Fisher Information Geometry of The Barycenter Map

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Abstract. We report Fisher information geometry of the barycenter map associated with Busemann function B_θ of an Hadamard manifold X and present its application to Riemannian geometry of X from viewpoint of Fisher information geometry. This report is an improvement of [I-Sat'13] together with a fine investigation of the barycenter map.

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1. BARYCENTER MAP

Let μ be a probability measure on the ideal boundary ∂X of X. A point $x \in X$ is called a *barycenter* of μ, when x is a critical point of the μ-average Busemann function on X;

$$\mathbf{B}_\mu(y) = \int_{\theta\in\partial X} B_\theta(y) d\mu(\theta), y \in X. \tag{1}$$

Denote by $\mathscr{P}^+ = \mathscr{P}^+(\partial X, d\theta)$ the space of probability measures $\mu = f(\theta)d\theta$ defined on ∂X satisfying $\mu \ll d\theta$ and with continuous density $f = f(\theta) > 0$. A point $x \in X$ is a barycenter of a measure μ if and only if the μ-average one-form $d\mathbf{B}_\mu(\cdot) = \int_{\theta\in\partial X} dB_\theta(\cdot)d\mu(\theta)$ vanishes at x.

We follow the idea given by [Douady-E], [Bes-C-G'95].

Theorem 1.1([I-Sat'14-2]). The function $\mathbf{B}_\mu$ admits for any $\mu \in \mathscr{P}^+$ a barycenter, provided (i) X satisfies the axiom of visibility and (ii) $B_\theta(x)$ is continuous in $\theta \in \partial X$.

X is said to satisfy the axiom of visibility, when any two ideal points θ, θ_1 of ∂X, $\theta \neq \theta_1$, can be joined with a geodesic in X (see [Eber-O]). In [Bes-C-G'95] the existence theorem is verified under the condtions that (i) B_θ satisfies $\lim_{x\to\theta_1} B_\theta(x) = +\infty$, when $\theta_1 \neq \theta$ and (ii) $B_\theta(\cdot)$ is continuous with respect to θ. The condition (i) can be replaced by the axiom of visibility (refer to [Ball-G-S]) to obtain Theorem 1.1.

For the uniqueness, we have:

Theorem 1.2([I-Sat'14-2],[I-Sat'14-3]). Assume (i) and (ii) in Theorem 1.1. If, for some $\mu_o \in \mathscr{P}^+$ the μ_o-average Hessian

$$(\nabla d\mathbf{B}_{\mu_o})_x(\cdot,\cdot) = \int_{\theta\in\partial X} (\nabla dB_\theta)_x(\cdot,\cdot) d\mu_o(\theta) \tag{2}$$

is positive definite on T_xX at any $x \in X$, then the existence of barycenter is unique for any $\mu \in \mathscr{P}^+$.

Thus, we obtain a map, called the *barycenter map*:

$$bar : \mathscr{P}^+ = \mathscr{P}^+(\partial X, d\theta) \to X,\, \mu \mapsto x,$$

where x is a barycenter of μ.

Notice that the differentiability of $\mathbf{B}_\mu$ is guaranteed when the Hessian of B_θ is uniformly bounded with respect to θ and (X,g) is of uniformly bounded Ricci curvature.

2. A FIBRE SPACE STRUCTURE OF $\mathscr{P}^+$ OVER X AND FISHER INFORMATION METRIC

It is easily shown that the map *bar* is regular at any μ, that is, the differential map

$$d\,bar_\mu : T_\mu \mathscr{P}^+ \to T_y X$$

is surjective(see [Bes-C-G'96]). Moreover the map *bar* is itself surjective and hence it yields a fibre space projection with fibre $bar^{-1}(x)$ over $x \in X$,

$$\begin{array}{c} \mathscr{P}^+(\partial X, d\theta) \\ \downarrow\, bar \\ X \end{array} \tag{3}$$

provided X carries Busemann-Poisson kernel $P(x,\theta)d\theta = \exp\{-qB_\theta(x)\}$, the fundamental solution of Dirichlet problem at the boundary ∂X, namely, Poisson kernel represented by $B_\theta(x)$ in an exponential form ($q = q(X) > 0$ is the volume entropy of X). An Hadamard manifold admitting Busemann-Poisson kernel turns out to be asymptotically harmonic ([Led],[I-Sat'11]), since ΔB_θ is constant for any θ.

