Koszul Information Geometry and Souriau Lie Group Thermodynamics

Frédéric Barbaresco[1]

[1] Thales Air Systems, Advanced Radar Concepts, Voie Pierre-Gilles de Gennes. 91470 Limours, France

[1] frederic.barbaresco@thalesgroup.com

Abstract. The François Massieu 1869 idea to derive some mechanical and thermal properties of physical systems from "Characteristic Functions", was developed by Gibbs and Duhem in thermodynamics with the concept of potentials, and introduced by Poincaré in probability. This paper deals with generalization of this Characteristic Function concept by Jean-Louis Koszul in Mathematics and by Jean-Marie Souriau in Statistical Physics. The Koszul-Vinberg Characteristic Function (KVCF) on convex cones will be presented as cornerstone of "Information Geometry" theory, defining Koszul Entropy as Legendre transform of minus the logarithm of KVCF, and Fisher Information Metrics as hessian of these dual functions, invariant by their automorphisms. In parallel, Souriau has extended the Characteristic Function in Statistical Physics looking for other kinds of invariances through co-adjoint action of a group on its momentum space, defining physical observables like energy, heat and momentum as pure geometrical objects. In covariant Souriau model, Gibbs equilibriums states are indexed by a geometric parameter, the Geometric (Planck) Temperature, with values in the Lie algebra of the dynamical Galileo/Poincaré groups, interpreted as a space-time vector, giving to the metric tensor a null Lie derivative. Fisher Information metric appears as the opposite of the derivative of Mean "Moment map" by geometric temperature, equivalent to a Geometric Capacity or Specific Heat. These elements has been developed by author in [10][11].

KOSZUL CHARACTERISTIC FUNCTION/ENTROPY BY LEGENDRE DUALITY

We define Koszul-Vinberg Hessian metric on convex sharp cone, and observe that the Fisher information metric of Information Geometry coincides with the canonical Koszul Hessian metric (given by Koszul forms). We also observe, by Legendre duality (Legendre transform of minus Koszul characteristic function logarithm), that we are able to introduce a *Koszul Entropy*, that plays the role of generalized Shannon Entropy.

Koszul-Vinberg Characteristic Function and Metric for convex sharp cone

Jean-Louis Koszul [1][2][3][4] and E. Vinberg have introduced an affinely invariant hessian metric on a sharp convex cone Ω through its characteristic function ψ. In the following, Ω is a sharp open convex cone in a vector space E of finite dimension on R (a convex cone is sharp if it does not contain any full straight line). In dual space E^* of E, Ω^* is the set of linear strictly positive forms on $\overline{\Omega} - \{0\}$. Ω^* is the dual cone of Ω and is a sharp open convex cone. If $\xi \in \Omega^*$, then the intersection $\Omega \bigcap \{x \in E / \langle x, \xi \rangle = 1\}$ is bounded. $G = Aut(\Omega)$ is the group of linear transform of E that preserves Ω. $G = Aut(\Omega)$ operates on Ω^* by $\forall g \in G = Aut(\Omega), \forall \xi \in E^*$ then $\tilde{g}.\xi = \xi \circ g^{-1}$

Koszul-Vinberg Characteristic function definition:

Let $d\xi$ be the Lebesgue measure on E^*, the following integral:

$$\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi \quad \forall x \in \Omega \tag{1}$$

*with Ω^* the dual cone is an analytic function on Ω, with $\psi_\Omega(x) \in \,]0,+\infty[$, called the **Koszul-Vinberg characteristic function** of cone Ω.*

The Koszul-Vinberg Characteristic Function has the following properties:

- ψ_Ω is analytic function defined on the interior of Ω and $\psi_\Omega(x) \to +\infty$ as $x \to \partial\Omega$

  If $g \in Aut(\Omega)$ then $\psi_\Omega(gx) = |\det g|^{-1} \psi_\Omega(x)$ and since $tI \in G = Aut(\Omega)$ for any $t > 0$, we have

  $$\psi_\Omega(tx) = \psi_\Omega(x)/t^n \tag{2}$$

- ψ_Ω is logarithmically strictly convex $(\phi_\Omega(x) = \log(\psi_\Omega(x))$ is strictly convex$)$

