Beyond Landau–Pollak and entropic inequalities: geometric bounds imposed on uncertainties sums

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Abstract. In this paper we propose generalized inequalities to quantify the uncertainty principle. We deal with two observables with finite discrete spectra described by positive operator-valued measures (POVM) and with systems in mixed states. Denoting by $p(A;\rho)$ and $p(B;\rho)$ the probability vectors associated with observables A and B when the system is in the state ρ, we focus on relations of the form $U_\alpha(p(A;\rho)) + U_\beta(p(B;\rho)) \geq \mathscr{B}_{\alpha,\beta}(A,B)$ where U_λ is a measure of uncertainty and $\mathscr{B}$ is a non-trivial state-independent bound for the uncertainty sum. We propose here:

(i) an extension of the usual Landau–Pollak inequality for uncertainty measures of the form $U_f(p(A;\rho)) = f(\max_i p_i(A;\rho))$ issued from well suited metrics; our generalization comes out as a consequence of the triangle inequality. The original Landau–Pollak inequality initially proved for nondegenerate observables and pure states, appears to be the most restrictive one in terms of the maximal probabilities;

(ii) an entropic formulation for which the uncertainty measure is based on generalized entropies of Rényi or Havrda–Charvát–Tsallis type: $U_{g,\alpha}(p(A;\rho)) = \frac{g(\sum_i [p_i(A;\rho)]^\alpha)}{1-\alpha}$. Our approach is based on Schur-concavity considerations and on previously derived Landau–Pollak type inequalities.

Keywords: Generalized uncertainty relations, Landau–Pollak type inequalities, entropic uncertainty relation, pure and mixed states, POVM
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INTRODUCTION

The uncertainty principle establishes in quantum mechanics that one cannot predict with certainty and simultaneously the outcomes of two incompatible measurements. First quantitative formulations of this principle made use of variances as uncertainty measures, exhibiting thus the existence of a lower bound for the product of the variances [1]. However, those formulations have some drawbacks. For instance, when dealing with discrete-spectrum observables, the so-called Heisenberg–Robertson–Schrödinger inequalities are state-dependent and the state-independent lower bound appears to be trivial. Thus, many authors attempt to propose alternative formulations, using more general measures of uncertainty rather than the variance. Among them one finds relations issued from geometrical concepts giving rise to generalizations of the Landau–Pollak inequality [2], or formulations making use of information-theoretic measures, typically entropies [3–9].

We provide here two formulations of the uncertainty principle valid in the general context of observables described by positive operator-valued measures (POVM) and for systems described by density operators. One of the formulations is a generalization of the Landau–Pollak inequality, whereas the other one is based on entropies. The former is not only an uncertainty formulation in itself, but also an ingredient for the proof of the latter approach. The entropic formulation that we derive is based on one-parameter generalized entropies including the well-known cases of Rényi and Tsallis ones. In both formulations, geometrical considerations (metrics and convexity arguments, respectively) are used to derive our inequalities.

PRELIMINARIES

We deal with two observables described by the POVM $A = \{A_i\}_1^{N_A}$ and $B = \{B_j\}_1^{N_B}$, respectively, i.e., A and B are sets of self-adjoint positive semi-definite operators acting on an N-dimensional Hilbert space $\mathscr{H}$, that satisfy the completeness relation $\sum_{i=1}^{N_A} A_i = I = \sum_{j=1}^{N_B} B_j$, where I is the identity operator on $\mathscr{H}$, and N_A and N_B are not necessarily equal one to the other, nor equal to N. Operators O_k ($O = A$ or B) represent the possible outcomes of observable O. Besides, we consider a system whose state is described by a density operator ρ acting on $\mathscr{H}$, where ρ is self-adjoint, positive semi-definite, with unit trace $\mathrm{Tr}\rho = 1$. The quantity

$$p(O;\rho) = [p_1(O;\rho) \cdots p_{N_O}(O;\rho)] \qquad \text{with} \qquad p_k(O;\rho) = \mathrm{Tr}(O_k \rho)$$

