Fisher Information Geometry of The Barycenter Map

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Aim of this talk: Fisher information geometry of barycenter map:

\[
\text{bar} : \mathcal{P}^+ (\partial X) \to X \quad \text{and}
\]

isometry problem related to a barycentrically associated map \( \varphi \) of \( X \);

\[
\begin{array}{cccc}
\mathcal{P}^+ (\partial X) & \xrightarrow{\Phi^\#} & \mathcal{P}^+ (\partial X) \\
\downarrow \text{bar} & & \downarrow \text{bar} \\
X & \xrightarrow{\varphi} & X
\end{array}
\]
Content:

1. Barycenter and barycenter map
2. Barycenter map and Fisher information metric
3. Barycentrically associated maps and isometries of \((X, g)\)
4. Poisson kernel
5. Quasi-isometries and quasi-geodesics
§1 Barycenter and barycenter map

Let \((X, g)\) be an Hadamard manifold, i.e., a simply connected, complete Riemannian manifold of curvature \(K \leq 0\), and \(\partial X\) the ideal boundary of \((X, g)\) by taking quotient

\[\partial X = \mathcal{R}_X / \sim,\]

where \(\mathcal{R}_X\) : the set of all geodesic rays on \(X\),
$\gamma, \sigma \in \mathcal{R}_X$ are asymp.equiv. ($\gamma \sim \sigma$) if

$$d(\gamma(t), \sigma(t)) < D \text{ for } t \geq 0.$$ 

For $\forall x \in X$ and $\theta \in \partial X \ \exists \gamma \in \mathcal{R}_X$ s.t.

$$\gamma(0) = x, \ [\gamma] = \theta$$

For $o \in X$, an arbitrary fixed point

$$S_oX \equiv \partial X; \ \ U \mapsto [\gamma], \ \gamma(t) = \exp_x tU,$$

$$S_oX := \{U \in T_oX \mid \|U\| = 1\}$$
\( \overline{X} := X \cup \partial X \) admits a cone topology, so \( \overline{X} \) is homeo to a unit closed ball.

\( d\theta \): the can.volume measure on \( \partial X \) via \( \partial X \equiv S_0X \).

- Let \( \mathcal{P}^+ (\partial X) = \mathcal{P}^+ (\partial X, d\theta) \) be the space of probability measures on \( \partial X \);

\[ \mathcal{P}^+ (\partial X) = \{ \mu = f(\theta)d\theta \ll d\theta ; \; f \in C^0(\partial X) > 0 \} \]

abs.conti.w.r.t.\( d\theta \) and having conti.positive density function. So any \( \mu \in \mathcal{P}^+ (\partial X) \) is
written as

\[ \mu(\theta) = f(\theta) d\theta, \theta \in \partial X, f(\theta) > 0. \]

**Def. 1.1. Normalized Busemann function**

\[ B_\theta(x) = \lim_{t \to \infty} \{ d(x, \gamma(t)) - t \}, \quad x \in X \]

is defined on \( X \) assoc.to \( \theta \in \partial X \);
where \( \gamma = \gamma(t) : \) the geodesic, \( \gamma(0) = o, [\gamma] = \theta. \)

**Note 1.1.** \( B_\theta(o) = 0. \quad B_\theta(\gamma(t)) = -t, \forall t. \)
\(|(\nabla B_\theta)_x| = 1, \forall x. \quad B_\theta \) is \( C^2 \)-convex \Rightarrow
Hessian $\nabla dB_\theta \geq 0$, $\forall \theta \in \partial X$.

**Ex. 1.1.** $(X, g) = \mathbb{R}H^n$, $n \geq 2$; the real hyperbolic space of curv.$\equiv 1$. From Poincaré unit ball model $\partial X \cong S^{n-1}(1)$. $o$: origin

$$B_\theta(x) = \log \frac{|x - \theta|^2}{1 - |x|^2}, \quad B_\theta(o) = 0,$$

$$\nabla dB_\theta(U, V) = \langle U, V \rangle - \langle U, \nabla B_\theta \rangle \langle V, \nabla B_\theta \rangle.$$

- Barycenter: Following the idea of
[DouadyE’86] and [Besson et al.’95],[Besson et al.’96] we have

**DEF.1.2.** Let $\mu \in \mathcal{P}^+(\partial X)$. A point $y \in X$ is called a **barycenter** of $\mu$,

if the $\mu$-average Busemann function

$B_\mu : X \to \mathbb{R}$;

$$B_\mu(x) = \int_{\theta \in \partial X} B_\theta(x) d\mu(\theta)$$  \hspace{1cm} (1)

is critical at $y$. 
• \( B_\mu(\cdot) \) is convex and \( B_\mu(o) = 0 \).