From the KVCF, could be introduced two forms defined by Koszul:

Koszul 1-form α: *The differential 1-form $\alpha = d\phi_\Omega = d\log\psi_\Omega = d\psi_\Omega / \psi_\Omega$* $\tag{3}$

is invariant by all automorphisms $G = Aut(\Omega)$ of Ω. If $x \in \Omega$ and $u \in E$ then

$$\langle \alpha_x, u\rangle = -\int_{\Omega^*} \langle \xi, u\rangle.e^{-\langle \xi, x\rangle} d\xi \quad and \quad \alpha_x \in -\Omega^* \tag{4}$$

and

Koszul 2-form β: The symmetric differential 2-form

$$\beta = D\alpha = d^2\log\psi_\Omega \tag{5}$$

is a positive definite symmetric bilinear form on E invariant under $G = Aut(\Omega)$. $D\alpha > 0$

This positivity is given by Schwarz inequality and

$$d^2\log\psi_\Omega(u,v) = \int_{\Omega^*} \langle \xi, u\rangle\langle \xi, v\rangle e^{-\langle \xi, u\rangle} d\xi \tag{6}$$

We can then introduce the Koszul metric based on previous definitions:

Koszul Metric: $D\alpha$ *defines a Riemanian structure invariant by $Aut(\Omega)$, and then the Riemanian metric is given by*

$$g = d^2\log\psi_\Omega$$

$$\left(d^2\log\psi(x)\right)(u) = \frac{1}{\psi(u)^2}\left[\int_{\Omega^*} F(\xi)^2 d\xi.\int_{\Omega^*} G(\xi)^2 d\xi - \left(\int_{\Omega^*} F(\xi).G(\xi)d\xi\right)^2\right] > 0 \tag{7}$$

with $F(\xi) = e^{-\frac{1}{2}\langle x,\xi\rangle}$ and $G(\xi) = e^{-\frac{1}{2}\langle x,\xi\rangle}\langle u,\xi\rangle$

This result is obtained using Schwarz inequality, $d\log\psi = \dfrac{d\psi}{\psi}$ and $d^2\log\psi = \dfrac{d^2\psi}{\psi} - \left(\dfrac{d\psi}{\psi}\right)^2$ where

$(d\psi(x))(u) = -\int_{\Omega^*} e^{-\langle x,\xi\rangle}\langle u,\xi\rangle d\xi$ and $\left(d^2\psi(x)\right)(u) = \int_{\Omega^*} e^{-\langle x,\xi\rangle}\langle u,\xi\rangle^2 d\xi$.

A diffeomorphism is used to define dual coordinate:

$$x^* = -\alpha_x = -d\log\psi_\Omega(x) \tag{8}$$

With $\langle df(x), u\rangle = D_u f(x) = \dfrac{d}{dt}\bigg|_{t=0} f(x+tu)$. When the cone Ω is symmetric, the map $x \mapsto x^* = -\alpha_x$ is a bijection and

an isometry with one unique fixed point (the manifold is a Riemannian Symmetric Space given by this isometry):

$$(x^*)^* = x \ , \ \langle x, x^*\rangle = n \ \text{and} \ \psi_\Omega(x)\psi_{\Omega^*}(x^*) = cste \tag{9}$$

x^* is characterized by $x^* = \arg\min\{\psi(y)/ y \in \Omega^*, \langle x, y\rangle = n\}$ and x^* **is the center of gravity of the cross section** $\{y \in \Omega^*, \langle x, y\rangle = n\}$ of Ω^*:

$$x^* = \int_{\Omega^*} \xi.e^{-\langle \xi,x\rangle}d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi \quad \text{and} \quad \langle -x^*, h\rangle = d_h \log \psi_\Omega(x) = -\int_{\Omega^*} \langle \xi, h\rangle e^{-\langle \xi,x\rangle}d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi \tag{10}$$