is a probability vector where $p_k(O;\rho)$ represents the probability of measuring the kth outcome of observable O when the system is in the state ρ [10]. In the context of observables with nondegenerate spectra, the operators write $O_k = |o_k\rangle\langle o_k|$ and are rank-one projectors where $\{|o_k\rangle\}_1^N$ is eigenbasis of observable O. In this case, important parameters are the transformation matrix T and the so-called overlap c,

$$T_{i,j} = \langle b_j | a_i \rangle \qquad \text{and} \qquad c = \max_{i,j} \left| \langle b_j | a_i \rangle \right| \in \left[\frac{1}{\sqrt{N}}, 1 \right],$$

the latter measuring the degree of incompatibility of the nondegenerate observables (e.g., $c = 1$ when they have a common eigenvector, and is minimal when they are complementary). Finally, a pure state is an element $|\Psi\rangle$ of $\mathscr{H}$ with unit norm, and the associated density operator writes $\rho = |\Psi\rangle\langle\Psi|$; any mixed state ρ can be written as a convex combination of (at most) N pure states [10].

In this general context of POVM description for quantum observables and systems in mixed states, the goal is then to quantify the simultaneous unpredictability on the pair of observables, through an inequality of the type

$$\mathscr{U}_\alpha\big(p(A;\rho)\big) + \mathscr{U}_\beta\big(p(B;\rho)\big) \geq \mathscr{B}_{\alpha,\beta}(A,B)$$

where $\mathscr{U}_\lambda$ is an uncertainty measure, parametrized by quantities or functions λ, and the bound $\mathscr{B}_{\alpha,\beta}(A,B)$ is state-independent and non trivial.

The basic features that we impose on the uncertainty measure $\mathscr{U}_\lambda$ are that (i) it is invariant under any permutation of the components of p, (ii) it is Schur-concave, and (iii) $\mathscr{U}_\lambda([1 \quad 0 \quad \cdots \quad 0]) = 0$. Thus, $\mathscr{U}_\lambda(p) \geq 0$ since $p \prec [1 \quad 0 \quad \cdots \quad 0]$ and is zero (minimal) when the probability distribution is $p_k = \delta_{k,i}$ for certain i, that is, the ith outcome appears with certainty so that the ignorance is zero. At the opposite, from the Schur-concavity, $\mathscr{U}_\lambda$ is maximal when p is uniform since $\left[\frac{1}{N} \quad \cdots \quad \frac{1}{N}\right] \prec p$, i.e., all outcomes appear with equal probability, so that the uncertainty is maximal.

EXTENDED LANDAU–POLLAK INEQUALITIES

We focus now on a particular measure of uncertainty that allows us for a generalization of Landau–Pollak inequality. We concentrate on the maximal probabilities

$$P_{A;\rho} = \max_i p_i(A;\rho) \qquad \text{and} \qquad P_{B;\rho} = \max_j p_j(B;\rho) \tag{1}$$

The uncertainty principle manifests through these quantities by the fact that not all pairs $(P_{A;\rho}, P_{B;\rho})$ are allowed. As an example, certainty in both observables $(P_{A;\rho}, P_{B;\rho}) = (1,1)$ is in general not possible. The restrictions can either be described by Landau–Pollak type inequalities, or by the allowed domain for $(P_{A;\rho}, P_{B;\rho})$, which is precisely the goal of the present section.

Let us consider *continuous decreasing* functions $f : [0;1] \to \mathbb{R}_+$, with $f(1) = 0$, and such that

$$f\left(|\langle\Psi|\Phi\rangle|^2\right) = d_f(|\Psi\rangle, |\Phi\rangle)$$

defines a metric for any two pure states $|\Psi\rangle$ and $|\Phi\rangle$. Some well-known cases are [10, 14] $f(x) = \arccos\sqrt{x}$ leading to Wootters metric, $f(x) = \sqrt{2(1-\sqrt{x})}$ leading to Bures metric, $f(x) = \sqrt{1-x}$ related to the root-infidelity metric. Then, the quantity

$$\mathscr{U}_f\big(p(O;\rho)\big) = f\left(P_{O;\rho}\right) \tag{2}$$

gives an uncertainty measure for the POVM O when the system is in state ρ. Indeed, $\mathscr{U}_f$ satisfies all requirements we imposed on an uncertainty measure.

