

Bounds on thermal efficiency from inference

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We consider reversible work extraction from two finite reservoirs of perfect gases with given initial temperatures T_+ and T_- , when the final values of the temperatures are known but they can be assigned to specific reservoirs only probabilistically. Using inference, we characterize the reduced performance resulting from this uncertainty. The estimates for the efficiency reveal that uncertainty regarding the exact labels reduces the maximal efficiency below the Carnot value, its minimum value is the well known Curzon-Ahlborn value: $1 - \sqrt{T_-/T_+}$. We also estimate the efficiency when even the value of temperature is not specified, by finding a suitable prior distribution for this problem. For the case of maximal uncertainty in the labels, we find the average estimate for efficiency drops to one-third value of Carnot limit. Using the concavity property of efficiency, we find the upper bound for the average estimate to agree with the CA-value upto two lowest order terms in the expansion near equilibrium.

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Prediction is concerned with deducing the consequences of a given set of causes. Inference, on the other hand, deals with guessing the plausible causes, given the consequences [1–3]. The questions of interest in the latter category, are also called inverse problems [4]. For example, the usual question in probability theory is: given a certain proportion of black and white balls in a closed bag, what is the probability that, in the next run, a white ball will be drawn? A related problem of inference could be that the results of some previous (finite) number of draws are given and one is required to guess the proportion of white/black balls in the bag [5, 6].

In a recent paper by the authors [7], it was shown how limited information in the form of label uncertainty impacts our expectations about the performance of heat cycles. Label uncertainty implies that there is a limited knowledge about the exact subsystem to which a certain control parameter is assigned. For example, consider a system specified by a set of quantities $\{X, Y, \dots\}$. In a multi-partite setup, we also label the subsystems with an index i . Then the properties of all constituent subsystems are identified if our labels are defined as $\{X_i, Y_i, \dots\}$. We consider a situation where the values of the individual parameters (X, Y, \dots) are known, but we are uncertain about the exact subsystem labels. The question is how to estimate the performance of the system based on this incomplete information. For simplicity, we discuss only a bipartite set up.

In the context of thermodynamics which concerns us here, we deal with constrained processes. The process follows the laws of thermodynamics under specified constraints. For example, in the process of work extraction from two finite-sized reservoirs whose initial temperatures are given to be T_+ and T_- , maximum work is extracted if the process is reversible. This means the total entropy of the reservoirs has to be conserved during the process. The work extraction is said to be optimal if the reservoirs finally obtain a common temperature, which is the minimum temperature possible in this set-up. Usually, it is assumed that the process is completely specified and we know the values of thermodynamic coordinates of each reservoir. Thus considering a quasi-static flow of heat, the temperature of the initially hot reservoir successively goes down and the other reservoir heats up. But as we show in Fig.1, if there is a limited information on the process, then there may be other feasible processes consistent with the constraints. In Ref. [7], we highlighted that such limited information leads to an expected behavior of the machine which is analogous to a physically irreversible model.

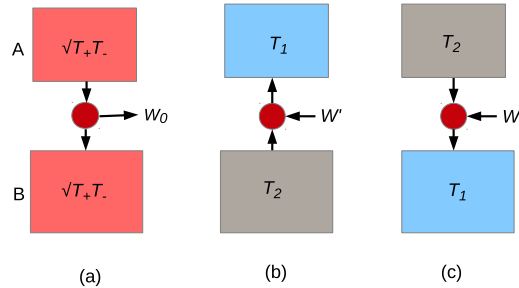


FIG. 1. A net amount of work W may be extracted in a reversible process by first extracting the optimal work W_0 and bringing the two systems to minimum common temperature $\sqrt{T_+ T_-}$ (a); then a part W' of this work is used back to drive heat reversibly either from system B to A (b) or from A to B (c), resulting in the same pair (T_1, T_2) of final temperatures, but with an opposite order. The net work extracted in each case is: $W_0 - W' = W$.

I. THE MODEL

A. Complete information

We first describe the thermodynamic model considered in Ref. [7]. Take the textbook example [8] of two *finite* ideal gas systems with a constant heat capacity C , at initial temperatures T_+ and T_- ($T_+ > T_-$), serving as the heat source and the sink, respectively. They are coupled via a reversible work source. At some stage, the initially hot reservoir has acquired a temperature T_1 and the initially cold reservoir a temperature T_2 . The amount of heat absorbed by the engine and the heat rejected to the sink, are given by $Q_{\text{in}} = C(T_+ - T_1)$ and $Q_{\text{out}} = C(T_2 - T_-)$, respectively. The total entropy change in the two reservoirs being zero, $\Delta S = C \ln(T_1/T_+) + C \ln(T_2/T_-) = 0$. This gives the following relation between the final reservoir temperatures

$$T_1 = \frac{T_+ T_-}{T_2}. \quad (1)$$

The work performed is $W = Q_{\text{in}} - Q_{\text{out}} = C(T_+ + T_- - T_1 - T_2)$. In the following, we set $C = 1$. Using Eq. (1), the efficiency $\eta = 1 - Q_{\text{out}}/Q_{\text{in}}$, can be written as: $\eta = 1 - T_2/T_+ = 1 - T_-/T_1$. We note that the work is optimal if the final temperatures obtained are: $T_1 = T_2 = \sqrt{T_+ T_-}$, and the efficiency for this process is $\eta = 1 - \sqrt{T_-/T_+}$, well-known as Ahlborn-Chambadal-Curzon-Novikov efficiency or briefly as CA-efficiency.