If we set $\Phi(x) = -\log \psi_\Omega(x)$, Gromov has observed that $x^* = d\Phi(x)$ is an injection where the closure of the image equals the convex hull of the support and the volume of this hull is the n-dimensional volume defined by the integral of the determinant of the hessian of this function $\Phi(x)$, where the map $\Phi \mapsto M(\Phi) = \int_\Omega \det(Hess(\Phi(x))).dx$ obeys non-trivial convexity relation given by the Brunn-Minkowsky inequality $[M(\Phi_1 + \Phi_2)]^{1/n} \geq [M(\Phi_1)]^{1/n} + [M(\Phi_2)]^{1/n}$.

Koszul Entropy and its barycenter

From this last equation, we can deduce the "**_Koszul Entropy_**" **defined as Legendre Transform of** $\Phi(x)$ **minus logarithm of Koszul-Vinberg characteristic function**:

$$\Phi^*(x^*) = \langle x, x^*\rangle - \Phi(x) \text{ with } x^* = D_x\Phi \text{ and } x = D_{x^*}\Phi^* \text{ where } \Phi(x) = -\log \psi_\Omega(x) \tag{11}$$

$$\Phi^*(x^*) = \langle (D_x\Phi)^{-1}(x^*), x^*\rangle - \Phi[(D_x\Phi)^{-1}(x^*)] \quad \forall x^* \in \{D_x\Phi(x)/ x \in \Omega\} \tag{12}$$

By definition of Koszul-Vinberg Characteristic function, and by use that $-\langle \xi, x\rangle = \log e^{-\langle \xi,x\rangle}$, we can write:

$$-\langle x^*, x\rangle = \int_{\Omega^*} \log e^{-\langle \xi,x\rangle}.e^{-\langle \xi,x\rangle}d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi \tag{13}$$

and

$$\Phi^*(x^*) = \langle x, x^*\rangle - \Phi(x) = -\int_{\Omega^*} \log e^{-\langle \xi,x\rangle}.e^{-\langle \xi,x\rangle}d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi + \log \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi$$

$$\Phi^*(x^*) = \left[\left(\int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi\right).\log \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi - \int_{\Omega^*} \log e^{-\langle \xi,x\rangle}.e^{-\langle \xi,x\rangle}d\xi\right] \Big/ \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi$$

$$\Phi^*(x^*) = \left[\log \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi - \int_{\Omega^*} \log e^{-\langle \xi,x\rangle}.\frac{e^{-\langle \xi,x\rangle}}{\int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi}d\xi\right]$$

$$\Phi^*(x^*) = \left[\log \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi.\left(\int_{\Omega^*} \frac{e^{-\langle \xi,x\rangle}}{\int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi}d\xi\right) - \int_{\Omega^*} \log e^{-\langle \xi,x\rangle}.\frac{e^{-\langle \xi,x\rangle}}{\int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi}d\xi\right] \text{ with } \int_{\Omega^*} \frac{e^{-\langle \xi,x\rangle}}{\int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi}d\xi = 1$$

$$\Phi^*(x^*) = \left[-\int_{\Omega^*} \frac{e^{-\langle \xi,x\rangle}}{\int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi}.\log\left(\frac{e^{-\langle \xi,x\rangle}}{\int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi}\right)d\xi\right] \tag{14}$$

In this last equation, $p_x(\xi) = e^{-\langle \xi,x\rangle} \Big/ \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi$ appears as a density, and the Legendre transform $\Phi^*()$ looks like the classical Shannon Entropy, named in the following **Koszul Entropy**:

$$\Phi^* = -\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \tag{15}$$

With $p_x(\xi) = e^{-\langle \xi,x\rangle} \Big/ \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi = e^{-\langle x,\xi\rangle - \log \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi} = e^{-\langle x,\xi\rangle + \Phi(x)}$ and $x^* = \int_{\Omega^*} \xi.p_x(\xi)d\xi$ (16)