The main results of this section are given by the following propositions that give, respectively, 1) lower bounds for $\mathscr{U}_f\big(p(A;\rho)\big) + \mathscr{U}_f\big(p(B;\rho)\big)$, 2) the most restricting function f for the family of uncertainty relations we derived, and 3) the most restrictive domain, corresponding to the generalized Landau–Pollak inequality.

Proposition 1 *Let $A = \{A_i\}_1^{N_A}$ and $B = \{B_j\}_1^{N_B}$ be two POVM sets describing observables A and B respectively, and acting on an N-dimensional Hilbert space $\mathscr{H}$. Then for an arbitrary density operator ρ acting on $\mathscr{H}$, the following relation holds:*

$$f\left(P_{A;\rho}\right) + f\left(P_{B;\rho}\right) \ge \max\left\{f\left(c_A^2\right) + f\left(c_B^2\right), f\left(c_{A,B}^2\right)\right\} \tag{3}$$

where the triplet of overlaps $\mathbf{c} = (c_A, c_B, c_{A,B})$ is given by

$$c_A = \max_i \left\|\sqrt{A_i}\right\|, \qquad c_B = \max_j \left\|\sqrt{B_j}\right\| \qquad \text{and} \qquad c_{A,B} = \max_{i,j} \left\|\sqrt{A_i}\sqrt{B_j}\right\| \tag{4}$$

In the case of nondegenerate observables, the triplet of overlaps is $\mathbf{c} = (1,1,c)$.

To prove the proposition, we proceed in four steps.

(i) For any observable and pure state $p_k(O;|\Psi\rangle\langle\Psi|) = \langle\Psi|O_k|\Psi\rangle \leq \||\Psi\rangle\| \, \|O_k|\Psi\rangle\| \leq \||\Psi\rangle\|^2 \, \|O_k\| = \|\sqrt{O_k}\|^2 \leq c_O^2$. This inequality remains valid for mixed states (just write a mixed state as a convex combination of pure-states density matrices). Taking the maximum over k gives $P_{O;\rho} \leq c_O^2$: from the decreasing property of f, $f(c_A^2) + f(c_B^2)$ lower bounds the uncertainty sum.

(ii) In the context where A_i and B_j are projectors, consider a pure state $|\Psi\rangle$ and for any operator O_k defines pure states as $|\psi_k^O\rangle = \frac{O_k|\Psi\rangle}{\|O_k|\Psi\rangle\|}$. Thus, $|\langle\Psi|\Psi_k^O\rangle|^2 = p_k(O_k, |\psi\rangle\langle\Psi|)$ and in this context of projectors, the lower bound $f(c_{A,B}^2)$ is a direct consequence of the triangle inequality satisfied by metric d_f applied on the triplet $|\psi_i^A\rangle$, $|\psi_j^B\rangle$ and $|\psi\rangle$ (for any i and j), followed by the use of the Cauchy–Schwartz inequality and the decreasing property of f.

(iii) Bound $f(c_{A,B}^2)$ remains valid for any POVM, which is proved via an extension of the Hilbert space as a direct sum $\mathscr{H} \oplus \mathscr{H}^{\text{aux}} \oplus \mathscr{H}^{\text{aux}}$, pure states $|\Phi\rangle = |\Psi\rangle \oplus |0\rangle \oplus |0\rangle$ and projectors

$$A_i' = \begin{pmatrix} A_i & \sqrt{A_i(I-A_i)} & 0 \\ \sqrt{A_i(I-A_i)} & I-A_i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_j' = \begin{pmatrix} B_j & 0 & \sqrt{B_j(I-B_j)} \\ 0 & 0 & 0 \\ \sqrt{B_j(I-B_j)} & 0 & I-B_j \end{pmatrix}$$

(iv) Finally bound $f(c_{A,B}^2)$ remains again valid dealing with mixed states, which is proved via a purification (or Schmidt decomposition), $|\Phi\rangle = \sum_{l=1}^N \sqrt{\rho_l}|l\rangle \otimes |l^{\text{aux}}\rangle \in \mathscr{H} \otimes \mathscr{H}^{\text{aux}}$ of the mixed states written under its diagonal form $\rho = \sum_{l=1}^N \rho_l |l\rangle\langle l|$, together with extended operators $O_k \otimes I$.