**Theorem 1.1 (Existence and Uniqueness)**

Let \((X, g)\) be an Hadamard manifold.

(i) Assume that \((X, g)\) satisfies the axiom of **visibility** and Busemann function \( B_\theta(x) \) is conti.w.r.t. \( \theta \in \partial X \).

Then any \( \mu \in \mathcal{P}^+(\partial X) \) admits a barycenter.

(ii) Assume some \( \mu_o \in \mathcal{P}^+(\partial X) \) the \( \mu \)-**average** 

Hessian \( \nabla dB_{\mu_o} > 0 \).

Then, for any \( \mu \in \mathcal{P}^+(\partial X) \) the existence of
barycenter is unique. Here

\[
(\nabla dB_\mu)_x(U, V) := \int_{\partial X} (\nabla dB_\theta)_x(U, V) d\mu(\theta)
\]

**DEF. 1.3 (EberleinO’Neil’73)**

An Hadamard manifold \((X, g)\) satisfies **axiom of visibility**, if, for any \(\theta, \theta_1 \in \partial X, \theta \neq \theta_1\), there exists a geodesic \(\gamma : (-\infty, +\infty)\) such that \([\gamma] = \theta, [\gamma^-] = \theta_1\). The axiom of visibility is
equiv. to

$$B_\theta(x) = +\infty, \text{ when } x \to \theta_1 \neq \theta$$

([Ballmann et al.’91])

**Remark.** Theorem 1.1 is a generalization of [Besson et al.’95, Appendice A].

We have thus a map, called **barycenter map**

$$\text{bar} : \mathcal{P}^+(\partial X, d\theta) \to X; \mu \mapsto y,$$

when $y$ is a barycenter of $\mu$. 
Remark. [Besson et al.’95] use barycenter to assert the Mostow rigidity of hyperbolic manifolds.

§2 Barycenter Map and Fisher Inf. Metric

The barycenter map induces a fibre space projection

\[ \mathcal{P}^+ (\partial X) \downarrow \text{bar} \]

\[ X \]
provided \((X, g)\) carries the Busemann-Poisson kernel \(P(x, \theta) = \exp\{-QB_\theta(x)\}\) \((Q > 0 : \text{volume entropy of } (X, g))\)

The fibre over \(x \in X\):

\[\overline{\mu}^{-1}(x) = \{\mu \in \mathcal{P}^+(\partial X), \overline{\mu}(\mu) = x\} .\]

The tangent space \(T_\mu \overline{\mu}^{-1}(x)\) is characterized;
\( \{ \tau \in T_\mu \mathcal{P}^+(\partial X), \int_{\partial X} (dB_\theta)_x(U) d\tau(\theta) = 0, \forall U \in T_x \mathcal{X} \} \)

and also as

\( \{ \tau \in T_\mu \mathcal{P}^+(\partial X), G_\mu(\tau, N^\mu_x(U)) = 0, \forall U \in T_x \mathcal{X} \} \).

Here \( G \) is the **Fisher inf. metric** on \( \mathcal{P}^+(\partial X) \);

\[
G_\mu(\tau, \tau_1) := \int_{\theta \in \partial X} \frac{d\tau}{d\mu}(\theta) \frac{d\tau_1}{d\mu}(\theta) \ d\mu(\theta),
\]

\( \tau, \tau_1 \in T_\mu \mathcal{P}^+(\partial X) \).
\[
\frac{d\tau}{d\mu}(\theta) = \frac{h(\theta)}{f(\theta)}
\] is the Radon-Nikodym derivative of \(\tau = h(\theta)d\theta\) w.r.t. \(\mu = f(\theta)d\theta\) and

\[
N^\mu_x : T_x X \rightarrow T_\mu \mathcal{P}^+(\partial X)
\]

\[
U \mapsto (dB_\theta)_x(U)d\mu(\theta)
\]

is an assoc.linear map.