We will call $p_x(\xi) = \dfrac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi}$ the Koszul Density, with the property that:

$$\log p_x(\xi) = -\langle x, \xi \rangle - \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi = -\langle x, \xi \rangle + \Phi(x) \tag{17}$$

and $E_\xi[-\log p_x(\xi)] = \langle x, x^* \rangle - \Phi(x)$ \hfill (18)

We can observe that:

$$\Phi(x) = -\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi = -\log \int_{\Omega^*} e^{-[\Phi^*(\xi) + \Phi(x)]} d\xi = \Phi(x) - \log \int_{\Omega^*} e^{-\Phi^*(\xi)} d\xi$$

$$\Rightarrow \int_{\Omega^*} e^{-\Phi^*(\xi)} d\xi = 1 \tag{19}$$

But the development is not achieved and we have to make appear x^* in $\Phi^*(x^*)$. For this objective, we have to write:

$$\log p_x(\xi) = \log e^{-\langle x, \xi \rangle + \Phi(x)} = \log e^{-\Phi^*(\xi)} = -\Phi^*(\xi)$$

$$\Rightarrow \Phi^* = -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi = \int_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = \Phi^*(x^*) \tag{20}$$

The last equality is true if and only if we have the following relation:

$$\int_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = \Phi^*\left(\int_{\Omega^*} \xi . p_x(\xi) d\xi \right) \text{ as } x^* = \int_{\Omega^*} \xi . p_x(\xi) d\xi \tag{21}$$

This condition could be written more synthetically:

$$E[\Phi^*(\xi)] = \Phi^*(E[\xi]) \ , \ \xi \in \Omega^* \tag{22}$$

The meaning of this relation is that "***Barycenter of Koszul Entropy is Koszul Entropy of Barycenter***".
This condition is achieved for $x^* = D_x \Phi$ taking into account Legendre Transform property:

$$\text{Legendre Transform}: \ \Phi^*(x^*) = \underset{x}{Sup}[\langle x, x^* \rangle - \Phi(x)]$$

$$\Rightarrow \begin{cases} \Phi^*(x^*) \geq \langle x, x^* \rangle - \Phi(x) \\ \Phi^*(x^*) \geq \int_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi \end{cases} \Rightarrow \begin{cases} \Phi^*(x^*) \geq E[\Phi^*(\xi)] \\ \text{equality for } x^* = \dfrac{d\Phi}{dx} \end{cases} \tag{23}$$

Relation of Koszul density with Maximum Entropy Principle

We will observe in this section, that Koszul density is solution of Maximum Entropy. Classically, the density given by Maximum Entropy Principle] is given by:

$$\underset{p_x(.)}{Max}\left[-\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi \right] \text{ such } \begin{cases} \int_{\Omega^*} p_x(\xi) d\xi = 1 \\ \int_{\Omega^*} \xi . p_x(\xi) d\xi = x^* \end{cases} \tag{24}$$

If we take $q_x(\xi) = e^{-\langle \xi, x \rangle} / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi = e^{-\langle x, \xi \rangle - \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi}$ such that:

$$\begin{cases} \int_{\Omega^*} q_x(\xi) . d\xi = \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi = 1 \\ \log q_x(\xi) = \log e^{-\langle x, \xi \rangle - \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} = -\langle x, \xi \rangle - \log \int_{\Omega^*} e^{-\langle x, \xi \rangle} d\xi \end{cases} \tag{25}$$

Then by using the fact that $\log x \geq (1 - x^{-1})$ with equality if and only if $x = 1$, we find the following:

$$-\int_{\Omega^*} p_x(\xi) \log \frac{p_x(\xi)}{q_x(\xi)} d\xi \leq -\int_{\Omega^*} p_x(\xi) \left(1 - \frac{q_x(\xi)}{p_x(\xi)}\right) d\xi \tag{26}$$

We can then observe that:

$$\int_{\Omega^*} p_x(\xi) \left(1 - \frac{q_x(\xi)}{p_x(\xi)}\right) d\xi = \int_{\Omega^*} p_x(\xi) d\xi - \int_{\Omega^*} q_x(\xi) d\xi = 0 \tag{27}$$

because $\int_{\Omega^*} p_x(\xi) d\xi = \int_{\Omega^*} q_x(\xi) d\xi = 1$.

