Transversely Hessian foliations and information geometry

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Abstract. A family of probability distributions parametrized by an open domain Λ in R^n defines the Fisher information matrix on this domain which is positive semi-definite. In information geometry the standard assumption has been that the Fisher information matrix is positive definite defining in this way a Riemannian metric on Λ. If we replace the "positive definite" assumption by "0-deformable" condition a foliation with a transvesely Hessian structure appears naturally. We develop the study of transversely Hessian foliations in view of applications in information geometry.

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Introduction – Information Geometry

The Fisher metric is one of the basic tools in information geometry. It is defined on an open domain in R^m which parametrizes the set of probability distributions under consideration. The assumpsions made lead to the study of affine manifolds with Riemannian metric, with both geometrical structures loosely related. The assumption that the defined operator is positive definite is rather strong, therefore we propose to study the consequences of a weaker condition. In this situation a foliation with a very particular transverse structure appears. Such foliated manifolds are natural generalizations of Hessian manifolds when these manifolds are considered as foliated by points.

Let Λ be a domain in R^m. We consider families of probability distributions on a set $\mathcal{X}$ parametrized by $\lambda \in \Lambda$.

$$\mathcal{P} = \{p(x;\lambda) | \lambda \in \Lambda\}$$

(1) Λ is a domain in R^m,

(2) $p(x;\lambda)$ for a fixed x is a smooth function in λ,

(3) the operation of integration with respect to x and differentiation with respect to λ are commutative.

Definition *Let $\mathcal{P} = \{p(x;\lambda) | \lambda \in \Lambda\}$ be. We set $l_\lambda = l(x;\lambda) = log p(x;\lambda)$ and denote by E_λ the expectation with respect to $p_\lambda = p(x;\lambda)$. Then a matrix $g = [g_{ij}(\lambda)]$ defined by*

$$g_{ij}(\lambda) = E_\lambda[\frac{\partial l_\lambda}{\partial \lambda^i}\frac{\partial l_\lambda}{\partial \lambda^j}] = \int_{\mathcal{X}} \frac{\partial l(x;\lambda)}{\partial \lambda^i}\frac{\partial l(x;\lambda)}{\partial \lambda^j} p(x;\lambda) dx$$

is called the Fisher information matrix.

Simple calculations show, see [7], that

$$g_{ij}(\lambda) = -E_\lambda[\frac{\partial^2 l_\lambda}{\partial\lambda^i\partial\lambda^j}].$$

The Fisher information matrix $g = [g_{ij}(\lambda)]$ is positive semi-definite on Λ:

$$\Sigma_{i.j} g_{ij}(\lambda) c^i c^j = \int_{\mathcal{X}} \{\Sigma_i c^i \frac{\partial l(x;\lambda)}{\partial\lambda^i}\}^2 p(x,\lambda) dx \geq 0.$$

In information geometry the standard assumption has been, cf. [7], p.105,:

(4) The Fisher information matrix $g = [g_{ij}(\lambda)]$ for a family of probability distributions $\mathcal{P} = \{p(x;\lambda) | \lambda \in \Lambda\}$ is **positive definite** on Λ.

We **weaken this condition** assuming only that the Fisher information matrix is a 0-deformable tensor field on Λ. Then a foliation appears in a very natural way, and under some mild assumptions it has a transverse Hessian structure. The main part of this note is devoted to a sketch of a theory of transversely Hessian foliation which can be applied to a classification of space of probability distributions in the non-regular case. Finally, we are going to discuss two problems. The first one (P1) consists in searching for an invariant which determines a statistical model $P(x,\lambda)$ up to an isomorphism. The second (P2) is the search for a necessary and sufficient condition condition ensuring that a model is locally isomorphic to an exponential model.