The tangent space $T_\mu bar^{-1}(x)$ of $bar^{-1}(x)$ is characterized as:

$$\{\tau \in T_\mu \mathscr{P}^+ \mid \int_{\theta \in \partial X} (dB_\theta)_x(U)\, d\tau(\theta) = 0,\, \forall U \in T_xX\},$$

so one gets:

Proposition 2.1. $\tau \in T_\mu \mathscr{P}^+$ belongs to $T_\mu bar^{-1}(x)$ if and only if

$$G_\mu\left(\tau, N_\mu(U)\right) = 0,\, \forall U \in T_xX \tag{4}$$

where G_μ is the Fisher information metric of $\mathscr{P}^+$ at μ and $N_\mu : T_xX \to T_\mu \mathscr{P}^+$ is a linear map defined by

$$\begin{array}{rcl} N_\mu : T_xX & \to & T_\mu \mathscr{P}^+ \\ U & \mapsto & (dB_\theta)_x(U)\, d\mu(\theta). \end{array} \tag{5}$$

From this we have:

Proposition 2.2. At any $\mu \in \mathscr{P}^+$ the tangent space $T_\mu \mathscr{P}^+$ admits an orthogonal direct sum decomposition into the vertical and horizontal subspaces as

$$T_\mu \mathscr{P}^+ = T_\mu bar^{-1}(x) \oplus ImN_\mu, x = bar(\mu), \tag{6}$$

with dim $ImN_\mu = \dim X$.

Definition 2.1([Am-N], [Fried] and [I-Sat'11]). A positive definite inner product G_μ on the tangent space $T_\mu \mathscr{P}^+$ is defined by:

$$G_\mu(\tau, \tau_1) = \int_{\theta \in \partial X} \frac{d\tau}{d\mu}(\theta) \frac{d\tau_1}{d\mu}(\theta) d\mu(\theta), \;\; \tau, \tau_1 \in T_\mu \mathscr{P}^+. \tag{7}$$

The collection $\{G_\mu \,|\, \mu \in \mathscr{P}^+\}$ provides a Riemannian metric on $\mathscr{P}^+$, called Fisher information metric G.

As G is viewed as a Riemannian metric on an infinite dimensional manifold $\mathscr{P}^+$, the Levi-Civita connection ∇ is given (see [Fried, p.276])

$$\nabla_{\tau_1} \tau = -\frac{1}{2} \left(\frac{d\tau}{d\mu}(\theta) \frac{d\tau_1}{d\mu}(\theta) - \int \frac{d\tau}{d\mu}(\theta) \frac{d\tau_1}{d\mu}(\theta) d\mu(\theta) \right) \mu, \tag{8}$$

at a point $\mu \in \mathscr{P}^+$ for constant vector fields τ, τ_1 on $\mathscr{P}^+$.

The space $\mathscr{P}^+$ with the metric G has then constant sectional curvature $\frac{1}{4}$ (refer to [Fried, Satz 2, §1]). By using formula (8) we have:

Theorem 2.3.([I-Sat'14-2],[I-Sat'14-3]) Let $\gamma(t)$ be a geodesic in $\mathscr{P}^+$ satisfying $\gamma(0) = \mu$ and $\gamma'(0) = \tau \in T_\mu \mathscr{P}^+$, where τ is a unit tangent vector; $G(\tau, \tau) = 1$. Then $\gamma(t)$ is described as

$$\begin{aligned} \gamma(t) &= \left(\cos \frac{t}{2} + \sin \frac{t}{2} \, \frac{d\tau}{d\mu}(\theta) \right)^2 d\mu(\theta) \\ &= \left(\cos^2 \frac{t}{2} + 2 \cos \frac{t}{2} \, \sin \frac{t}{2} \, \frac{d\tau}{d\mu}(\theta) + \sin^2 \frac{t}{2} \, \left(\frac{d\tau}{d\mu} \right)^2 (\theta) \right) d\mu(\theta). \end{aligned} \tag{9}$$

Note that the geodesic lies inside of $\mathscr{P}^+$ as far as the density maintains positivity with respect to $\theta \in \partial X$.