We can then deduce that:

$$-\int_{\Omega^*} p_x(\xi) \log \frac{p_x(\xi)}{q_x(\xi)} d\xi \leq 0 \Rightarrow -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi \leq -\int_{\Omega^*} p_x(\xi) \log q_x(\xi) d\xi \tag{28}$$

If we develop the last inequality, using expression of $q_x(\xi)$:

$$-\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi \leq -\int_{\Omega^*} p_x(\xi) \left[-\langle x, \xi \rangle - \log \int_{\Omega^*} e^{-\langle x, \xi \rangle} d\xi\right] d\xi \tag{29}$$

$$-\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi \leq \left\langle x, \int_{\Omega^*} \xi.p_x(\xi) d\xi \right\rangle + \log \int_{\Omega^*} e^{-\langle x, \xi \rangle} d\xi \tag{30}$$

If we take $x^* = \int_{\Omega^*} \xi.p_x(\xi) d\xi$ and $\Phi(x) = -\log \int_{\Omega^*} e^{-\langle x, \xi \rangle} d\xi$, then we deduce that the Koszul density

$q_x(\xi) = e^{-\langle \xi, x \rangle} / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi = e^{-\langle x, \xi \rangle - \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi}$ is the Maximum Entropy solution constrained by $\int_{\Omega^*} p_x(\xi) d\xi = 1$ and

$$\int_{\Omega^*} \xi.p_x(\xi) d\xi = x^* : -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi \leq \langle x, x^* \rangle - \Phi(x) \tag{31}$$

$$-\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi \leq \Phi^*(x^*) \tag{32}$$

We have then observed that Koszul Entropy provides density of Maximum Entropy:

$$p_{\bar{\xi}}(\xi) = \frac{e^{-\langle \xi, \Theta^{-1}(\bar{\xi}) \rangle}}{\int_{\Omega^*} e^{-\langle \xi, \Theta^{-1}(\bar{\xi}) \rangle} d\xi} \quad \text{with} \quad x = \Theta^{-1}(\bar{\xi}) \quad \text{and} \quad \bar{\xi} = \Theta(x) = \frac{d\Phi(x)}{dx} \tag{33}$$

where $\bar{\xi} = \int_{\Omega^*} \xi.p_{\bar{\xi}}(\xi) d\xi$ and $\Phi(x) = -\log \int_{\Omega^*} e^{-\langle x, \xi \rangle} d\xi$ \tag{34}

SOURIAU LIE GROUP THERMODYNAMICS

Souriau [6][7][8][9] has defined Gibbs canonical ensemble on Symplectic manifold M for a Lie group action on M. In classical statistical mechanics, a state is given by the solution of Liouville equation on the phase space, the partition function. As Symplectic manifolds have a completely continuous measure, invariant by diffeomorphisms, the Liouville measure λ, all statistical states will be the product of Liouville measure by the scalar function given by the generalized partition function $e^{\Phi - \beta.U}$ defined by the generalized energy U (the moment that is defined in dual of Lie Algebra of this dynamical group) and the geometric temperature β, where Φ is a normalizing constant such the mass of probability is equal to 1, $\Phi = -\log \int_M e^{-\beta.U} d\omega$. Jean-Marie Souriau then generalizes the Gibbs equilibrium state to all Symplectic manifolds that have a dynamical group. To ensure that all integrals, that will be defined, could converge, *the canonical Gibbs ensemble is the largest open proper subset (in Lie algebra) where these*

integrals are convergent. This canonical Gibbs ensemble is convex. The derivative of Φ , $Q = \dfrac{\partial \Phi}{\partial \beta}$ is equal to the mean value of the energy U (heat in thermodynamic). The minus derivative of this generalized heat Q , $-\dfrac{\partial Q}{\partial \beta}$ is symmetric and positive (it is a generalization of heat capacity). Entropy s is then defined by Legendre transform of Φ, $s = \beta.Q - \Phi$. If this approach is applied for the group of time translation, this is the classical thermodynamic theory. But *Souriau has observed that if we apply this theory for non-commutative group (Galileo or Poincaré groups), the symmetry has been broken*. *Classical Gibbs equilibrium states are no longer invariant by this group*. This symmetry breaking provides new equations, discovered by Jean-Marie Souriau.

