Structure, Probabiliy, Entropy

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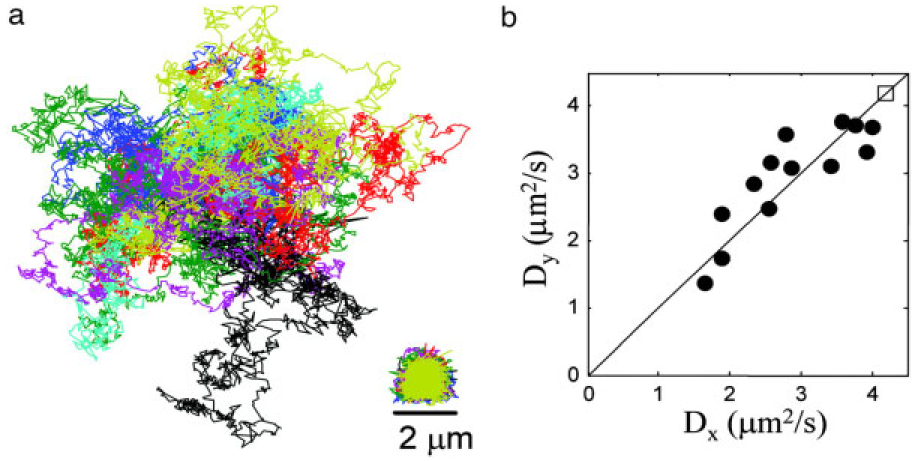
Rituparna, a king of Ayodhya said $\approx 5\,000$ years ago:

I of dice possess the science and in numbers thus am skilled.

More recently, ≈ 150 years ago, James Clerk Maxwell said:

The true logic of this world is the calculus of probabilities.

All the mathematical sciences are founded on relations between physical laws and laws of numbers.



... small compound bodies...
are set in perpetual motion
by the impact of invisible blows...

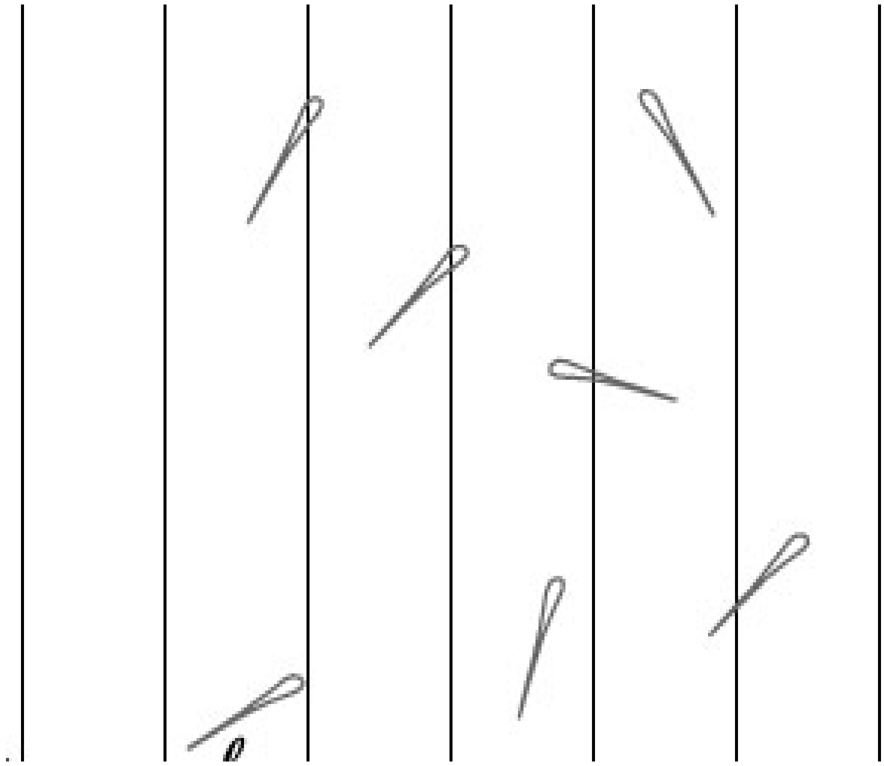
.