Now in the usual analysis, T_+ and T_- are the fixed initial values of the temperatures and due to relation (1), we may regard all the expressions as functions of only one of the two temperatures, T_1 or T_2 . Thus the work performed can be rewritten as

$$W(T_2) = \left(T_+ + T_- - T_2 - \frac{T_+ T_-}{T_2} \right), \quad (2)$$

with a similar expression in terms of T_1 .

It is important to note that just from the work expression, Eq. (2), it is not apparent as to which temperature is chosen as the variable. We have to look at the expression for the heat exchanged to assess the label of a specific reservoir. So an exact knowledge about the temperatures has two pieces of information: i) the individual values of the temperatures and ii) the labels for the reservoirs to which a value is assigned. So T_2 denotes the temperature value of the particular reservoir (label 2).

B. Limited information: Label uncertainty

Suppose a controller of the process who knows the final thermodynamic coordinates or temperatures of the reservoirs, invites us to play a game of guessing and promises to reveal the values (T_1, T_2) of the temperatures related by Eq. (1), but *not* the reservoir to which a specific value belongs. In our terminology, we are facing a situation where there is label uncertainty.

Now given an incomplete information, the task ahead of us is to perform inference and obtain the estimates on the work performed, the efficiency of the process and so on. As mentioned above, the work expression is symmetric over the labels for the individual reservoirs. Thus given one of the final temperature values, say T , the work expression will be $W(T) = T_+ + T_- - T - T_+ T_- / T$. Then the range of possible values for the final temperature T has to be fixed from the available prior information. In particular, we invoke the fact that the extracted work satisfies: $W \geq 0$, so that T is allowed to take values in the range $[T_-, T_+]$.

To illustrate how our estimates are affected as the degree of our belief changes, suppose further that we are given probability with numerical value γ ($0 \leq \gamma \leq 1$), for the hypothesis that one of the disclosed values, say T , belongs to the initially hot reservoir (henceforth labeled A). The parameter γ has been assumed independent of the T value.

We also know that *if* one of the final temperatures is T , the corresponding value for the other reservoir definitely is $T_+ T_- / T$. The final temperature of reservoir A therefore is: (i) T with probability γ and (ii) $T_+ T_- / T$ with probability $(1 - \gamma)$. Then upon knowing the value

T , the expected final temperature of reservoir A may be defined as:

$$\bar{T}_A = \gamma T + (1 - \gamma) \frac{T_+ T_-}{T}. \quad (3)$$

This is our estimate for the final temperature of reservoir A. Correspondingly, for the other reservoir B we have:

$$\bar{T}_B = (1 - \gamma) T + \gamma \frac{T_+ T_-}{T}. \quad (4)$$

Now we use these values to estimate further other quantities relevant to the performance of the engine. Our estimate of the heat absorbed by the engine from reservoir A is $Q_{\text{in}} = T_+ - \bar{T}_A$. Similarly, the estimate for the heat rejected to reservoir B will be: $Q_{\text{out}} = \bar{T}_B - T_-$. The estimate for work, $W = Q_{\text{in}} - Q_{\text{out}}$, turns out to be $W = T_+ + T_- - T - T_+ T_- / T$, which is equal to the actual work performed. In particular, the estimate for work is independent of the parameter γ , showing that the work is not affected by label uncertainty.

Now we normalise all the temperatures, relative to the initial temperature of reservoir A, and define $\theta = T_- / T_+$. Then the expected work for a given value of $\tau = T / T_+$ is

$$W(\tau) = 1 + \theta - \tau - \frac{\theta}{\tau}. \quad (5)$$

Also, the estimate for the efficiency $\eta = W / Q_{\text{in}}$ is given by

$$\eta_\gamma(\tau) = \frac{\tau + \theta\tau - \tau^2 - \theta}{\tau - \gamma\tau^2 - (1 - \gamma)\theta}, \quad (6)$$

which is modified due to label uncertainty.

It was shown in [7] that the inferred efficiency has an upper bound which in case of label uncertainty, is less than the Carnot efficiency. This can be interpreted to imply that limited information makes us expect a reduced or irreversible performance for an otherwise reversible process. The maximum value of efficiency can be shown to be

$$\eta_\gamma^* = \frac{1 - \theta}{1 + \sqrt{4\gamma(1 - \gamma)\theta}}. \quad (7)$$

It is interesting to note that the minimum value of this bound is the CA-efficiency $1 - \sqrt{\theta}$. Moreover, this minimum value is obtained for $\gamma = 1/2$. This corresponds to the case of maximal label uncertainty i.e. when we are completely uncertain about the label of a given temperature. According to Laplace's principle of insufficient reason [11], if we do not have any prior information about preference for a specific label of a given temperature, then we must assign $\gamma = 1/2$. In the following, we shall perform analysis for this case only.