For each temperature β, Jean-Marie Souriau has introduced a tensor f_β, equal to the sum of cocycle f and Heat coboundary (with [.,.] Lie bracket):

$$f_\beta(Z_1, Z_2) = f(Z_1, Z_2) + Q.ad_{Z_1}(Z_2) \quad \text{with} \quad ad_{Z_1}(Z_2) = [Z_1, Z_2] \tag{35}$$

This tensor f_β has the following properties:

- f_β is a symplectic cocycle (we refer to books of Sympectic geometry for cocycle definition)

- $\beta \in Ker\, f_\beta$

- The following symmetric tensor g_β, defined on all values of $ad_\beta(.)$ is positive definite:

$$g_\beta([\beta, Z_1], [\beta, Z_2]) = f_\beta(Z_1, [\beta, Z_2]) \tag{36}$$

These equations are universal, because they are not dependent of the symplectic manifold but only of the dynamical group G, its symplectic cocycle f, the temperature β and the heat Q. Souriau called this model "***Lie Groups Thermodynamics***". We can read in his paper this prophetical sentence "***Peut-être cette thermodynamique des groups de Lie a-t-elle un intérêt mathématique***". He explains that for dynamic Galileo group (rotation and translation) with only one axe of rotation, this thermodynamic theory is the theory of centrifuge where the temperature vector dimension is equal to 2 (sub-group of invariance of size 2), used to make "butter", "uranium 235" and "ribonucleic acid". The physical meaning of these 2 dimensions for vector-valued temperature are "thermic conduction" and "viscosity". Souriau said that the model unifies "heat conduction" and "viscosity" (Fourier and Navier equations) in the same theory of irreversible process. Souriau has applied this theory in details for relativistic ideal gas with Poincaré group for dynamical group.

We will give in the following the two others main theorems of Souriau on this "Lie Group Thermodynamics".

Souriau Theorem:
Let Ω be the largest open proper subset of $\mathfrak{g}$, Lie algebra of G, such that $\displaystyle\int_M e^{-\beta.U(\xi)} d\omega$ and $\displaystyle\int_M \xi.e^{-\beta.U(\xi)} d\omega$ are convergent integrals, this set Ω is convex and is invariant under every transformation $\overline{a}_\mathfrak{g}$, where $a \mapsto \overline{a}_\mathfrak{g}$ is the adjoint representation of G. Then, the variables are changed according to:

- $\beta \to \overline{a}_\mathfrak{g}(\beta)$ (37)

- $\Phi \to \Phi - \theta(a^{-1})\beta = \Phi + \theta(a).\overline{a}_\mathfrak{g}(\beta)$ (38)

- $s \rightarrow s$ $\hspace{6cm}$ (39)

- $Q \rightarrow \overline{a}_{\mathfrak{g}^*}(Q) + \theta(a) = \overline{a}_{\mathfrak{g}_\theta^*}(Q)$ $\hspace{4cm}$ (40)

- $\varsigma \rightarrow \overline{a}_M^+(\varsigma)$ $\hspace{5.5cm}$ (41)

where θ is the cocycle associated with the group G and the moment, and $\overline{a}_M^+(\varsigma)$ is the image under $\overline{a}_M$ of the probability measure ς.