**Prop. 2.1** The tangent space \(T_\mu \mathcal{P}^+(\partial X)\) admits an orthogonal direct sum w.r.t. \(G\);

\[
T_\mu \mathcal{P}^+(\partial X) = T_\mu \text{bar}^{-1}(x) \oplus \text{Im} N^\mu_x, \ x = \text{bar}(\mu),
\]
\[(\dim \text{Im} N_x^\mu = \dim X)\]
\[T_\mu \bar{a}r^{-1}(x): \text{vert. subsp.} \quad \text{Im} N_x^\mu: \]
\[\text{hor. subsp. (contributing the normal bundle of the subspace } \bar{a}r^{-1}(x)\text{).}\]

- Geometric properties of the Fisher inf.metric $G$
Prop. 2.2 (Friedrich’91). Levi-Civita connection is
\[ \nabla^G_{\tau_1} \tau = -\frac{1}{2} \left( \frac{d\tau_1}{d\mu}(\theta) \frac{d\tau}{d\mu}(\theta) - \int_{\partial X} \frac{d\tau_1}{d\mu} \frac{d\tau}{d\mu} d\mu \right) \mu \]
for constant vector fields \( \tau, \tau_1 \) on \( \mathcal{P}^+(\partial X) \).

Theorem 2.1 (Friedrich’91). \( (\mathcal{P}^+(\partial X), G) \) is a space form of constant curvature \( \frac{1}{4} \), but not geodesically complete.
Theorem 2.2 (I-Satoh’14-1). Let $\mu(t)$ be a geodesic in $t$, of $\mu(0) = \mu$ and $\dot{\mu}(0) = \tau$ a unit vector. Then $\mu(t)$ is written as

$$
\mu(t) = \left\{ \cos \frac{t}{2} + \sin \frac{t}{2} \frac{d\tau}{d\mu}(\theta) \right\}^2 d\mu(\theta).
$$

So, every geodesic is periodic (period $2\pi$). The length $\ell$ of a geodesic seg. joining $\mu$, $\mu_1$:

$$
\cos \frac{\ell}{2} \leq \int_{\partial X} \sqrt{\frac{d\mu_1}{d\mu}(\theta)} d\mu(\theta) =: D_f(\mu \| \mu_1)
$$
: the $f$-divergence, $f(u) = \sqrt{u}$ ([A-N’00]).

**Theorem 2.3** (I-Satoh’14-1). Let

$\mu \in \bar{bar}^{-1}(x)$ and $\tau \in T_\mu \bar{bar}^{-1}(x)$ unit tangent vector. Then the geodesic $\mu(t) = \exp_\mu t\tau$ belongs entirely to $\bar{bar}^{-1}(x)$ if and only if $H_\mu(\tau, \tau) = 0$, where $H_\mu$ is the second fundamental form of the submanifold $\bar{bar}^{-1}(x)$ at $\mu$.

**Theorem 2.4** (I-Satoh’14-2) For any $\mu$, $\mu_1 \in \mathcal{P}^+(\partial X)$, $\mu \neq \mu_1$ there exists a unique
geodesic $\mu(t)$ s.t. $\mu(0) = \mu$, $\mu(d) = \mu_1$, $d$ is defined by \( \cos \frac{d}{2} = \int \sqrt{\frac{d\mu_1}{d\mu}} d\mu(\theta) = D_f(\mu || \mu_1) \).

- For a homeo $\Phi : \partial X \rightarrow \partial X$ its push-forward $\Phi_\# : \mathcal{P}^+(\partial X) \rightarrow \mathcal{P}^+(\partial X)$ is defined by

\[
(\Phi_\# \mu)(A) := \mu(\Phi^{-1} A)
\]

for any Borel set $A$ of $\partial X$, or

\[
\int_{\theta \in \partial X} h(\theta) d(\Phi_\# \mu)(\theta) := \int_{\theta \in \partial X} h(\Phi(\theta)) d\mu(\theta)
\]
for any measurable function $h = h(\theta)$. See [Villani’03].

**Theorem 2.5** (Friedrich’91). Every push-forward $\Phi^\#$ is an isometry w.r.t. $G$;

$$G_{\Phi^\#\mu}(\Phi^\#\tau, \Phi^\#\tau_1) = G_\mu(\tau, \tau_1), \tau, \tau_1 \in T_\mu\mathcal{P}^+(\partial X).$$

§3 **Barycentrically assoc.maps and isometries of** $(X, g)$

**Prop.3.1** Let $\varphi$ be an isometry of $(X, g)$. Then

$$\text{bar}(\hat{\varphi}^\#\mu) = \varphi(\text{bar}(\mu)), \forall \mu$$
Busemann cocycle formula w.r.t. a Riemannian isometry $\varphi$ of $(X, g)$

$$B_\theta(\varphi x) = B_{\hat{\varphi}^{-1}\theta}(x) + B_\theta(\varphi o) \quad \forall (x, \theta) \in X \times \partial X$$