II. AVERAGE ESTIMATE OF EFFICIENCY AND BOUNDS

Finally, we consider the situation in which apart from label uncertainty, even the value of temperature is also uncertain and it is only known to lie in the permissible range $[T_-, T_+]$, or in terms of normalised temperatures, $[\theta, 1]$. So to quantify our degrees of belief in the likely values of temperature τ , we assign a prior probability distribution $p(\tau)$ to it. In [7], we used a uniform prior, which reflects maximum ignorance about the probable value. One can also argue in favour of Jeffreys' prior for this problem, given as $p(\tau) \propto 1/\tau$.

The crux of the argument leading to Jeffreys' prior is this. Since, we assume maximum label uncertainty, our degrees of belief in the likely values of one temperature, is equivalent to that for the other temperature. So we assume the same form of prior distribution function for both, i.e. p . However, there is a definite one-to-one relation between the two temperatures, Eq. (1). Using this fact alongwith $p(T_1)dT_1 = p(T_2)dT_2$, we arrive at the Jeffreys' form [12–15]. In terms of the normalised temperature, the prior is given by $p(\tau) = [\tau \ln(1/\theta)]^{-1}$. The average value of τ is $\bar{\tau} = (1 - \theta)/\ln(1/\theta)$.

The estimated value of the efficiency is obtained by averaging over the prior distribution as $\bar{\eta} = \int \eta(\tau) p(\tau) d\tau$, where $\eta(\tau)$ is from Eq. (6) with $\gamma = 1/2$. Explicitly, we obtain

$$\bar{\eta} = 2 + \frac{\sqrt{1-\theta}}{\ln \theta} \ln \left[\frac{1 + \sqrt{1-\theta}}{1 - \sqrt{1-\theta}} \right]. \quad (8)$$

For close to equilibrium, $\theta \approx 1$, the average estimate of efficiency behaves as $(1 - \theta)/3$. This gives the lower bound for efficiency in the present scenario with maximum label uncertainty. Further, the upper bound (ub) for average efficiency can be found by invoking the fact that the efficiency as given in Eq. (6) is a concave function of τ . So one can use Jensen's inequality to bound the average value of efficiency from above. Applying the inequality, we have

$$\bar{\eta} \leq \eta(\bar{\tau}) = \bar{\eta}_{\text{ub}}. \quad (9)$$

The maximum value of the efficiency has been earlier shown to be equal to the CA-efficiency. The upper bound for the average value is clearly less than this value, but closely follows it, as shown in Fig. 2.

One can expand the upper bound in terms of $(1 - \theta)$, as follows:

$$\bar{\eta}_{\text{ub}} = \frac{1}{2}(1 - \theta) + \frac{1}{8}(1 - \theta)^2 + \frac{17}{288}(1 - \theta)^3 + O[1 - \theta]^4. \quad (10)$$

Note that the first two terms in the expansion match with the expansion of CA-efficiency, $1 - \sqrt{\theta}$.

Concluding, we have considered reversible heat engines with limited information as uncertainty in the labels of the temperatures, as well as in the values of the temperatures. It

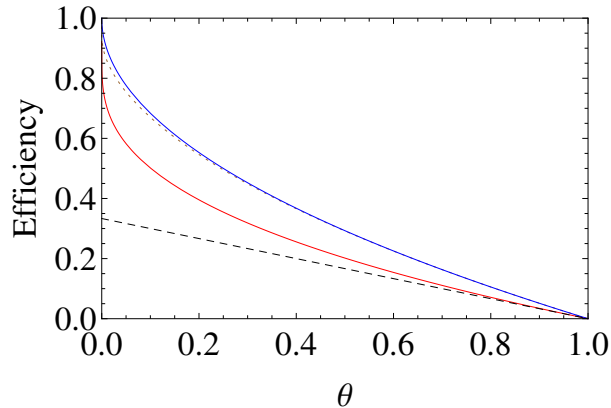


FIG. 2. Comparison of different efficiencies versus the ratio of initial reservoir temperatures, θ . The lower, dashed straight line is $(1 - \theta)/3$. Next to it, the average estimate of efficiency, Eq. (8), is bounded from below by this value. CA efficiency which is the maximum value of efficiency, is the topmost curve. Closely following it, the dotted line is the upper bound for the average estimate, derived from Jensen's inequality.

has been earlier shown [7] that with maximum label uncertainty, the efficiency with given values of temperatures, has the maximum value equal to the CA expression. In case of unknown temperature values, one can find an estimate of efficiency by averaging over the prior distribution which is argued to be Jeffreys' prior. We find that the average estimate is bounded from below by one-third of Carnot limit. By invoking Jensen's inequality we found that the upper bound for the average estimate behaves like CA-efficiency upto two lowest order terms near equilibrium.

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