The movement mounts up
from the atoms
and gradually emerges
to the level of our senses.

Articulated by...

Titus Lucretius in 50 BCE
and expressed in numbers by

THIELE (1880), BACHELIER (1900), EINSTEIN (1905), SMOLUCHOWSKI (1906), WIENER (1923).



Symmetry in Randomness.

Most (all?) of the classical mathematical probability theory is grounded on *(quasi)invariant Haar(-like) measures.*

(The year 2000 was landmarked by the discovery of *conformally in-*

variant probability measures in spaces of curves in Riemann surfaces parametrized by increments of *Brownian's processes* via the *Schram-Loewner evolution equation.*)

The canonized formalisation of probability, inspired by Buffon's needle (1733) and implemented by Kolmogorov (1933) reads:

Any kind of randomness in the world can be represented (modeled) geometrically by a subdomain Y *in the unit square* $\blacksquare$ *in the plane. You drop a points to* $\blacksquare$, *you count hitting* Y *for an* ***event*** *and define the probability of this event as* $area(Y)$.

(This *set theoretic* frame conceptually is similar to André Weil's *uni-*

versal domains from his 1946 book *Foundations of Algebraic Geometry.*)

If there is not enough symmetry and one can not *postulate* equiprobability (and/or something of this kind such as *independence*) of certain "events", then the advance of the classical calculus stalls, be it mathematics, physics, biology, linguistic or gambling.

On Randomness in Languges.

The notion of a probability of a sentence is an entirely useless one, under any interpretation of this term [that you find in 20th century textbooks"].

Naum Chomsky.

An essential problem with probability is a mathematical definition of ”events” the probabilities of which are being measured.

A particular path to follow is suggested by Boltzmann’s way of thinking about statistical mechanics – his ideas invite a use of

non-standard analysis

and of Grothendieck’s style

category theoretic language.

Also, the idea of probability in *languages* and in *mathematics of learning* deviates from Kolmogorov-Buffon ■.

Five Alternative Avenues for Ideas of Probability and Entropy.

1. Entropy via Grothendieck Semigroup.

2. Probality spacers as covariant functors

3. Large deviations and Non-Standard analysis for classical and quantum entropies.

4. Linearized Measures, Probabilities and Entropies.

5. Combinatorial Probability with Limited Symmetries.

"Naive Physicist's" Entropy

... pure thought can grasp reality... .

Albert Einstein.

...exceedingly difficult task of our time is to work on the construction of a new idea of reality.... .

Wolfgang Pauli.

A system $\mathcal{S}$ is an infinite ensemble of infinitely small *mutually equal* "states". The logarithm of the properly normalised number of these states is *(mean statistical Boltzmann) entropy of* $\mathcal{S}$.

The "space of states" of $\mathcal{S}$ is NOT a mathematician's "set", it is "something" that depends on a class of *mutually equivalent* imaginary experimental protocols.

Detectors of Physical States: Finite Measure Spaces. A finite measure space $P = \{p\}$ is a finite set of "atoms" with a positive function denoted $p \mapsto |p| > 0$, thought of as $|p| = mass(p)$.

$|P| = \sum_p |p|$: the (total) *mass* of P.

If $|P| = 1$, then P is called a *prob-*

ability space.

Reductions and $\mathcal{P}$. A map $P \xrightarrow{f} Q$ is a *reduction* if the *q-fibers* $P_q = f^{-1}(q) \subset P$ satisfy $|P_q| = |q|$ for all $q \in Q$.

(Think of Q as a "plate with windows" through which you "observe" P. What you see of the states of P is what "filters" through the windows of Q.)

Finite measure spaces P and reductions make a nice category $\mathcal{P}$. All morphisms in this category are epimorphisms, $\mathcal{P}$ looks very much as a partially ordered set (with $P \succ Q$ corresponding to reductions $f : P \to Q$ and few, if any, reductions between given P and Q); but it is advantages to treat $\mathcal{P}$ as a general category.

