Reparameterization invariant distance on the space of curves in the hyperbolic plane

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Abstract

We focus on the study of time-varying paths in the two-dimensional hyperbolic space, and our aim is to define a reparameterization invariant distance on the space of such paths. We adapt the geodesical distance on the space of parameterized plane curves given by Bauer et al. in [1] to the space $\mathrm{Imm}([0,1],\mathbb{H})$ of parameterized curves in the hyperbolic plane. We present a definition which enables to evaluate the difference between two curves, and show that it satisfies the three properties of a metric. Unlike the distance of Bauer et al., the distance obtained takes into account the positions of the curves, and not only their shapes and parameterizations, by including the distance between their origins.

Introduction

Todays surface radars face a new challenge: detecting small, low-altitude targets, melted in strongly inhomogeneous environments, which lead to the non-stationarity of the radar signal. The ultimate aim of the approach presented in this article is to define a CFAR (Constant False Alarm Rate) detector optimized for the hypothesis of a non-stationary signal.

The detector yet to define is based on a statistical analysis of the radar signal, which we study as a time-varying path in a differential manifold. We assume that the signal is locally stationary, and we represent each stationary portion by an autoregressive process, the parameters of which -or equivalently, the covariance matrix of which- we are looking to estimate. Depending on the chosen representation -coefficients of the autoregressive model or covariant matrix- a stationary portion of the time-varying radar signal can be seen as a path in the corresponding manifold -the Poincaré disk or the space of Toeplitz matrices. We are then confronted with the study of oriented paths -or curves- in differential manifolds, and a key point is to be able to compute the distance between two curves.

Here, we place ourselves in the two-dimensional hyperbolic space, of which the Poincaré disk is a possible representation, and study the curves which lie in that space.

Objective

Our aim is to find a **satisfying definition of distance between two curves in the hyperbolic plane**. We want our distance to be invariant under reparameterization, that is we want to induce a distance on the shape space.

Distance on the space of parameterized curves in $\mathbb{H}$

Bauer et al. suggested such a metric on the space of plane curves in [1]. They first define a Riemannian metric G on the space $\mathcal{P} = \mathrm{Imm}(S^1,\mathbb{R}^2)$ of parameterized plane curves, which is reparameterization invariant and therefore induces a Riemannian metric on the shape space $\mathcal{S} = \mathrm{Imm}(S^1,\mathbb{R}^2)/\mathrm{Diff}(S^1)$. The geodesical distances in $\mathcal{P}$ and in $\mathcal{S}$ are then linked by the following property

$$dist(C_0, C_1) = \inf_{\phi\in \mathrm{Diff}(S^1)} dist(c_0, c_1\circ\phi),$$

where C_0 and C_1 are the natural projections of c_0 and c_1 from $\mathcal{P}$ onto $\mathcal{S}$. The simplest example of a reparamerization invariant metric on the space $\mathrm{Imm}(S^1,\mathbb{R}^2)$ is the L^2-metric

$$G_c(h,k) = \int_{S^1} \langle h, k\rangle ds,$$

where $c \in \mathrm{Imm}(S^1,\mathbb{R}^2)$ is a curve, $h, k \in T_c\mathrm{Imm}(S^1,\mathbb{R}^2)$ are infinitesimal deformations and we integrate over arc-length ds in order to have the reparamerization invariance. Unfortunately, the geodesic distance induced by this metric on the shape space vanishes, as was shown in [2]. That is why Bauer et. al look into the family of first-order Sobolev metrics

$$G_c^{a,b}(h,h) = \int_{S^1} a^2\langle D_s h, n\rangle^2 + b^2\langle D_s h, v\rangle^2 ds.$$

They show that it can be obtained as the pullback of the L^2-metric in the space $C^\infty(S^1,\mathbb{R}^3)$ of curves in space, by a certain transformation $R^{a,b}$. In the case of plane curves, the image of this transformation is the set of curves with values in a certain cone $\mathcal{C}^{a,b}$. Therefore the geodesic distance corresponding to the metric G is simply the pointwise distance between the image curves in that cone

$$dist(c,d) = \int_0^{2\pi} dist_{\mathcal{C}^{a,b}}(R(c)(\theta), R(d)(\theta))d\theta.$$

We try to adapt this distance to the space of curves in the hyperbolic plane.