TRANSVERSELY HESSIAN FOLIATIIONS

We continue the study of a family of probability distributions described in Introduction. Assume that the tensor g is 0-deformable. Then there exists a torsion-free connection ∇, cf. [6], for which the tensor g is parallel, i.e., $\nabla g = 0$, and the distribution $kerg$:

$$kerg = \{v \in TM \colon g(v,v) = 0\} = \{v \in TM \colon g(v,w) = 0 \ \ \forall w \in TM\}$$

is parallel with respect to ∇ as $0 = \nabla_X g(Y,Z) = Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$ where $X,Z \in TM$ and $Y \in kerg$. As $g(Y,Z) = 0$ and we get that $g(\nabla_X Y, Z) = 0$ for any $Z \in TM$, which ensures that $\nabla_X Y \in kerg$ provided that $Y \in kerg$. As the connection ∇ is torsion-free, $0 = T(X,Y) = \nabla_X Y + \nabla_Y X - [X,Y]$. If $X,Y \in kerg$ then $\nabla_X Y \in kerg$ and $\nabla_Y X \in kerg$. Therefore for any $X,Y \in kerg$ $[X,Y] \in kerg$, the distribution $kerg$ is involutive, and it defines a foliation $\mathcal{F}$. The pseudo Riemannian metric g induces a (normal) Riemannian metric g^N in the normal bundle $N(M,\mathcal{F})$. The connection ∇ defines a connection ∇^N in the normal bundle $N(M,\mathcal{F})$ which is the Levi-Civita connection of the metric g^N.

The induced metric will be called foliated if $Xg(Y,Z) = 0$ for any vector field X tangent to the foliation $\mathcal{F}$ and any foliated vector fields Y and Z orthogonal to $\mathcal{F}$. It is the case if $\nabla_X Y \in T\mathcal{F}$ for any vector field X tangent to the foliation $\mathcal{F}$ and any foliated vector fields Y, i.e. a vector field whose flow preserves the foliation. As the connection ∇ is torsion-free this condition is equivalent to $\nabla_Y X \in T\mathcal{F}$ for any vector $Y \in TM$ and any vector field $X \in T\mathcal{F}$. More information and equivalent conditions see [8].

The flat connection D should be related in some way to the foliation $\mathcal{F}$. If we assume a similar condition for D, i.e. $D_Z W \in T\mathcal{F}$ for any vector $Z \in TM$ and any vector field $W \in T\mathcal{F}$, then $D^N_X[Y] = [D_X Y]$ where $[Y]$ denotes the section of the normal bundle defined by the vector field Y. The connection D^N is transversally projectable (foliated) if $D^N_X Y$ is a foliated section for any foliated section Y and a foliated vector field X, i.e. an infinitesimal automorphism of $\mathcal{F}$.

Let $\mathcal{F}$ be a foliation of codimension q on a manifold M of dimension m.The pair $(M,\mathcal{F})$ is called a foliated manifold. The dimension of its leaves is p, i.e. $p+q=m$. Assume that the foliation $\mathcal{F}$ is transversely affine, cf. [5, 9]. A Riemannian metric $\hat{g}$ is bundle-like if for any adapted chart $\varphi = (x^1,...,x^p,y^1,...,y^q)$ $g = \Sigma^p_{ij=1} g_{ij}(x,y)dx^i dx^j + \Sigma^q_{a,b=1} g_{ab}(y)dy^a dy^b$. A bundle-like metric $\hat{g}$ is said to be transversely Hessian if the horizontal part g of the metric is expressed by the formula

$$g = \Sigma^q_{i.j=1} \frac{\partial^2 h}{\partial y^i \partial y^j} dy^i dy^j$$

Remark. In principle h need not be basic, i.e. it may depend on variables x_i, we just assume that it does not.

Definition *We say that the foliation is transversely Hessian if it is transversely affine and it admits a bundle-like metric which is transversely Hessian.*

Therefore a transversely Hessian foliation is at the same time a Riemannian foliation and a transversely affine foliation althought both structures have not much in common.