We observe that the entropy s is unchanged, and Φ is changed but with linear dependence to β, with consequence that Fisher Information Geometry metric is unchanged by dynamical group:

$$I(\overline{a}_{\mathfrak{g}}(\beta)) = -\frac{\partial^2\left[\Phi - \theta(a^{-1})\beta\right]}{\partial\beta^2} = -\frac{\partial^2\Phi}{\partial\beta^2} = I(\beta) \hspace{3cm} (42)$$

In previous notation, $a \mapsto \overline{a}_{\mathfrak{g}}$ the adjoint representation of G can be written:

$$\overline{a}_{\mathfrak{g}}(Z) = \delta\left[a \times b \times a^{-1}\right] \text{ with } b = e, \ \delta b = Z \text{ and } \delta a = 0 \hspace{2cm} (43)$$

$a \mapsto \overline{a}_{\mathfrak{g}}$ defines an action of G on its Lie algebra $\mathfrak{g}$, with $\overline{a}_{\mathfrak{g}}$ is called the adjoint representation, that is a linear representation of G on its Lie algebra $\mathfrak{g}$. This Souriau theorem if based on invariance property of Liouville measure.
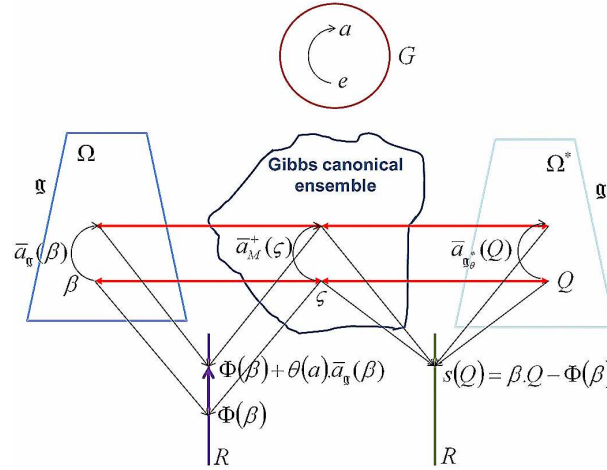

FIGURE 1. Souriau figure on Lie Groups Thermodynamics

We will synthetize in Table 1 results of previous chapters with Koszul Hessian Structure of Information Geometry and the Souriau model of Statistical Physics with the general concepts of geometric temperature, heat and capacity. Analogies between models will deal with characteristic function, Entropy, Legendre Transform, density of probability, dual coordinate systems, Hessian Metric and Fisher metric. As $Q = \dfrac{\partial\Phi}{\partial\beta}$, we observe that the Information

Geometry metric $I(\beta) = -\dfrac{\partial^2\Phi(\beta)}{\partial\beta^2} = -\dfrac{\partial Q}{\partial\beta}$ could be considered as a *generalization of "Heat Capacity"*. Souriau

called it K the *"Geometric Capacity"*. When $\beta = \dfrac{1}{kT}$, $K = -\dfrac{\partial Q}{\partial\beta} = -\dfrac{\partial Q}{\partial T}\left(\dfrac{\partial\frac{1}{kT}}{\partial T}\right) = \dfrac{1}{kT^2}\dfrac{\partial Q}{\partial T}$, then this geometric

capacity is related to calorific capacity. Q is related to the mean, and K is related to the variance of U:

$$I(\beta) = -\frac{\partial Q}{\partial\beta} = \text{var}(U) = \int_M U(\xi)^2 . p_\beta(\xi)d\omega - \left(\int_M U(\xi) . p_\beta(\xi)d\omega\right)^2 \hspace{2cm} (44)$$

