A Special Form of SPD Covariance Matrix for Interpretation and Visualization of data manipulated with Riemannian Geometry

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Summary

Currently the Riemannian geometry of symmetric positive definite (SPD) matrices is gaining momentum as a powerful tool in a wide range of engineering applications such as image, radar and biomedical data signal processing.

If the data is not natively represented in the form of SPD matrices, often we may summarize them in such form, typically estimating covariance matrices of the data. However, once we manipulate the covariance matrices on the Riemannian manifold we lose the representation in the original data space. *For instance, we can evaluate the geometric mean of a set of covariance matrices, but not the geometric mean of the data generating the covariance matrices*, the space of interest in which the geometric mean can be interpreted. As a consequence, Riemannian information geometry is often perceived by non-experts as a "black-box" tool and this perception prevents a wider adoption in the scientific community.

Hereby we show that this limitation can be overcome by constructing a special form of SPD matrix embedding both the covariance structure of the data and the data itself. Incidentally, whenever the original data can be represented in the form of a generic data matrix (not even square), this special SPD matrix describes exhaustively and uniquely the data up to second-order statistics.

This is achieved embedding the covariance structure of both the rows and columns of the data matrix, allowing naturally a wide range of possible applications and bringing us over and above just an interpretability issue.

Method

Given realization of data in the form of an RxC matrix

$$X_i \in \Re^{\mathrm{RxC}}$$

consider first the following data expansion:

$$Y_i = \begin{pmatrix} X_i & \alpha I_{(\mathrm{R})} \\ \alpha I_{(\mathrm{C})} & X_i^T \end{pmatrix} \in \Re^{\mathrm{TxT}}, \alpha > 0$$

where T=R+C and I is the identity matrix. Now, consider its associated Gram matrix (covariance structure) scaled by $1/2\alpha$ such as

$$C_i = \tfrac{1}{2\alpha} Y_i Y_i^T = \begin{pmatrix} \tfrac{1}{2\alpha}\left(X_i X_i^T + \alpha^2 I_{(\mathrm{R})}\right) & X_i \\ X_i^T & \tfrac{1}{2\alpha}\left(X_i^T X_i + \alpha^2 I_{(C)}\right) \end{pmatrix} \in \Re^{\mathrm{TxT}}, \alpha > 0.$$

Now, any manipulation of C_i on the Riemannian manifold will enforce a unique corresponding manipulation on its block X_i, allowing the interpretation of our manipulations in the original space of the data matrix X_i, whatever it represents.

Example:

moving images along geodesics in the SPD manifold

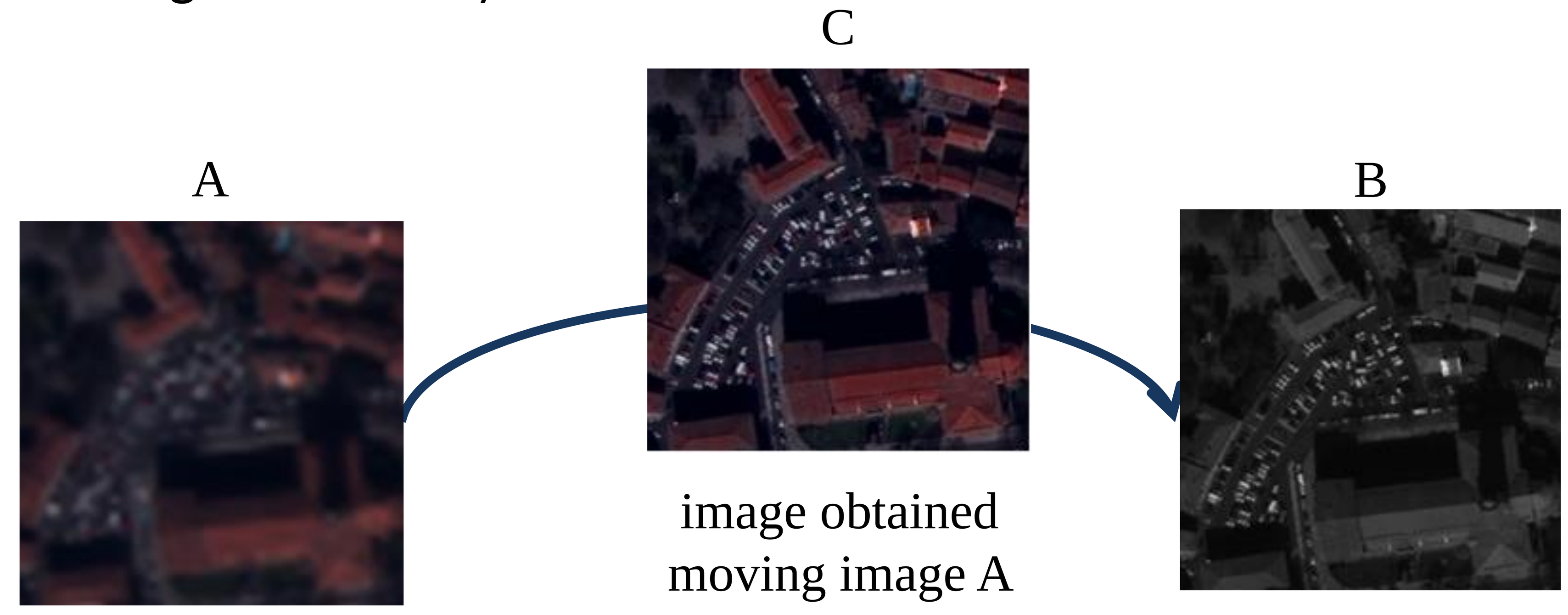
An image can be represented by a generic (not SPD) matrix, however once we represent images X_i by matrices C_i we can manipulate them on the SPD manifold.

In this example the problem is: given image A (a low-resolution color image) and B (an high-resolution image), get image C (a color image with high resolution).

C

A

B



color image at low-resolution

image obtained moving image A toward image B along the geodesic connecting them in the SPD manifold of matrices C_i.

B&W image at high-resolution