Corollary 2.4.([I-Sat'14-2],[I-Sat'14-3]) Every geodesic in $\mathscr{P}^+$ is periodic, of period 2π. The length ℓ of a geodesic segment joining two probability measures μ and μ_1 of $\mathscr{P}^+$ is given by:

$$\cos \frac{\ell}{2} \leq \int_{\partial X} \sqrt{\frac{d\mu_1}{d\mu}(\theta)} d\mu(\theta) = \int_{\partial X} \sqrt{\frac{d\mu}{d\mu_1}(\theta)} d\mu_1(\theta) \tag{10}$$

and equality " $=$ " in (10) holds provided at least $\cos(\frac{\ell}{2}) + \sin(\frac{\ell}{2}) \frac{d\tau}{d\mu}(\theta) > 0$ for any θ.

For these see also [Fried, p. 279]. The integration in RHS of (10) is the f-divergence

$$D_f(\mu||\mu_1) = \int f(\frac{d\mu_1}{d\mu})d\mu, f(u) = \sqrt{u} \tag{11}$$

in statistical models (refer to [Am-N, p. 56]).

The formula (9), an improvement of the formula given by T. Friedrich (refer to [Fried, p.279]), can then assert the following:

Corollary 2.5.([I-Sat'14-2],[I-Sat'14-3]) Let μ, $\mu_1 \in \mathscr{P}^+$, $\mu \neq \mu_1$. Then, there exists a unique geodesic $\mu(t)$ such that $\mu(0) = \mu$, $\mu(d) = \mu_1$, where $d > 0$ is defined by

$$\cos\frac{d}{2} = \int_\theta \sqrt{\frac{d\mu_1}{d\mu}(\theta)}d\mu(\theta) = \mathbf{D}_f(\mu||\mu_1). \tag{12}$$

Corollary 2.6.([I-Sat'14-2],[I-Sat'14-3]) Let $\gamma(t) = \exp_\mu t\tau$ be a geodesic satisfying $\gamma(0) = \mu$ and $\gamma'(0) = \tau$. Then γ is entirely contained in the fibre $bar^{-1}(x)$ over $x = bar(\mu)$ if and only if τ satisfies at μ

$$G_\mu(\nabla_\tau\tau, N_\mu(U)) = 0, \forall U \in T_xX. \tag{13}$$

The equation (13) implies that the tangent vector τ is a totally geodesic vector with respect to the second fundamental form H, i.e., τ satisfies $H(\tau,\tau) = 0$ at μ, since the image $Im\, N_\mu$ of the linear map N_μ distributes a normal bundle of $bar^{-1}(x)$ at the measure μ. Here, $H_\mu(\tau,\tau_1) := (\nabla_\tau\tau_1)^\perp$ at μ.

Example 2.1. Let o be the base point for ∂X, $\dim X \geq 2$ such that $\partial X \cong S_oX$ and $bar(\mu) = o$ for the canonical measure $\mu = d\theta \in \mathscr{P}^+$. Identify $(dB_\theta)_o$ with $-\sum_i \theta^i e_i$, $\theta^i \in \mathbf{R}$, with respect to an orthonormal basis $\{e_i\}$ of T_oX. Define $\tau = \frac{1}{c}\theta^i\theta^j d\theta$, $i \neq j$ a vector tangent to $\mathscr{P}^+$(c is a constant normalizing τ as a unit). Then $\tau \in T_\mu bar^{-1}(o)$ is seen and $\gamma(t) = \exp_\mu t\tau$ is a geodesic which is, from Corollary 2.6, contained in $bar^{-1}(o)$ for t, provided at least the density function is positive. In fact, the τ satisfies (13).

