The entropy-based quantum metric

- R.B., Entropy, 16 (7) 3878-3888 (2014)

- R.B., Y. Alhassid, H. Reinhardt, Physics Reports 131, 1-146 (1986), §3.

- R.B. Springer proceedings on geometry of information 035136-518 (2013)

I Quantum structures (finite systems)

1. "<u>Observables</u>" $\hat{O}$: Physical non-comuting random variables

C^* algebra , $\hat{O} = \hat{O}^\dagger$

(Hamiltonian matrices in Hilbert space) dim^n: <u>n</u>

2. "<u>States</u>" $\hat{D}$: encode information about $\hat{O}$

$\hat{O} \longmapsto \langle \hat{O} \rangle =$ q-expectation value

Linear correspondence, $\langle \hat{O} \rangle$ real, $\langle \hat{O}^2 \rangle \geqslant 0$

Scalar product $\boxed{\langle \hat{O} \rangle = (\hat{O}; \hat{D}) = \mathrm{Tr}\, \hat{O}\hat{D}}$

($\hat{D}$ = Hermitean matrix, $\hat{D} \geqslant 0$, $\mathrm{Tr}\,\hat{D} = 1$)

Dual vector spaces $\left[\begin{array}{ll} \hat{O} & \text{with algebra (and } \hat{O} = \hat{O}^\dagger) \\ \hat{D} & \text{with positivity (and } \hat{D} = \hat{D}^\dagger, \mathrm{Tr}\,\hat{D} = 1) \end{array} \right.$

Only scalar products are physically meaningful
(independent of representation)

? <u>Metric</u> in space of states $\hat{D}$? = <u>Density matrices</u>

3. von Neumann entropy $\boxed{S(\hat{D}) = -\mathrm{Tr}\,\hat{D}\ln\hat{D}}$ (1932)

- Thermodynamic entropy if $\hat{D}$ = thermo equilibrium state
- Information in quantum measurements

Shannon (1948) $S(p) = -\sum_i p_i \ln p_i$

Brillouin (1957) $S(\hat{D})$ = missing information when only $\hat{D}$ is known

Jaynes (1958) MaxEnt for assigning unbiased state

Justifications $\begin{cases} \text{Axiomatic (concave, subadditive, invariant)} \\ \text{Equiprobability for ensemble} \Rightarrow \text{MaxEnt} \end{cases}$

R.B. 8th MaxEnt + R.B. and N. Balazs, Ann. Phys. 179, 97 (1987)

- Only physically meaningful quantities : $(\underline{\hat{D};\hat{O}})$ and $\underline{S(\hat{D})}$

→ Rich geometric structure

⇒ Concavity $d^2S \leq 0$ suggest natural metric:

$$\boxed{ds^2 = -d^2S = \mathrm{Tr}\, d\hat{D}\, d\ln\hat{D}}$$ Distance between neighbouring states

- Metric tensor = Hessian of $S(\hat{D})$
- ds^2 = non-commutative extension of Fisher's $-d^2S(p) = \sum_i \frac{dp_i^2}{p_i}$

II Use for quantum information

- **q-bits** ($n=2$) Positive 2×2 matrices $\hat{D} = \frac{1}{2}(\hat{I} + \vec{r}\cdot\hat{\vec{\sigma}})$ (Pauli matrices

  parametrised by $\vec{r} = \langle \hat{\vec{\sigma}} \rangle$, vector in unit 3d-sphere

  $$ds^2 = \frac{1}{1-r^2}\left(\frac{\vec{r}\cdot d\vec{r}}{r}\right)^2 + \frac{1}{2r}\ln\frac{1+r}{1-r}\left\|\frac{\vec{r}\times d\vec{r}}{r}\right\|^2$$ singular near pure states

- **Nearly complete randomness** $\left\|\hat{D} - \frac{\hat{I}}{n}\right\| \ll \frac{1}{n}$

  $ds^2 \sim n\,\mathrm{Tr}\, d\hat{D}^2$ (as for Fisher metric) Euclidean

- **Loss of information by mixing** $\frac{1}{2}(\hat{D}' + \hat{D}'') = \hat{D}$ $\quad \delta\hat{D} \equiv \hat{D}' - \hat{D}''$

  $$\boxed{\Delta S = S(\hat{D}) - \frac{1}{2}\left[S(\hat{D}') + S(\hat{D}'')\right] \sim \frac{\delta s^2}{8}}$$

