INTRODUCTION

Information geometry is approached here by considering the statistical model of multivariate normal distributions $\mathcal{M}$ as a Riemannian manifold with the natural metric provided by the Fisher information matrix. This differential geometric approach to probability theory, introduced by C. Rao in 1945, has been recently applied to different domains such as statistical inference, mathematical programming, image processing and radar signal processing [4]. Explicit forms for the Fisher-Rao distance associated to this metric and for the geodesics of general distribution models are usually very hard to determine. In the case of general multivariate normal distributions lower and upper bounds have been derived. We approach here some of these bounds and introduce a new one discussing their tightness in specific cases.

PRELIMINARES

For multivariate normal distributions,

$$p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{n/2} |\Sigma|^2} 
\exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right),$$

where $x = (x_1, \ldots, x_n)$, $\mu = (\mu_1, \ldots, \mu_n)$ is the mean vector and $\Sigma$ is the covariance matrix, we consider the statistical model $\mathcal{M} = (p_\theta, \theta = (\mu, \Sigma) \in \Theta = \mathbb{R}^n \times P_+(\mathbb{R}^n))$, where $P_+(\mathbb{R}^n)$ is the space of positive-definite order $n$ symmetric matrices, is an $n(n+1)/2$-dimensional manifold. A natural Riemannian structure [1] can be provided by the Fisher information matrix:

$$g_i(\theta) = E_p \left( \frac{\partial}{\partial \theta_i} \log p(x; \theta) \frac{\partial}{\partial \theta_j} \log p(x; \theta) \right),$$

where $E_p$ is the expected value with respect to the distribution $p_\theta$.

The Fisher-Rao distance between two distributions $p_\theta$ and $p_\phi$ is then given by the shortest length, $D(\theta, \phi) = \inf_{\gamma} \int_{\Sigma} \sqrt{\text{det}(\gamma)} \text{d}x$, where $\gamma(x, y) = x'g_{ij}y$ is the inner product defined on $\mathcal{M}$. The Fisher-Rao infinitesimal arc-length can be expressed as

$$ds^2 = d\mu^T \Sigma^{-1} d\mu + \frac{1}{2} \text{tr}(\Sigma^{-1} d\Sigma)^2. \quad (1)$$

An explicit expression for the distance between two general normal multivariate distributions is very hard obtain.

Closed forms:

- In the case $n = 1$, a closed form for the Fisher-Rao distance is known via an association with the classical model of the hyperbolic plane.
- The submanifold $\mathcal{M}_0$ where the covariance matrix is diagonal, $\Sigma = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$. The Fisher matrix [3]:

$$g_i(\theta) = \begin{pmatrix}
\frac{1}{\sigma_1^2} & 0 & \cdots & 0 \\
0 & \frac{1}{\sigma_2^2} & \cdots & 0 \\
0 & 0 & \cdots & \frac{1}{\sigma_n^2}
\end{pmatrix},$$

$$d\mu_0(1, 2) = \sum_{i=1}^n \log |B_i|^2,$$

$$B_i = \begin{pmatrix}
\frac{\sigma_i}{\sigma_1} & -\frac{\sigma_i}{\sigma_2} \\
\frac{\sigma_i}{\sigma_1} & \frac{\sigma_i}{\sigma_2}
\end{pmatrix} - \begin{pmatrix}
\frac{1}{\sigma_1^2} & 0 \\
0 & \frac{1}{\sigma_2^2}
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sigma_1} & 0 \\
0 & \frac{1}{\sigma_2}
\end{pmatrix}$$

- The submanifold $\mathcal{M}_0$ where $\Sigma$ is constant,

$$d\mu_0(3, 4) = \sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} \Sigma^{-1} \begin{pmatrix} \mu_1 - \mu_2 \\ \mu_2 - \mu_1 \end{pmatrix}.$$