3. BARYCENTRICALLY ASSOCIATED MAPS AND ISOMETRIES OF X

A Riemannian isometry φ of X transforms every geodesic into a geodesic and hence induces naturally a map $\hat\varphi : \partial X \to \partial X$, a homeomorphism with respect to the cone topology. Further, the normalized Busemann function admits a cocycle formula ([Gui-L-T]);

$$B_\theta(\varphi x) = B_{\hat\varphi^{-1}\theta}(x) + B_\theta(\varphi o), \forall (x,\theta) \in X \times \partial X \tag{14}$$

(o is the normalization point of B_θ).

Proposition 3.1 (Equivariant action formula [Bes-C-G'95, (5.1)]).

$$\begin{aligned} bar \circ \hat\varphi_\sharp &= \varphi \circ bar, \text{ namely} \\ bar(\hat\varphi_\sharp\mu) &= \varphi(bar(\mu))\ \forall \mu \in \mathscr{P}^+, \end{aligned} \tag{15}$$

where $\Phi_\sharp : \mathscr{P}^+ \to \mathscr{P}^+$ is the push-forward of a homeomorphism Φ of ∂X;

$$\int_{\theta\in\partial X} h(\theta)\, d[\Phi_\sharp\mu](\theta) = \int_{\theta\in\partial X} (h\circ\Phi)(\theta)\, d\mu(\theta) \tag{16}$$

for any function $h = h(\theta)$ on ∂X (see [Vill, p.4]).

So, we consider the situation converse of Proposition 3.1 as

Definition 3.1. Let $\Phi : \partial X \to \partial X$ be a homeomorphism of ∂X. Then, a bijective map $\varphi : X \to X$ is called *barycentrically associated* to Φ, when φ satisfies the relation $bar\circ\Phi_\sharp = \varphi\circ bar$ in the diagram

$$\begin{array}{ccc} \mathscr{P}^+(\partial X, d\theta) & \stackrel{\Phi_\sharp}{\longrightarrow} & \mathscr{P}^+(\partial X, d\theta) \\ \downarrow bar & & \downarrow bar \\ X & \stackrel{\varphi}{\longrightarrow} & X \end{array} \tag{17}$$

So, an isometry φ is a map barycentrically associated to $\Phi = \hat{\varphi}$.

Let $bar : \mathscr{P}^+ \to X$ be the barycenter map. Then, with respect to a homeomorphism $\Phi : \partial X \to \partial X$ and a bijective map $\varphi : X \to X$ we obtain the following ([I-Sat'14], [I-Sat'14-2],[I-Sat'14-3])

Theorem 3.2. Assume that a pair (Φ,φ) with $\varphi \in C^1$ satisfies: (a) $bar(\Phi_\sharp\mu) = \varphi(bar(\mu)), \forall\mu \in \mathscr{P}^+,$ and (b) $\Theta(\varphi(x)) = \Phi_\sharp(\Theta(x)), \forall x \in X$;

$$\begin{array}{ccc} \mathscr{P}^+(\partial X, d\theta) & \stackrel{\Phi_\sharp}{\longrightarrow} & \mathscr{P}^+(\partial X, d\theta) \\ \uparrow \Theta & & \uparrow \Theta \\ X & \stackrel{\varphi}{\longrightarrow} & X \end{array} \tag{18}$$

Then, φ must be a Riemannian isometry of X.

Here, $\Theta : X \to \mathscr{P}^+; y \mapsto P(y,\theta)d\theta$ is a map associated with a Busemann-Poisson kernel $P(x,\theta) = \exp\{-qB_\theta(x)\}$. For the definition of Poisson kernel refer to [Sch-Y] and [Bes-C-G'95] and see also [I-Sat'14-2] for the definition of Busemann-Poisson kernel.