The fact that the foliation is transversely affine is equivalent to the existence of a foliated connection D in the normal bundle $N(M,\mathcal{F})$. Denote by the same letter g the Riemannian metric induced by the horizontal part of $\hat{g}$ in the normal bundle. The metric g and the connection D are related by the normal Codazzi equations if $D_X g(Y,Z) = D_Y g(X,Z)$ for any foliated sections X,Y,Z of the normal bundle.

Normal Codazzi structure on a foliated manifold $(M,\mathcal{F})$ is a pair (D,g) consisting of a foliated connection D and a foliated metric g in the normal bundle satisfying the equation $D_X g(Y,Z) = D_Y g(X,Z)$ for any foliated sections X,Y,Z of the normal bundle.

Having a normal Codazzi structure (D,g) on a foliated manifold $(M,\mathcal{F})$ we can define a new foliated torsionfee connection D' by the formula $Xg(Y,Z) = g(D_X Y,Z) + g(Y,D'_X Z)$ for any foliated sections X,Y,Z of the normal bundle. The connection D' is called the dual connection of the connection D with respect to the metric g.

Let ∇ be the normal Levi-Civita connection of the normal foliated Riemannian metric g in $N(M,\mathcal{F})$. ∇ is a foliated connection. The difference $\gamma = \nabla - D$ is also a foliated tensor. Moreover, as both connections are torsion-free, $\gamma_X Y = \gamma_Y X$.

A normal foliated Hessian structure (D,g) is said to be of Koszul type if there exists a closed *basic* form ω such that $g = D\omega$. Like in the standard case we have the following:

Theorem 1 *Let $(M,\mathcal{F})$ be a foliated manifold. Assume that $\mathcal{F}$ is transversely affine with foliated flat connection D and g is a foliated metric on $(M,\mathcal{F})$. Then the following conditions are equivalent*

i) g is a normal Hessian metric;
ii) $D_X g(Y,Z) = D_Y g(X,Z)$ *for any foliated sections* X,Y,Z *of the normal bundle;*
iii) $\frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{kj}}{\partial y^i}$
iv) $g(\gamma_X Y,Z) = g(Y,\gamma_X Z)$
v) $\gamma_{ijk} = \gamma_{jik}$

Let $\mathcal{F}$ be a transversely affine foliation on a manifold M, let $N(M,\mathcal{F})$ be its normal bundle. It admits a foliation $\mathcal{F}_N$ of the same dimension as $\mathcal{F}$ but of codimension $2q$.

Let $\varphi = (x^1,...,x^p,y^1,...,y^q)$ be an adapted chart. Then on the normal bundle $N(M,\mathcal{F})$ we have an adapted chart $\varphi^N = (x^1,...,x^p,y^1,...,y^q,dy^1,...,dy^q)$, of course the coordinates x^i are identified with $x^i p$ where p is the projection onto the base M in the normal bundle. Moreover, putting $z^i = y^i + \sqrt{-1}dy^i$ we define transversely holomorphic coordinate system on the normal bundle. Therefore if the foliations $\mathcal{F}$ is tranversely affine the foliation $\mathcal{F}_N$ is tranversely holomorphic for a foliated complex structure J_N. Moreover, on the normal bundle we define a normal Riemannian metric g^T by the local formula

$$g^T = \Sigma_{ij}^q g_{ij} p dz^i d\bar{z}^j$$

In this case we have the following proposition

Proposition 1 *Let* $(M,\mathcal{F})$ *be a foliated manifold, and g be a foliated metric. Then the following conditions are equivalent:*
(1) g is a foliated Hessian metric on $(M,\mathcal{F},D)$
(2) g^T *is a Kälerian metric on* $N(M,\mathcal{F},J_N)$

DUAL FOLIATED CONNECTIONS

Let $(M,\mathcal{F},D,g)$ be a tranversely Hessian foliated manifold. Let ∇ be the Levi-Civita connection in the normal bundle of $\mathcal{F}$ associated to the Riemannian metric g. Then the connection $D' = 2\nabla - D$ is a flat tranversely projectable connection in the normal bundle and $Xg(Y,Z) = g(D'_X Y,Z) + g(Y,D'_X Z)$. Moreover, $(M,\mathcal{F},D',g)$ is a transversely Hessian foliated manifold. D' is called the dual connection of the transversely Hessian foliated manifold $(M,\mathcal{F},D,g)$.