See [Givarchi et al.'97]. Here $\hat{\varphi}: \partial X \to \partial X$ is an extension of $\varphi$;

$$\hat{\varphi}(\theta) := [\varphi \circ \gamma], \quad \gamma(0) = o, \quad [\gamma] = \theta \quad \text{and then}$$

$$B_\mu(\varphi x) = B_{\hat{\varphi}^{-1}_\mu}(x) + B_\mu(\varphi o)$$

$$\forall (x, \mu) \in X \times \mathcal{P}^+(\partial X).$$
So one gets Prop. 3.1.

We consider the **following situation:**

Let $\Phi$ be a homeo of $\partial X$. The push-forward $\Phi_\#$ yields a bijective map $\varphi : X \to X$ satisfying

$$\bar{\varphi} \circ \Phi_\# = \varphi \circ \bar{\psi}$$
We call such a \( \varphi \) a map, **barycentrically associated** to \( \Phi \).

**Lemma 3.1.** The composition \( \varphi \circ \varphi_1 \) of maps \( \varphi, \varphi_1 \) barycentrically assoc.to \( \Phi, \Phi_1 \), resp. is also barycentrically assoc.to \( \Phi \circ \Phi_1 \). \( \varphi^{-1} \) is barycentrically assoc.to \( \Phi^{-1} \).
§4 Poisson kernel
Consider the Dirichlet problem at the $\partial X$:

$$\Delta u = 0 \quad \text{in } X, \quad u|_{\partial X} = f,$$

$$f = f(\theta) \in C(\partial X) : \text{ a given data}$$

Def. 4.1. A function $P(x, \theta)$ of $(x, \theta)$ $\in X \times \partial X$ is called Poisson kernel, when (i) it is the fundamental solution of the Dirichlet
problem at the $\partial X$ s.t. the $u$ is described as

$$u = u(x) = \int_{\partial X} P(x, \theta) f(\theta) d\theta$$

(ii) **(Positivity and normalization)** $P(x, \theta) > 0$ for any $(x, \theta)$ and $P(o, \theta) = 1$ for any $\theta$ (iii) $\lim_{x \to \theta_1} P(x, \theta) = 0$, $\forall \theta, \theta_1 \in \partial X, \theta_1 \neq \theta$. See [SchoenYau’94].

**Remark 4.1.** Damek-Ricci spaces (including rank one symmetric spaces of non-cpt type)
admit a Poisson kernel described specifically as

\[ P(x, \theta) = \exp\{-QB_\theta(x)\}. \]

in terms of the Busemann function and the volume entropy \( Q > 0 \), See [Besson et al.'95], [I-Satoh’10], [I-Satoh’11], [I-Satoh’14], [I-Satoh’14-1]. We call such a Poisson kernel as **Busemann-Poisson kernel**, a fusion of harmonic measure and Patterson-Sullivan measure.
Remark 4.2. An Hadamard manifold admitting Busemann-Poisson kernel must be asymptotically harmonic, that is, $\Delta B_\theta = -Q$, (see [Ledrappier’90]) so $B_\theta$ and $B_\mu$ turn out to be a smooth function on $X$ by elliptic regularity.

Example 4.1. $(X, g) = \mathbb{R}H^n$, $n \geq 2$.

$$P(x, \theta) = \left( \frac{1 - |x|^2}{|x - \theta|^2} \right)^{n-1}$$
**Lemma 4.1.** Let \((X, g)\) admit Busemann-Poisson kernel. Then, with the assumptions in Th.1.1. we have

(i) \(\mu_x := P(x, \theta) d\theta \in \mathcal{P}^+(\partial X)\) is a probability measure, parametrized in \(x\) for which \(\bar{\text{ar}}(\mu_x) = x\). See [Besson et al.'95].

(ii) For any \(\mu = \mu_x\), the \(\mu\)-average Hessian \(\nabla dB_\mu(\cdot, \cdot)\) is positive definite everywhere on \(X\)

\[
\nabla dB_{\mu_x}(\cdot, \cdot) = Q \ G_{\mu_x}(N_{x}^{\mu_x}(\cdot), N_{x}^{\mu_x}(\cdot)).
\]
• From Th.1.1 the uniqueness of barycenter is guaranteed.