A way to look at inequalities (3) is that they restrict the domain allowed for the pair of maximal probabilities $\left(P_{A;\rho}\,,\,P_{B;\rho}\right)$. More precisely, notice first that if $P_{A;\rho} \leq c_{A,B}^2$, then $f(P_{A;\rho}) \geq f(c_{A,B}^2)$ and thus if and only if the pair $(P_{A;\rho}, P_{B;\rho})$ is within $(c_{A,B}^2\,;\,1]^2$ bound $f(c_{A,B}^2)$ can restrict the uncertainty that writes down as

$$P_{B;\rho} \leq f^{-1}\left(f(c_{A,B}^2 - f(P_{A;\rho})\right) \qquad \text{for} \qquad P_{A;\rho} \in (c_{A,B}^2\,;\,1] \tag{5}$$

(and similarly exchanging A and B). The question is then to determine which function f leads to the most restrictive inequality (5). The answer is given by

Proposition 2 *Within the whole family of uncertainty inequalities given by Proposition 1, the strongest restriction for the pair of maximal probabilities* $\left(P_{A;\rho}, P_{B;\rho}\right)$*, rewritten as inequality* (5)*, corresponds to Wootters case* $f(x) = w(c) = \arccos\sqrt{x}$.

To prove this proposition, let us fix $P_{A;\rho} \equiv \cos^2\theta$ and let us introduce $\gamma = \arccos c$. Assume then that there is a metric (function) f such that $f^{-1}\left(f(c) - f(\cos^2\theta)\right) < w^{-1}\left(w(c) - w(\cos^2\theta)\right) = \cos^2(\gamma - \theta)$, which rewrites $f(\cos^2\theta) + f(\cos^2(\gamma-\theta)) < f(\cos^2\gamma)$. We can check that for two orthogonal states $|\psi_1\rangle$ and $|\Psi\rangle$, the triplet $|\phi\rangle = \cos\theta|\psi_1\rangle + \sin\theta|\Psi\rangle$, $|\psi_2\rangle = \cos\gamma|\psi_1\rangle + \sin\gamma|\Psi\rangle$ and $|\psi_1\rangle$ violate the triangle inequality applied to d_f, proving that $f^{-1}\left(f(c) - f(\cos^2\theta)\right) < w^{-1}\left(w(c) - w(\cos^2\theta)\right)$ is not possible.

Propositions 1 and 2 lead directly to the following one, describing the restricted allowed domain for the couple of maximal probabilities $(P_{A;\rho}, P_{B;\rho})$.

Proposition 3 *In the context of Proposition 1, the pair of maximal probabilities* $(P_{A;\rho}, P_{B;\rho})$ *is constrained to the domain*

$$\mathbb{D}_{\mathrm{LP}}(\mathbf{c}) = \left\{(P_A, P_B) \in \left[\frac{1}{N_A}\, ;\, c_A^2\right] \times \left[\frac{1}{N_B}\, ;\, c_B^2\right] : P_B \le \cos^2(\gamma_{A,B} - \arccos\sqrt{P_{A;\rho}})\right\} \tag{6}$$

$$\textit{with} \qquad \gamma_A \equiv \arccos c_A, \quad \gamma_B \equiv \arccos c_B, \quad \gamma_{A,B} \equiv \arccos c_{A,B} \tag{7}$$

If $\gamma_A + \gamma_B \ge \gamma_{A,B}$*, the allowed domain becomes* $\left[\frac{1}{N_A}\, ;\, c_A^2\right] \times \left[\frac{1}{N_B}\, ;\, c_B^2\right]$.

GENERALIZED ENTROPIC UNCERTAINTY RELATIONS

We consider now as measure of uncertainty *generalized* (g,λ)*-entropies*:

$$\mathscr{U}_{g,\lambda}(p) = G_\lambda(p) \equiv \frac{g\left(\sum_k p_k^\lambda\right)}{1-\lambda} \tag{8}$$

where $\lambda \ge 0$ is called *entropic index* and function g does not depend on λ, is continuous, differentiable and strictly increasing on $\mathbb{R}_+$ with $g(1) = 0$ and $g'(1) = 1$. From these conditions, the limiting case $\lambda \to 1$ is well defined and gives Shannon entropy [11], namely $G_1(p) \equiv H(p) = -\sum_k p_k \ln p_k$. Then, G_λ generalizes Shannon entropy, the index λ playing the role of a "magnifying glass": when $\lambda < 1$ the contribution of the tails of the distribution are stressed and, conversely, when $\lambda > 1$ the leading probabilities are stressed.