A first definition

We consider the transformation $R^{a,b} : \mathrm{Imm}([0,1],\mathbb{H}) \to C^\infty([0,1], T\mathbb{H}\times\mathbb{R}_+)$:

$$R^{a,b}(c) = ||\dot c||^{1/2}\left(a\begin{pmatrix} v \\ 0\end{pmatrix} + \sqrt{4b^2-a^2}\begin{pmatrix} 0\\ 1\end{pmatrix}\right)$$

For a given curve $c \in \mathrm{Imm}([0,1],\mathbb{H})$, the image $R^{a,b}(c)(t)$ of each point $c(t)$ belongs to a cone placed above $c(t)$, and so we use parallel transport to bring every image vector in the same cone. For each $t\in[0,1]$, we first send the image vectors $R(c)(t)$ and $R(d)(t)$ of both curves on the image cones based at their respective origins $\mathcal{C}_{c(0)}$ et $\mathcal{C}_{d(0)}$, and then we parallel transport one of the two vectors from one cone onto the other, for example from $\mathcal{C}_{d(0)}$ onto $\mathcal{C}_{c(0)}$, along the geodesic γ that connects them, as illustrated in figure 1. Once both vectors are in the same cone, we compute the distance between them in the cone that contains them. We add the length of the geodesic γ in order to take into account the relative positions of the curves.

$$dist(c,d) = \sqrt{d_\gamma^2(c,d) + \ell^2(\gamma)}, \tag{1}$$

where

$$d_\gamma^2(c,d) = \int_0^1 dist^2_{\mathcal{C}_{c(0)}}\left(P_c^{t\to 0}(R(c)(t)), P_\gamma^{1\to 0}\circ P_d^{t\to 0}(R(d)(t))\right)\, dt.$$

Results

- We find that the parallel transport of a vector $u \in T_{c(t)}\mathbb{H}$ along a curve $c = (x,y)$ to its origin $c(0)$ is obtained by a rotation of angle $b(t)$ coupled with homothety of ratio $k(t)$, with

$$\begin{cases} b(t) = \int_0^t \frac{\dot x(\tau)}{y(\tau)}d\tau \quad \text{and} \quad k(t) = \frac{y(t)}{y(0)} & \text{(Poincaré half-plane)},\\[2ex] b(t) = \int_0^t \frac{x\dot y - y\dot x}{1-(x^2+y^2)}(\tau)\,d\tau \quad \text{and} \quad k(t) = \sqrt{\frac{1-r(t)^2}{1-r(0)^2}} & \text{(Poincaré disk)}.\end{cases}$$

- The distance on the cone $\mathcal{C}^{a,b} \in T\mathbb{H}\times\mathbb{R}_+$ is the same as the distance on the cone $\mathcal{C}^{a,b}$ of $\mathbb{R}^3$ given by Bauer et al.
- The function defined by (1) is invariant under a *single* reparameterization,

$$dist(c\circ\phi, d\circ\psi) = dist(c,d),$$

which assures the symmetry of the distance on the shape space.
- The function defined by (1) does not verify the triangular inequality, and is therefore not a distance function. This can be seen with the example shown in figure 2, of a triangle of geodesics and portions d and e of geodesics obtained by parallel transporting another portion c of geodesic along γ_1 and then γ_2 respectively. Then we have $d_{\gamma_1}(c,d) = d_{\gamma_2}(d,e) = 0$ but $d_{\gamma_3}(c,e) > 0$, and if we make c long enough, then $d_{\gamma_3}(c,e)$ will outgrow the difference of length between $\gamma_1\hat{\gamma}_2$ and γ_3.