• Let \( \Theta : X \to \mathcal{P}^+(\partial X) ; x \mapsto \mu_x \) be the canonical map, called **Poisson kernel map**.

  It holds \( \text{bar} \circ \Theta = \text{id}_X \).

So, \( \Theta : X \to \mathcal{P}^+(\partial X) \) enjoys a cross section of the fibration \( \mathcal{P}^+(\partial X) \to X \).

**Theorem 4.1 (I-Satoh’14, I-Satoh’14-1).** Let \( (X, g) \) admit Busemann-Poisson kernel. With
the assumptions in Th.1.1.

Let \( \varphi : X \to X \) be a barycentrically assoc. to a homeo \( \Phi : \partial X \to \partial X \) \( (\bar{\text{bar}} \circ \Phi^\# = \varphi \circ \bar{\text{bar}}) \). If \( \varphi \) is of \( C^1 \) and moreover satisfies

\[
\Phi^\# \circ \Theta = \Theta \circ \varphi;
\]

\[
P^+(\partial X) \xrightarrow{\Phi^\#} P^+(\partial X)
\]

(3)
then $\varphi$ is a Riemannian isometry of $(X, g)$ whose $\partial X$-extension $\hat{\varphi} = \Phi$.

**Theorem 4.2** (ItohSatoh’14-1) Let $(X, g)$ be with the assumptions in Th.1.1 and admit Busemann-Poisson kernel.

Let $\Phi$ be a homeo of $\partial X$ and $\varphi : X \to X$ be a $C^1$-bijective map with surjective $d\varphi_x$ at $\forall x \in X$.

Then $\Phi^\# \circ \Theta = \Theta \circ \varphi$ implies $\bar{\text{bar}} \circ \Phi^\# = \varphi \circ \bar{\text{bar}}$. 
§5 Quasi-isometries and quasi-geodesics

Theorem 5.1.
Let \((X_0, g_0)\) be a Damek-Ricci space. Let \((X, g)\) be an Hadamard manifold which is quasi-isometric to \((X_0, g_0)\). Then, an arbitrary isometry \(\psi\) of \((X_0, g_0)\) induces a homeo \(\Phi\) of the ideal boundary \(\partial X\) of \((X, g)\).

Refer to [Berndt et al.'91] for definition and geometric properties of Damek-Ricci spaces.
Definition 5.1 Let \((X_1, d_1), (X_2, d_2)\) be metric spaces. A map \(f : X_1 \to X_2\) is a \((\lambda, k)\)-quasi-isometric map (or, simply, quasi-isometric map), if \(\exists \lambda \geq 1, k \geq 0\) s.t.

\[
\frac{1}{\lambda} d_1(x, x') - k \leq d_2(fx, fx') \leq \lambda d_1(x, x') + k
\]

A quasi-isometric map is a generalization of an isometric, or homothetic map.

- A curve \(c : \mathbb{R} \to X\) is a quasi-geodesic, if \(c\) is a quasi-isometric map. A geodesic is
quasi-geodesic. A quasi-isometric map \( f : (X_o, g_o) \to (X, g) \) maps a geodesic \( \gamma : \mathbb{R} \to X_o \) into a quasi-geodesic \( f \circ \gamma : \mathbb{R} \to X \). Moreover, it holds that let \( \varphi : X \to X \) be a quasi-isometric and \( \gamma : \mathbb{R} \to X \) be a quasi-geodesic. Then the curve \( \varphi \circ \gamma : \mathbb{R} \to X \) is quasi-geodesic.

**Definition 5.2** Let \((X, d)\) be a metric space. A geod.triangle \( \Delta = [xyz] \) in \( X \) is called \( \delta\)-**thin**, 


for a $\delta \geq 0$, if for any point $p$ on the side $[xy]$

$$d(p, [xz] \cup [yz]) < \delta.$$ 

We call $(X, d)$ $\delta$-hyperbolic, or Gromov-hyperbolic, if for a $\delta \geq 0$ all geod. triangles are $\delta$-thin. [Bourdon’95], [Coornaert et al.’80].

**Example 5.1.** $(X, g) = RH^n, n \geq 2$ is $\delta$-hyperbolic with $\delta = \log 3$. See [Coornaert et al.’80].
Example 5.2. A Damek-Ricci space is $\delta$-hyperbolic with certain $\delta$. This is from [Knieper’12] and [Anker et al.’96].

References

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