  $\delta s =$ distance

- **Pure state limit** $S(\hat{D}') \to 0$, $S(\hat{D}'') \to 0$ $\quad$ loss $\Delta S \to S(\hat{D})$

  $$\begin{cases} 8\,\Delta S \sim \delta\varphi^2 \ln(4/\delta\varphi) \\ \delta s^2 \sim \delta\varphi^2 \ln(4\sqrt{\pi}/\delta\varphi) \end{cases} \qquad \delta\varphi^2 \equiv 2\,\mathrm{Tr}\,\delta\hat{D}^2$$

  distance $ds^2 = -d^2 S$ measures still loss of information

NB: $\delta s^2_{BH} = 4\,\delta s^2_{FS} \sim \delta\varphi^2 \neq 8\,\Delta S$

Discrepancy of Bures-Helstrom and Fubini-Study with information loss

III Conjugation states $\leftrightarrow$ observables

- Mapping obs. $\to$ states $\boxed{\hat{D} = \dfrac{e^{\hat{X}}}{\mathrm{Tr}\, e^{\hat{X}}}}$
- Contravariant and covariant variations $d\hat{D}$ and $d\hat{X}$ $\boxed{ds^2 = \mathrm{Tr}\, d\hat{D}\, d\hat{X}}$
- Metric in the space of observables $\hat{X}$

$$\boxed{ds^2 = -d^2 S(\hat{D}) = d^2 \ln \mathrm{Tr}\, e^{\hat{X}}} = \mathrm{Tr} \int_0^1 d\xi\, \hat{D}\, e^{-\xi \hat{X}}\, d\hat{X}\, e^{\xi \hat{X}}\, d\hat{X} - \left(\mathrm{Tr}\, \hat{D}\, d\hat{X}\right)^2$$

identified with the Bogoliubov-Kubo-Mori metric

- Legendre transform $S(\hat{D}) \equiv -\mathrm{Tr}\, \hat{D} \ln \hat{D} \longleftrightarrow \boxed{F(\hat{X}) \equiv \ln \mathrm{Tr}\, e^{\hat{X}}}$

$$dS = -\mathrm{Tr}\, \hat{X}\, d\hat{D} \quad ; \quad dF = \mathrm{Tr}\, \hat{D}\, d\hat{X} \quad ; \quad S = F - \mathrm{Tr}\, \hat{D}\hat{X}$$

- Equilibrium statistical mechanics provides geometric structure of thermodynamics for $\hat{D}$ = Boltzmann-Gibbs

Example: Canonical equilibrium $\hat{X} = -\beta \hat{H}$ ($-\beta$ = coordinate of $\hat{X}$)

$\hat{D} = \dfrac{e^{-\beta \hat{H}}}{\mathrm{Tr}\, e^{-\beta \hat{H}}}$ ($U = \langle \hat{H} \rangle$ = coordinate of $\hat{D}$ in map $\hat{H} \mapsto \langle \hat{H} \rangle$)

F = Massieu potential $\underset{\text{(von Neumann)}}{S}$ = entropy $= F + \beta U$

$$dS = \beta\, dU \; ; \; dF = -U\, d\beta \; ; \; ds^2 = dU\, d\beta \quad \text{metric}$$

IV Use in non-equilibrium (quantum) statistical dynamics

- Choice of relevant observables $\{\hat{A}_k\}$: follow only $\boxed{a_k(t) = \mathrm{Tr}\, \hat{D}(t)\, \hat{A}_k}$
- Reduced description = eliminate information about irrelevant observables : replace $\hat{D}(t)$ by the simpler reduced state $\hat{D}_R(t)$
- $\hat{D}_R(t)$ retains information about $a_k(t)$: $\boxed{\mathrm{Tr}\, \hat{D}_R(t)\, \hat{A}_k = a_k(t)}$
  but otherwise most random — MaxEnt criterion for $\hat{D}_R(t)$
- Yields $\hat{D}_R(t) = \dfrac{e^{\hat{X}_R}}{\mathrm{Tr}\, e^{\hat{X}_R}}$, $\hat{X}_R(t) = \sum_k \lambda_k(t)\, \hat{A}_k$ Surface of relevant states $\mathcal{R}$
  Conjugate parametrizations $\{a_k\} \leftrightarrow \{\lambda_k\}$ of surface $\mathcal{R}$
- Relevant entropy $S(\hat{D}_R)$ = lack of information if only $\{a_k\}$ are known
  Examples : Boltzmann entropy ; non-equilibrium thermodynamic entropy
- Method of projection : dynamics of $\hat{D}_R(t)$ instead of $\hat{D}(t)$
- Use of metric $ds^2 = -d^2S$
  1. Its restriction to the surface $\mathcal{R}$ defines a metric for thermodynamics in non-equilibrium cases
  2. With this metric, $\hat{D} - \hat{D}_R$ is perpendicular at $\hat{D}_R$ to the surface $\mathcal{R}$: $\hat{D}_R$ is the best approximation to $\hat{D}$ on $\mathcal{R}$

V Properties of the metric (Riemannian)

- In a Hilbert space representation when $\hat{D}$ is diagonal

$$ds^2 = \sum_{ij} \frac{\ln D_i - \ln D_j}{D_i - D_j} dD_{ij} \, dD_{ji}$$

Diagonal variations reduce to $\frac{(dD_{ii})^2}{D_{ii}}$ (~ Fisher)