In particular, we employ here two families of measures. For $f(x) = \ln x$ and $f(x) = x-1$, entropy G_λ is respectively Rényi entropy R_λ [12] or Havrda–Charvát–Tsallis entropy S_λ [13]:

$$R_\lambda(p) = \frac{\ln\left(\sum_k p_k^\lambda\right)}{1-\lambda} \qquad \text{and} \qquad S_\lambda(p) = \frac{1-\sum_k p_k^\lambda}{\lambda-1} \tag{9}$$

One can easily check that G_λ satisfies the required properties, and also that $G_\lambda(p)$ is a decreasing function vs λ when p is fixed.

The study of entropic formulations to quantify the uncertainty principle is not new and has been addressed in various contexts [3]. However, the problem of finding optimal bounds still remains open in many cases. Moreover, many available results correspond to conjugated indices (in the sense of Hölder: $\frac{1}{2\alpha} + \frac{1}{2\beta} = 1$) as they are based on the Riesz–Thorin theorem. To fix notation, we define by $\mathscr{C} = \left\{(\alpha,\beta) \in \left(\frac{1}{2}\, ;+\infty\right)^2 : \beta = \frac{\alpha}{2\alpha-1}\right\}$ the so-called conjugacy curve. Then, $\underline{\mathscr{C}}$ denotes the domain of positive indices "below" this curve, while $\overline{\mathscr{C}}$ denotes the domain of positive indices "above" this curve. For the

Shannon entropy $(\alpha,\beta)=(1,1)$, nondegenerate observables and pure states, Deutsch obtained a first bound [4], improved by Maassen and Uffink [5] to $\mathscr{B}^{MU}(c)=-2\ln c$ and later on by de Vicente and Sanchez-Ruiz [6]. For the Rényi entropy, $(\alpha,\beta)\in\mathscr{C}$, bound $\mathscr{B}^{MU}(c)$ remains valid and was extended by Rastegin to the case of mixed states and POVM [7]. For $(\alpha,\beta)\in\underline{\mathscr{C}}$, the bound $\mathscr{B}^{MU}(c)$ remains valid due to the decreasing property of Rényi entropy with the index. Finally, for $\beta=\alpha$, Puchała, Rudnicki and Życzkowski (PRZ) in Ref. [8] derived recently a series of $N-1$ bounds depending on the transformation matrix T by using majorization technique. We denote by $\mathscr{B}^{PRZ}_{\alpha;\ln}(T)$ the greatest of those bounds. Moreover, a refined bound depending also on the second larger element of T was proposed recently by Coles and Piani [9].

Here, we extend these results in the following way:

Proposition 4 *In the conditions of Proposition 1, for generalized entropies of the form* (8)*, with any pair of entropic indices $(\alpha,\beta)\in\mathbb{R}^2_+$, the following uncertainty relation holds:*

$$G_\alpha\big(p(A,\rho)\big)+G_\beta\big(p(B,\rho)\big)\geq\mathscr{B}_{\alpha,\beta;g}(\mathbf{c}) \tag{10}$$

with $\mathbf{c}=\big(c_A,c_B,c_{A,B}\big)$ & $\big(\gamma_A,\gamma_B,\gamma_{A,B}\big)$) Eqs. (4)-(7)*, and the lower bound expresses as*

$$\mathscr{B}_{\alpha,\beta;g}(\mathbf{c})=\begin{cases}\mathscr{D}_{\alpha;g}(\gamma_A)+\mathscr{D}_{\beta;g}(\gamma_B) & \text{if}\quad \gamma_{A,B}\leq\gamma_A+\gamma_B\\[2ex] \displaystyle\min_{\theta\in[\gamma_A,\gamma_{A,B}-\gamma_B]}\big(\mathscr{D}_{\alpha;g}(\theta)+\mathscr{D}_{\beta;g}(\gamma_{A,B}-\theta)\big) & \text{otherwise}\end{cases} \tag{11}$$