Remark 3.1. If X admits a Busemann-Poisson kernel, then Θ gives a cross section of the fibre space $\mathscr{P}^+ \to X$, since $bar(\mu_x) = x$ for $\mu_x = P(x,\theta)d\theta$ ([Bes-C-G'95, (5.1)]), and moreover, every $\mu \in \mathscr{P}^+$ admits a unique barycenter from Theorem 1.2, since it holds

$$\int_{\partial X} (\nabla dB_\theta)_x(U,V)d\mu_x(\theta) = q\int_{\partial X} (dB_\theta)_x(U)(dB_\theta)_x(U)d\mu_x(\theta), U,V \in T_xX \tag{19}$$

that is

$$(\nabla d\, \mathbf{B}_{\mu_x})_x(U,V) = q\, G_{\mu_x}\left(N_{\mu_x}(U), N_{\mu_x}(V)\right) \tag{20}$$

($q > 0$ is the volume entropy of X) and at any $y \in X$

$$(\nabla d\, \mathbf{B}_{\mu_x})_y(U,U) \geq C\,(\nabla d\, \mathbf{B}_{\mu_y})_y(U,U) \tag{21}$$

for some constant $C > 0$, depending on x, y. From these, the μ_x-average Hessian $\nabla d\, \mathbf{B}_{\mu_x}$ turns out to be positive definite everywhere.

With respect to the conditions (a) and (b) of Theorem 3.2 we have

Theorem 3.3. Let X be an Hadamard manifold. Assume that X satisfies assumptions (i) and (ii) of Theorem 1.1 and moreover admits a Busemann-Poisson kernel. Let $\Phi : \partial X \to \partial X$ be a homeomorphism. If a bijective map $\varphi : X \to X$ is C^1 with surjective differential $d\varphi_x$, $\forall x \in X$, then the condition (b), namely, $\Theta(\varphi(x)) = \Phi_\sharp(\Theta(x))$, $\forall x \in X$, implies (a), namely, $bar(\Phi_\sharp \mu) = \varphi(bar(\mu))$, $\forall \mu \in \mathscr{P}^+$.

4. DAMEK-RICCI SPACES AND MOTIVATION

A Damek-Ricci space is a solvable Lie group, an **R**-extension of a generalized Heisenberg group and carries a left invariant Riemannian metric and further provides a space on which harmonic analysis is developed ([Ank-D-Y],[Dam-R]). For precise definition and differential geometry of Damek-Ricci space, refer to [Bern-T-V]. Damek-Ricci spaces are Hadamard manifolds whose typical examples are complex hyperbolic, quaternionic hyperbolic and Cayley hyperbolic spaces as strictly negatively curved ones, except for real hyperbolic spaces ([Dotti],[Lanz]). Any Damek-Ricci space satisfies the axiom of visibility and has θ-continous Busemann function (refer to [I-Sat'10] for these) . Moreover, it admits a Busemann-Poisson kernel (see [I-Sat'10]) so that it satisfies (i) and (ii) of Theorem 1.1, and Theorem 1.2. The most important implication of Damek-Ricci spaces is that they provide the counterexample of Lichnerowicz conjecture of non-compact harmonic manifold version (refer to [Bern-T-V]).

So, relating to this, our motivation is to characterize Damek-Ricci spaces from a viewpoint of geometry, since only a Lie group characterization of Damek-Ricci space is known from Heber's theorem ([Heb]). A Damek-Ricci space turns out recently to be Gromov-hyperbolic, whereas it admits zero sectional curvature (see [I-Sat'14-2]) for this and refer to [Coo-D-P], [Bourd], [Kniep] for the Gromov hyperbolicity).

Thus, we pose the following. Let X_o be a Damek-Ricci space and X an Hadamard manifold, quasi-isometric to X_o and assume that if X admits a Busemann-Poisson kernel, then, is X isometric, or homothetic to X_o as a Riemannian manifold ? At least, from this assumption, we have that any Riemanian isometry of X_o induces a homeomorphism of ∂X of X (for the detail, see [I-Sat'14-2]). From this fact, we have faced our central theme, namely, differential geometry of a map being associated barycentrically to a homeomorphism of ∂X, as discussed in sections 1 and 3, where we answered partially to the above question.

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