Lemma 1 *Let g be a foliated metric on the normal bundle. Let* (D,D') *be a Codazzi pair with respect to g. Then if one of the connections is foliated so is the other.*

Lemma 2 *Let D be a torsion-free connection and let g be a Riemannian metric in the normal bundle. Define a new connection* D' *by*

$$Xg(Y,Z) = g(D_X Y,Z) + g(Y,D'_X Z)$$

for any foliated sections X,Y,Z*. Then the following conditions (1)-(3) are equivalent:*
(1) the connection D' *is torsion free,*
(2) the pair (D,g) *satisfies the Codazzi equation*

$$D_X g(Y,Z) = D_Y g(X,Z),$$

(3) let ∇ be the Levi-Civita connection for g, and let $\gamma_X Y = \nabla_X Y - D_X Y$. Then g and γ satisfy

$$g(\gamma_X Y, Z) = g(Y, \gamma_X Z).$$

If the pair (D, g) satisfies the Codazzi equation, then the pair (D', g) also satisfies this equation and

$$D' = 2\nabla - D,$$

$$D_X g(Y, Z) = 2g(\gamma_X Y, Z)$$

Remarks. Proposition 1 asserts that a flat connection D and a Riemannian metric g in the normal bundle of a foliation $\mathcal{F}$ form a normal Hessian structure iff they satisfy the Codazzi equation. The same is true for foliated structures.

Definition *A pair (D, g), g being a Riemannian metric in the normal bundle and D a torsion-free connection in this vector bundle is called a normal Codazzi structure if it satisfies the Codazzi equation for any foliated sections X, Y, Z :*

$$D_X g(Y, Z) = D_Y g(X, Z).$$

If both objects are foliated, then the pair (D, g) is called a *foliated Codazzi structure.* A foliated manifold $(M, \mathcal{F})$ is called *a foliated Codazzi manifold,* and is denoted $(M, \mathcal{F}; D, g)$. If g is a foliated Riemannian metric and D a foliated connecction, then the pair is called a *foliated Codazzi manifold.*

For a normal Codazzi structure (D, g) the connection defined by

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D'_X Z)$$

is called the dual normal connection of D with respect to g, and the pair (D', g) is said to be the *dual normal Codazzi structure* of (D, g). If $(M, \mathcal{F}; D, g)$ is a foliated Codazzi manifold, so is $(M, \mathcal{F}; D', g)$.

A foliated Codazzi structure is said to be of constant curvature c if the curvature tensor R_D of the connection D satisfies

$$R_D(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y)$$

for any sections X, Y, Z of the normal bundle.

Proposition 2 *Let $(M, \mathcal{F}; D, g)$ be a foliated Codazzi manifold, and (D', g) be its dual Codazzi structure.*

(1) Denoting by R_D and $R_{D'}$ the curvature tensors of D and D', respectively, we have

$$g(R_D(X, Y)Z, W) + g(Z, R_{D'}(X, Y)W) = 0,$$

for any sections X, Y, Z, W of the normal bundle.

(2) If (D, g) is a Codazzi structure of constant curvature c, then (D', g) is also a Codazzi structure of constant curvature c.

Proposition 3 *A foliated Codazzi structure (D,g) is of constant curvature 0 iff the foliation $\mathcal{F}$ is transverely Hessian with the transverse structure given by the pair (D,g) .*

NORMAL (FOLIATED) CURVATURES

Let (D,g) be a normal Hessian structure on a foliated manifold $(M,\mathcal{F})$.i.e. a Hessian structure on the normal bundle $N(M,\mathcal{F})$. Let $\gamma = \nabla - D$ be the difference tensor of the normal Levi-Civita connection of g and D. A normal tensor field Q of type (1,3) defined as $Q = D\gamma$ is called the *normal Hessian curvature tensor* of (D,g).

