$$

using

$$\hat{\nabla}^{(\hat{\alpha})}_{\frac{\partial}{\partial t}} \Phi_*(\frac{\partial}{\partial x^i}) - \hat{\nabla}^{(\hat{\alpha})}_{\frac{\partial}{\partial x^i}} \Phi_*(\frac{\partial}{\partial t}) = \Phi_*([\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}]) = 0.$$

Since $\nabla^{(\alpha)}$ preserves $g^{(\alpha)}$, setting

$$g^{(\alpha)}(X_t, Y) = h^{(\hat{\alpha})}(\Phi_*(\frac{\partial}{\partial t}), \Phi_*(Y)), \quad X_t, Y \in \Gamma(TM)$$

we have

$$\frac{d}{dt}E(\phi_t) = \int_M \sum_{i,j=1}^m g^{(\alpha)ij}\{\frac{\partial}{\partial x^i}g^{(\alpha)}(X_t, \frac{\partial}{\partial x^j}) - h^{(\hat{\alpha})}(\Phi_*(\frac{\partial}{\partial t}), \hat{\nabla}^{(\hat{\alpha})}_{\frac{\partial}{\partial x^i}}\Phi_*(\frac{\partial}{\partial x^j}))\}\, v_{g^{(\alpha)}}$$

$$= \int_M \sum_{i,j=1}^m g^{(\alpha)ij}\{g^{(\alpha)}(\nabla^{(\alpha)}_{\frac{\partial}{\partial x^i}}X_t, \frac{\partial}{\partial x^j}) + g^{(\alpha)}(X_t, \nabla^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})$$

$$-h^{(\hat{\alpha})}(\Phi_*(\frac{\partial}{\partial t}), \hat{\nabla}^{(\hat{\alpha})}_{\frac{\partial}{\partial x^i}}\Phi_*(\frac{\partial}{\partial x^j}))\}\, v_{g^{(\alpha)}}$$

$$= \int_M div^{(\alpha)}(X_t)\, v_{g^{(\alpha)}} + \int_M \sum_{i,j=1}^m g^{(\alpha)ij}\{h^{(\hat{\alpha})}(\Phi_*(\frac{\partial}{\partial t}), \Phi_*(\nabla^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}))$$

$$-h^{(\hat{\alpha})}(\Phi_*(\frac{\partial}{\partial t}), \hat{\nabla}^{(\hat{\alpha})}_{\frac{\partial}{\partial x^i}}\Phi_*(\frac{\partial}{\partial x^j}))\}\, v_{g^{(\alpha)}}$$

$$= -\int_M h^{(\hat{\alpha})}(\Phi_*(\frac{\partial}{\partial t}), \sum_{i,j=1}^m g^{(\alpha)ij}\{\hat{\nabla}^{(\hat{\alpha})}_{\frac{\partial}{\partial x^i}}\Phi_*(\frac{\partial}{\partial x^j}) - \Phi_*(\nabla^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})\})\, v_{g^{(\alpha)}}$$

by the similar technique in [3], [4]. In the above, we use the Green's formula relative to $(g^{(\alpha)}, \nabla^{(\alpha)})$, i. e.,

$$\int_M div^{(\alpha)}(X)\, v_{g^{(\alpha)}} \equiv \int_M \sum_{i,j=1}^m g^{(\alpha)ij}g^{(\alpha)}(\nabla^{(\alpha)}_{\frac{\partial}{\partial x^i}}X, \frac{\partial}{\partial x^j})\, v_{g^{(\alpha)}} = 0, \quad X \in \Gamma(TM),$$

which is given by the similar technique on the proof of the Green's formula relative to (g, ∇^{LC}).

Then it holds that

$$\frac{d}{dt}\big|_{t=0}E(\phi_t) = -\int_M h^{(\hat{\alpha})}(\hat{X}, \sum_{i,j=1}^m g^{(\alpha)ij}\{\hat{\nabla}^{(\hat{\alpha})}_{\frac{\partial}{\partial x^i}}\Phi_*(\frac{\partial}{\partial x^j}) - \Phi_*(\nabla^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})\})\, v_{g^{(\alpha)}},$$

where we set

$$\hat{X} = \Phi_*(\frac{\partial}{\partial t})\big|_{t=0}.$$

Then

$$\sum_{i,j=1}^m g^{(\alpha)ij}\{\hat{\nabla}^{(\hat{\alpha})}_{\frac{\partial}{\partial x^i}}\Phi_*(\frac{\partial}{\partial x^j}) - \Phi_*(\nabla^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})\} = 0$$

if and only if it hold that $(d/dt)|_{t=0}E(\phi_t) = 0$ for any $\hat{X} \in \Gamma(\phi_*^{-1}TN)$. Hence the next theorem holds.