"Nearly diagonal" terms $|D_i - D_j| \ll D_i + D_j$ yield

$$\simeq \frac{2}{D_i + D_j} dD_{ij} \, dD_{ji}$$ (Bures - Helstrom) [not physical]

- Taking biorthogonal bases $\hat{\Omega}^\nu$ for observables and $\hat{\Sigma}_\mu$ for states (with $\mathrm{Tr}\, \hat{\Omega}^\nu \hat{\Sigma}_\mu = \delta^\nu_\mu$), regarding them as elements of two dual vector spaces, coordinates are $\hat{O} = O_\nu \hat{\Omega}^\nu$, $\hat{D} = D^\mu \hat{\Sigma}_\mu$

Metric tensor: $g^{\mu\nu} = \frac{\partial^2 F}{\partial X_\mu \partial X_\nu}$, $g_{\mu\nu} = -\frac{\partial^2 S}{\partial D^\mu \partial D^\nu}$ (Hessian)

Christoffel: $\Gamma_{\mu\nu\rho} = -\frac{1}{2} \frac{\partial^3 S}{\partial D^\mu \partial D^\nu \partial D^\rho}$

→ Geodesics and finite distances

→ Curvatures (Riemann, Ricci, scalar) The space of quantum states is curved (Classical = flat)

- Fully worked out for a <u>q-bit</u> $(n=2)$

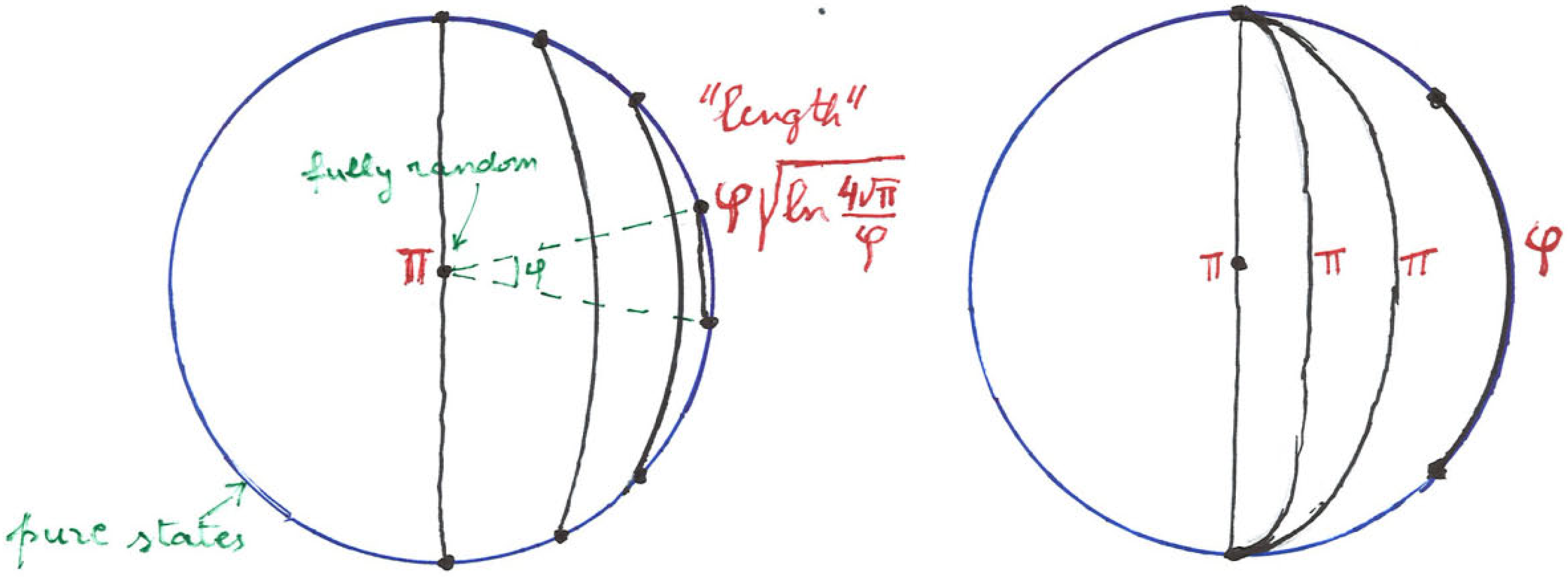
→ <u>Scalar curvature</u> R = function of r (degree of polarisation)

$r \ll 1,\ R \sim -\frac{10}{9} r^2$ (vanishes at $r=0$)

$r \to 1,\ R \sim -\frac{2}{(1-r)\,|\ln(1-r)|}$ (infinite for pure states)

$\Big\}$ negative

→ <u>Geodesics</u> in sphere $r \leqslant 1$



$ds^2 = -d^2 S$

$n=2$

<u>Physical metric!</u>

Bures

$n=2$