If the structure is foliated so is its normal Hessian curvature tensor. The components Q^i_{jkl} of Q with respect to an adapted foliated affine coordinate system $(x^1,..x^p,y^1,...,y^q)$ are given by

$$Q^i_{jkl} = \frac{\partial \gamma^i_{jl}}{\partial y^k}.$$

Wa have standard results for these curvaturies of the connections in the normal bundle $N(M,\mathcal{F})$, cf.[7].

Proposition 4 *A) Let $g_{ij} = \frac{\partial^2 \varphi}{\partial y^i y^j}$. Then for $i,j,k,l = 1,...q$.*

(1) $Q_{ijkl} = \frac{1}{2}\frac{\partial^4 \varphi}{\partial y^i y^j y^k y^l} - \frac{1}{2} g^{rs} \frac{\partial^3 \varphi}{\partial y^i y^k y^r}\frac{\partial^3 \varphi}{\partial y^j y^l y^s}$.

(2) $Q_{ijkl} = Q_{klij} = Q_{kjil} = Q_{ilkj} = Q_{jilk}$.

B) Let R be the normal Riemannian curvature tensor for the normal Riemannian metric g. Then

$$R_{ijkl} = \frac{1}{2}(Q_{ijkl} - Q_{jikl}).$$

C) Let R^N be the normal Riemannian curvature tensor for the normal Kählerian metric g^N of the foliated manifold $(N(M,\mathcal{F}),\mathcal{F}^N)$. Then

$$R^N_{ijkl} = \frac{1}{2} Q_{ijkl} \circ \pi$$

where π is the natural projection $N(M,\mathcal{F}) \to M$.

Let ω be the normal volume form defined by g. We define a closed 1-form α and a symmetric bilinear normal form β by

$$D_X \omega = \alpha(X)\omega, \quad \beta = D\alpha.$$

The forms α and β are called the first normal Koszul form nad the second normal Koszul form, respectively, of the normal Hessian structure (D,g) on a foliated manifold $(M,\mathcal{F})$.

Proposition 5 *(1) $\alpha(X) = Tr_N \gamma_X$,*

(2) $\alpha_i = \frac{1}{2}\frac{\partial logdet|g_{kl}|}{\partial x^i} = \gamma^r_{ri}$

(3) $\beta_{ij} = \frac{\partial \alpha_i}{\partial x^j} = \frac{1}{2}\frac{\partial^2 logdet|g_{kl}|}{\partial x^i \partial x^j} = Q^r{}_{rij} = Q_{ij}{}^r{}_r$.

Proposition 6 *Let $\mathbf{R}^N_{ij}$ be the normal Ricci tensor for the normal Kählerian metric g^N of the foliated manifold $(N(M,\mathcal{F}),\mathcal{F}^N)$. Then $\mathbf{R}^N_{ij} = -\frac{1}{2}\beta_{ij}\circ\pi$.*

If the normal Hessian structure (D,g) satisfies the condition $\beta = \lambda g, \quad \lambda = \frac{\beta^i_i}{q}$ then the normal Hessian structure is called Einstein-Hessian.

Theorem 2 *Let $(M,\mathcal{F})$ be a foliated manifold, and (D,g) be a normal Hessian structure. Let (J^N,g^N) be the normal Kähler structure of the foliated manifold $(N(M,\mathcal{F}),\mathcal{F}^N)$ induced by (D,g). Then the following conditions are equivalent*
(1) (D,g) is Einstein-Hessian;
(2) (J^N,g^N) is Einstein-Kählerian.