$$\text{where}\qquad \mathscr{D}_{\lambda;g}(\theta)\equiv\frac{1}{1-\lambda}f\left(\left\lfloor\frac{1}{\cos^2\theta}\right\rfloor\big(\cos^2\theta\big)^\lambda+\left(1-\left\lfloor\frac{1}{\cos^2\theta}\right\rfloor\cos^2\theta\right)^\lambda\right) \tag{12}$$

To prove this proposition we proceed in two steps: (i) minimization $G_{\lambda,\min}(P)=\min_p G_\lambda(p)$ subject to $P=\max_k p_k$ leading to $G_\alpha(p(A;\rho))+G_\beta(p(B;\rho))\geq G_{\alpha,\min}\big(P_{A;\rho}\big)+G_{\beta,\min}\big(P_{B;\rho}\big)$; (ii) minimization of the left-hand side subject to the Landau-Pollak inequality.

For the first step, one can checked that vector $p=[P\quad\cdots P\quad 1-MP\quad 0\quad\cdots\quad 0]$ with $M=\lfloor 1/P\rfloor$ majorizes all probability vectors satisfying the constraints, and thus gives the minimal entropy from the Schur-concavity property. For the second step, from the Schur-concavity one can show that $G_{\lambda,\min}(P)$ is a decreasing function of P, the result coming from studying what happens in the Landau-Pollak domain (6) (fixing alternatively $P_{A;\rho}$ and $P_{B;\rho}$).

SOME ILLUSTRATIONS

Here, we present some illustrations of the uncertainty relations derived. To this end, we draw randomly POVM pairs A and B (nondegenerate and degenerated cases). For any given pair of POVM, we draw randomly states ρ, and we calculate the uncertainty sums and the corresponding bounds given in proposition 1 and in proposition 4. Moreover, to illustrate proposition 3, we also draw the cloud of points $\big(P_{A;\rho}\,,P_{B;\rho}\big)$ for fixed POVM pairs, together with their allowed domain $\mathbb{D}_{\mathrm{LP}}(\mathbf{c})$.

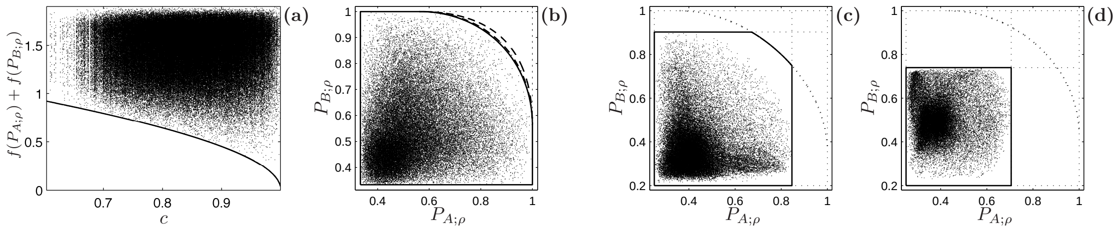

FIGURE 1. Extended Landau–Pollak formulations. (a) Uncertainty sum $f(P_{A;\rho})+f(P_{B;\rho})$ (points), and the bound $f(c^2)$ (solid line) in the nondegenerate case with $N=3$; (b)-(d) Domain $\mathbb{D}_{\mathrm{LP}}$ that corresponds to Wootters metric (solid line), the Bures metric (dashed-dotted line) and the root-infidelity metric (dashed line); the points represent the pairs $(P_{A,\rho},P_{B,\rho})$ with (A,B) fixed. Nondegenerate case (b) with $c=.75$; degenerate case ($N_A=4$ and $N_B=5$) (c) with $\mathbf{c}=(0.92\,,0.95\,,0.60)$ and (d) with $\mathbf{c}=(0.84\,,0.86\,,0.84)$.