	Koszul Information Geometry Model	*Souriau Lie Groups Thermodynamics Model*
Characteristic function	$\Phi(x) = -\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \quad \forall x \in \Omega$	$\Phi(\beta) = -\log \int_{M} e^{-\beta.U(\xi)} d\omega \quad \forall \beta \in \mathfrak{g}$
Entropy	$\Phi^*(x^*) = -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi$	$s = -\int_{M} p(\xi) \log p(\xi) d\omega$
Legendre Transform	$\Phi^*(x^*) = \langle x, x^* \rangle - \Phi(x)$	$s(Q) = \beta.Q - \Phi(\beta)$
Density of probability	$p_x(\xi) = e^{-\langle x, \xi \rangle + \Phi(x)} = \dfrac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi}$	$p_\beta(\xi) = e^{-\beta.U(\xi) + \Phi(\beta)} = \dfrac{e^{-\beta.U(\xi)}}{\int_{M} e^{-\beta.U(\xi)} d\omega}$
Dual Coordinate Systems	$x \in \Omega \text{ and } x^* \in \Omega^*$ $x^* = \int_{\Omega^*} \xi.p_x(\xi) d\xi = \dfrac{\int_{\Omega^*} \xi.e^{-\langle \xi, x \rangle} d\xi}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi}$ $x^* = \dfrac{\partial \Phi(x)}{\partial x} \text{ and } x = \dfrac{\partial \Phi^*(x^*)}{\partial x^*}$	$\beta \in \mathfrak{g} \text{ and } Q \in \mathfrak{g}^*$ $Q = \int_{M} U(\xi).p_\beta(\xi) d\omega = \dfrac{\int_{M} U(\xi) e^{-\beta.U(\xi)} d\omega}{\int_{M} e^{-\beta.U(\xi)} d\omega}$ $Q = \dfrac{\partial \Phi}{\partial \beta} \text{ and } \beta = \dfrac{\partial s}{\partial Q}$
Hessian Metric	$ds^2 = -d^2\Phi(x)$	$ds^2 = -d^2\Phi(\beta)$
Fisher metric	$I(x) = -E_\xi\left[\dfrac{\partial^2 \log p_x(\xi)}{\partial x^2}\right]$ $I(x) = -\dfrac{\partial^2 \Phi(x)}{\partial x^2} = \dfrac{\partial^2 \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi}{\partial x^2}$	$I(\beta) = -E_\xi\left[\dfrac{\partial^2 \log p_\beta(\xi)}{\partial \beta^2}\right]$ $I(\beta) = -\dfrac{\partial^2 \Phi(\beta)}{\partial \beta^2} = \dfrac{\partial^2 \log \int_{M} e^{-\beta.U(\xi)} d\omega}{\partial \beta^2}$ $I(\beta) = -\dfrac{\partial^2 \Phi(\beta)}{\partial \beta^2} = -\dfrac{\partial Q}{\partial \beta} = K \text{ Souriau Capacity}$

REFERENCES

1. Koszul J.L., Variétés localement plates et convexité, Osaka J. Math. , n°2, p.285-290, 1965
2. Koszul J.L., Domaines bornées homogènes et orbites de groupes de transformations affines, Bull. Soc. Math. France 89, pp. 515-533., 1961
3. Koszul J.L., Ouverts convexes homogènes des espaces affines, Math. Z. 79, pp. 254-259., 1962
4. Koszul J.L., Déformations des variétés localement plates, .Ann Inst Fourier, 18 , 103-114., 1968
5. Souriau J.M., Définition covariante des équilibres thermodynamiques, Suppl. Nuov. Cimento, n°1, pp. 203–216, 1966
6. Souriau J.M., Structure of dynamical systems, volume 149 of Progress in Mathematics. Birkhäuser Boston Inc., Boston, MA, 1997. A symplectic view of physics
7. Souriau J.M., Thermodynamique et géométrie. In Differential geometrical methods in mathematical physics, II, vol. 676 of Lecture Notes in Math., pages 369-397. Springer, Berlin, 1978
8. Souriau J.M., Géométrie Symplectique et physique mathématique, CNRS 75/P.785, Dec. 1975
9. Souriau J.M., Mécanique classique et géométrie symplectique, CNRS CPT-84/PE-1695, Nov. 1984
10. Barbaresco F., Eidetic Reduction of Information Geometry through Legendre Duality of Koszul Characteristic Function and Entropy, Geometric Theory of Information, Springer, 2014
11. Barbaresco F., Koszul Information Geometry and Souriau Geometric Temperature/Capacity of Lie Group Thermodynamics, Special Issue "Information, Entropy and their Geometric Structures", MDPI Entropy, n°16, pp. 4521-4565, August 2014.
http://www.mdpi.com/1099-4300/16/8/4521/pdf