Normal Hessian sectional curvature
Let Q^N be the normal Hessian curvature tensor. The formula

$$\hat{Q}^{Nik}(\xi) = \hat{Q}^{Ni}{}_j{}^k{}_l\xi_{jl}$$

defines an endomorphism $\hat{Q}^N$ on the space of symmetric contravariant normal two tensors, i.e., on $\otimes^2 N(M,\mathcal{F})$

$$q^N(\xi) = \frac{<\hat{Q}^N(\xi),\xi>}{<\xi,\xi>}$$

for any $\xi\in\otimes^2 N(M,\mathcal{F})$ and $<,>$ the inner product defined by the normal Riemannian metric g of the foliated Hessian structure. We say that the foliated Hessian structure is of constant normal sectional curvature if q^N is a constant function on $\otimes^2 N(M,\mathcal{F})$.

Proposition 7 *The normal Hessian sectional curvature of a foliated Hessian structure $(M,D,g;\mathcal{F})$ is constant and equal to c iff*

$$\hat{Q}_{ijkl} = \frac{c}{2}\{\hat{g}_{ij}\hat{g}_{kl} + \hat{g}_{il}\hat{g}_{jk}\}$$

Corollary 3 *Let $(M,D,g;\mathcal{F})$ be a foliated Hessian structure. Then the following two conditions are equivalent:*
(1) the Hessian normal sectional curvature is a constant c;
(2) the holomorphic normal sectional curvature of $(N(M,\mathcal{F}),J^N,g^N;\mathcal{F}^N)$ is constant and equal to -c.

Corollary 4 *Let $(M,D,g;\mathcal{F})$ be a foliated Hessian structure of constant normal Hessian sectional curvature c. Then the foliated Riemannian manifold $(M,g;\mathcal{F})$ is a foliated Riemannian manifold modelled on a space form of constant section curvature $-\frac{c}{4}$.*

MODELS

We use the notion of *exponential model* in the sense of [1], That means the following: $\lambda = (\lambda_1,..,\lambda_m)$ are affine coordinate functions in R^m and x stands for the random variable. Then (Θ, P) is exponential if there exists a map $(c, F, \psi)(\lambda, x) = (c(x), F_1(x), .., F_m(x)) \in R^{m+1}$ and a smooth real function $\psi(\lambda)$ such that

$$P(\lambda, x) = exp(c(x) + \sum F_j(x)\lambda_j - \psi(\lambda).$$

Let (Λ, P) and (Λ', P') be two statistical models for a measurable set (Ξ, Ω) and for (Ξ', Ω'), respectively. A morphism of model is a map assigning to$(\lambda, x) \in \Lambda \times \Xi$ the element $\phi(\lambda, \phi^*(x)) \in \Lambda' \times \Xi'$ subject to two requirements:
(r1): ϕ is differentiable, ϕ^* is measurable.
(r2) : $P'(\phi(\lambda, \phi^*(x))) = P(\lambda, x)$.

Let us denote by $\mathcal{L}(\Lambda)$ the affine space of linear connections in Λ and by $\mathcal{N}(\Lambda)$ the subspace of locally flat connections in Λ .

To a model $P(x, \lambda)$ we associate a mapping q which is defined on $\mathcal{L}(\Lambda)$ by the formula

$$q(\nabla)(x, \lambda) = \nabla^2(l(x, \lambda)).$$

On the right hand side we have a 2-linear form defined on the tangent space $T_\lambda \Lambda$.

The discussion of the property (P1) leads to the following:

Theorem 5 *The mapping q determines $P(x, \lambda)$ up to an isomorphism.*

Each $\nabla \in \mathcal{N}(\Lambda)$ defines a cochain complex $C^*(\nabla)$ whose homogeneous cochains of a positif degree k are contravariant tensors of degree k in Λ, [2], [3], [4]. We denote by d_∇ the coboundary operator in $C^*(\nabla)$. Let g be the Fisher information metric of Λ.

Theorem 6 *The model space Λ is isomorphic to an exponential model iff the Fisher information metric g is d_∇-closed for a $\nabla \in \mathcal{N}(\Lambda)$.*

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