A better definition of the distance

In order to overcome the problem highlighted by the example of figure 2, we modify the previous definition in the following way :

$$dist(c,d) = \inf_{\substack{\gamma \text{ path of } \mathbb{H}\\ \gamma(0)=c(0),\,\gamma(1)=d(0)}} \sqrt{d_\gamma^2(c,d) + \ell^2(\gamma)}. \tag{2}$$

Results

- The function defined by (2) is a distance function (equality of indiscernibles, symmetry, triangular inequality). The infimum over the paths connecting the two origins of the curve make the example of figure 2 no longer a problem.
- The interesting properties of the precedent definition still hold.

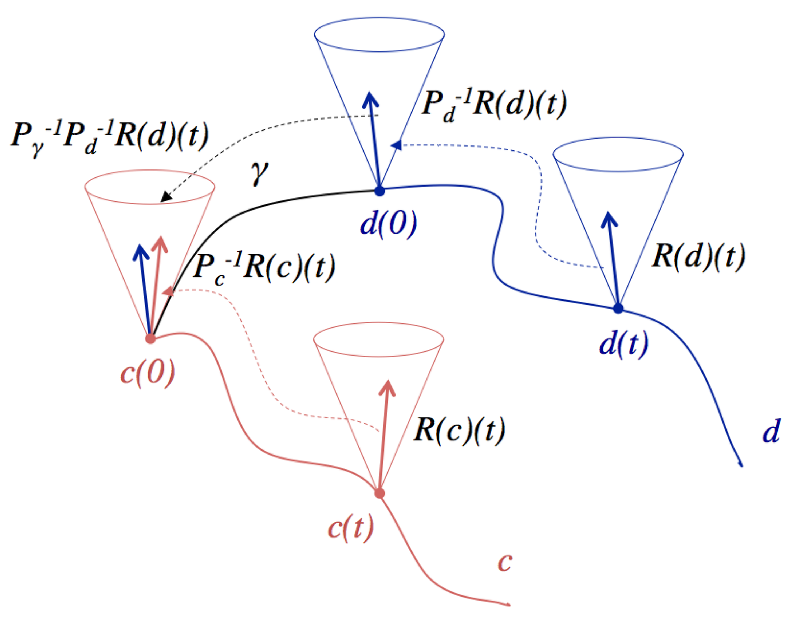

Figure 1: Illustration of our distance

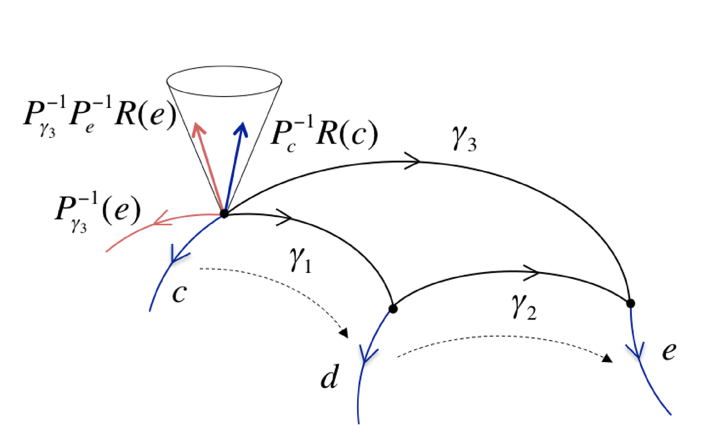

Figure 2: Examples of curves for which distance (1) does not verify the triangular inequality

Conclusions

- We were able to define a first function which measures the disparity between two curves of the two-dimensional hyperbolic space. It takes into account not only the shape and parameterization, but also the difference of position. Two curves that differ only by translation will not be considered the same.
- As this function failed to verify the triangular inequality, we modified its definition to obtain a second function which proved to be a distance function.
- Since it is invariant by a single reparameterization, it will induce a distance function on the shape space.
- The key point now resides in whether we can equip the space of parameterized curves $\mathrm{Imm}([0,2\pi],\mathbb{H})$ with a Riemannian structure, i.e. whether the distance presented in this article corresponds to the geodesical distance of a certain scalar product.

Acknowledgements

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References

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