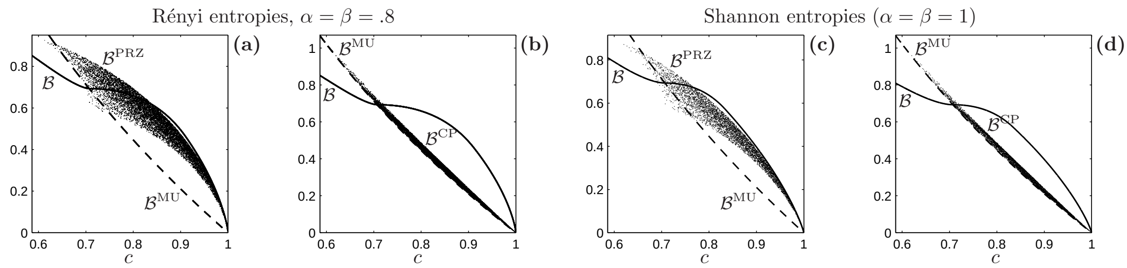

FIGURE 2. Entropic formulations. Our bound $\mathscr{B}$ (given by Proposition 4), Maassen–Uffink bound $\mathscr{B}^{\mathrm{MU}}$, Puchała et al. bound $\mathscr{B}^{\mathrm{PRZ}}$ and Coles and Piani bound $\mathscr{B}^{\mathrm{CP}}$ vs the overlap c for the case of nondegenerate observables with $N=3$.

Figures 1-(a) represents the Landau–Pollak type uncertainties sum $f(P_{A;\rho})+f(P_{B;\rho})$ in the nondegenerate context, versus the overlap c, compared to the lower bound $f(c^2)$. Only the Wootters metric is considered ($f(x)=w(x)=\arccos\sqrt{x}$). Figures 1-(b) to (d) represent the allowed domain for $\left(P_{A;\rho}\,,P_{B;\rho}\right)$ for fixed POVM pairs, (b) in the nondegenerate context (in this case the Bures metric and the root-infidelity metric are also represented), (c) and (d) for degenerated cases. These curves illustrate both proposition and let also suggest that $\mathscr{D}_{\mathrm{LP}}$ is optimal in the sense that given a triplet of overlaps $\mathbf{c}$, there is a POVM pair so that the cloud of points issued from the ensemble outcomes seems to full the domain. This assertion remains however to be proved.

Figures 2-(a) to (d) represent the Rényi entropies sum $R_\alpha(p(A;\rho))+R_\alpha(p(B;\rho))$ in the nondegenerate context, versus the overlap c, compared to the lower bound $\mathscr{B}_{\alpha,\alpha;\ln}(c)$. Our bound is compared to the most well known bound, due to Maassen–Uffink, and to more recent bounds depending on the whole transformation T, namely the bounds due to Puchała et al. and that due to Coles and Piani. Although not represented here, similar results are obtained with the Tsallis entropy. The figures illustrate that in many cases, our bound improves that of the literature, even that which depend on the whole transformation T. The optimality of the proposed bound (in the sense of fixed $\mathbf{c}$ and/or fixed POVM pair) remain to be further investigated.

CONCLUSIONS

We derive a family of uncertainty relations in the most general context of observables described by POVM sets and for mixed quantum states. In a first part, the obtained relations extend and generalize the well-known Landau–Pollak inequality. The key point in our treatment is that the measures of uncertainty are given in terms of a metric, which satisfies the triangle inequality. Moreover, within the family of metrics considered, it comes out that Wootters metric, leading to the usual Landau–Pollak inequality (its extension to mixed states and POVM descriptions) is the most restrictive among the family of inequalities we obtain. From these propositions, we determine the allowed domain for the pair of maximal probabilities corresponding to two observables.

A direct consequence of our results is that a previous work [6] dealing with generalized entropies of probability vectors extends very easily in the most general case of POVM representations of observables. This extension, given in the second part, is obtained by minimizing the entropy sum for a fixed maximal probability, and thus minimizing this minimal entropies subject to the obtained Landau–Pollak inequality. The first part is solved using Schur-convexity properties of the generalized entropy we employed.

Finally, both formulations are illustrated via simulations. As a perspective, the tightness of the bounds/domains remain to be fully solved, and the geometrical structure of the most restrictive domain (overlap or POVMs fixed) remain to be studied.

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