Theorem 1 For flat statistical manifolds (M, ∇, g) and $(N, \hat{\nabla}, h)$, a map $\phi : M \to N$ is a harmonic map relative to $(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})$ if and only if

$$\tau_{(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})}(\phi) \equiv \sum_{i,j=1}^m g^{(\alpha)ij}\{\hat{\nabla}^{(\hat{\alpha})}_{\frac{\partial}{\partial x^i}}\phi_*(\frac{\partial}{\partial x^j}) - \phi_*(\nabla^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})\} = 0. \tag{5}$$

We define the *tension field of ϕ relative to* $(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})$ by the left side equation of (5). Then $\tau_{(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})}(\phi)$ coincides with $\Delta_{(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})}\phi$ the *Laplacian of ϕ relative to* $(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})$.

$$\Delta_{(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})}\phi \equiv \sum_{i,j=1}^{m} g^{(\alpha)ij}\{\hat{\nabla}^{(\hat{\alpha})}_{\frac{\partial}{\partial x^i}}\phi_*(\frac{\partial}{\partial x^j}) - \phi_*(\nabla^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})\} \tag{6}$$

Let $\phi(x) = (\phi^1(x), \cdots, \phi^n(x))$, $x \in M$. Then definition (6) is described as

$$(\Delta_{(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})}\phi)^\gamma = \sum_{i,j=1}^{m} g^{(\alpha)ij}\{\frac{\partial^2\phi^\gamma}{\partial x^i\partial x^j} - (1-\alpha)\sum_{k=1}^{m}\Gamma^{LC}{}^k_{ij}\frac{\partial\phi^\gamma}{\partial x^k}$$

$$+ \sum_{\delta,\beta=1}^{n}(1-\hat{\alpha})\hat{\Gamma}^{LC}{}^\gamma_{\delta\beta}\frac{\partial\phi^\delta}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\}, \quad \gamma = 1, \cdots, n.$$

Thus we have

Corollary 1 For flat statistical manifolds (M, ∇, g) and $(N, \hat{\nabla}, h)$, a map $\phi : M \to N$ is a harmonic map relative to $(\nabla^{(\alpha)}, \hat{\nabla}^{(\hat{\alpha})})$ if and only if

$$\sum_{i,j=1}^{m} g^{(\alpha)ij}\{\frac{\partial^2\phi^\gamma}{\partial x^i\partial x^j} - (1-\alpha)\sum_{k=1}^{m}\Gamma^{LC}{}^k_{ij}\frac{\partial\phi^\gamma}{\partial x^k} + \sum_{\delta,\beta=1}^{n}(1-\hat{\alpha})\hat{\Gamma}^{LC}{}^\gamma_{\delta\beta}\frac{\partial\phi^\delta}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\} = 0, \tag{7}$$

$$\gamma = 1, \cdots, n; \quad \textit{the Euler-Lagrange equations}.$$

Remark 1 An affine harmonic map is a harmonic map relative to $(\nabla^{LC}, \hat{\nabla}^{(1)})$. It is in the case of $\alpha = 0, \hat{\alpha} = 1$ [5].

Remark 2 The gradient map on a Hessian domain is a harmonic map relative to $(\nabla^{LC}, \hat{\nabla}^{(-1)})$ for some conditions. It is in the case of $\alpha = 0, \hat{\alpha} = -1$ [2], [6].

Remark 3 An α-affine harmonic map induced by the gradient map on a Hessian domain is a harmonic map relative to $(\nabla^{LC}, \hat{\nabla}^{(\alpha)})$ for some conditions. [7].

Remark 4 For a statistical manifold, a geodesic with respect to the primal (resp. dual) coordinate system is a harmonic map relative to $(\nabla^{(1)}, \hat{\nabla}^{(1)})$ (resp. $(\nabla^{(-1)}, \hat{\nabla}^{(-1)})$) on a 1-dimensional statistical manifold to an n-dimensional statistical manifold, where $n \geq 2$.

Remark 5 A Kähler affine harmonic map satisfies Eq. (8) of $\alpha = 1, \hat{\alpha} = 0$, and generally does not satisfies Eq. (7) [8].

$$\sum_{i,j=1}^{m} g^{ij}\{\frac{\partial^2\phi^\gamma}{\partial x^i\partial x^j} - (1-\alpha)\sum_{k=1}^{m}\Gamma^{LC}{}^k_{ij}\frac{\partial\phi^\gamma}{\partial x^k} + \sum_{\delta,\beta=1}^{n}(1-\hat{\alpha})\hat{\Gamma}^{LC}{}^\gamma_{\delta\beta}\frac{\partial\phi^\delta}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\} = 0, \tag{8}$$

Remark 6 In our previous papers, we defined a harmonic map between statistical submanifolds of flat statistical manifolds by Eq. (8), and not by Eq. (7). We showed conditions for harmonicity of maps between level surfaces of a Hessian domain in the case of $\alpha = \hat{\alpha}$ on Eq. (7) [7], [9], [10].

CONCLUSION

We describe first variations of harmonic maps relative to α-connections of statistical manifolds. It remains that we investigate applications to affine geometry, Kählerian geometry, information geometry, and so on.

The volume forms being parallel under α-connections are investigated in [11], [12]. We should evolve theory of harmonic maps in the aspect of their geometric theory with parallel forms.

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