Why Category? There is a subtle but significant conceptual difference between writing $P \succ Q$ and $P \xrightarrow{f} Q$. Physically speaking, there is no a priori given "attachment" of Q to P, an abstract "$\succ$" is meaningless, it must be implement by a particular operation f. (If one keeps track of "protocol of attaching Q to P", one arrives at the concept of 2-*category*.)

The f-notation, besides being more precise, is also more flexible. For example one may write $ent(f)$ but not $ent(\succ)$ with no P and Q in the notation.

Grothendieck Semigroup $Gr(\mathcal{P})$, *Bernoulli isomorphism* $Gr(\mathcal{P}) =$

$[1,\infty)^{\times}$ *and Entropy.*

Superadditivity of Entropy.

Functorial representation of infinite probability spaces X by sets of finite partitions of X, that are sets $mor(X \to P)$, for all $P \in \mathcal{P}$ and defining Kolmogorov's dynamical entropy in these terms.

Fisher metric and von Neumann's Unitarization of Entropy.

Hessian $h = Hess(e)$, $e = e(p) = \sum_{i\in I} p_i \log p_i$, on the simplex $\triangle(I)$ is a Riemannian metric on $\triangle(I)$ where the *real moment* map $M_{\mathbb{R}}$: $\{x_i\} \to \{p_i = x_i^2\}$ is, up to 1/4-factor, an *isometry* from the positive "quadrant" of the unit Euclidean sphere onto $(\triangle(I), h)$.

P: positive quadratic forms on the Euclidean space $\mathbb{R}^n$,

Σ: orthonormal frames $\Sigma = (s_1, ..., s_n)$,

$\underline{P}(\Sigma) = (\underline{p}_1, ..., \underline{p}_n)$, $\underline{p}_i = P(s_i)$,

$$ent_{VN(P)} = ent(P) = \inf_{\Sigma} ent(\underline{P}(\Sigma)).$$

Lanford-Robinson, 1968. *The function $P \mapsto ent(P)$ is concave on the space of density states:*

$$ent\left(\frac{P_1 + P_2}{2}\right) \geq \frac{ent(P_1) + ent(P_2)}{2}.$$

Indeed, the classical entropy is a concave function on the simplex of probability measures on the set I, that is $\{p_i\} \subset \mathbb{R}_+^I$, $\sum_i p_i = 1$, and infima of familes of concave functions are concave.

Spectral definition/theorem:
$ent_{VN}(P) = ent_{Shan}(spec((P))$.

Symmetrization as Reduction and Quantum Superadditivity.

LIEB-RUSKAI, 1973.
H and G: compact groups of unitary transformations of a finite dimensional Hilbert space S
P a state (positive semidefinite Hermitian form) on S.
If the actions of H and G commute,
then the von Neumann entropies of the G- and H-averages of P satisfy

$$ent(G*(H*P)) - ent(G*P) \leq ent(H*P) - ent(P).$$

On Algebraic Inequalities. Besides "unitarization" some Shannon inequalities admit linearization, where the first non-trivial instance of this is the following

linearized Loomis-Whitney 3*D-isoperimetric inequality* for ranks of bilinear forms associated with a 4-linear form $\Phi = \Phi(s_1, s_2, s_3, s_4)$ where we denote $|...| = rank(...)$:

$$|\Phi(s_1, s_2 \otimes s_3 \otimes s_4)|^2 \leq$$

$$|\Phi(s_1 \otimes s_2, s_3 \otimes s_4)| \cdot |\Phi(s_1 \otimes s_3, s_2 \otimes s_4)| \cdot$$

$$\cdot |\Phi(s_1 \otimes s_4, s_2 \otimes s_3)|$$

Measures defined via Cohomology and Parametric Packing Prob-

lem.

Entropy serves for the study of "ensembles" $\mathcal{A} = \mathcal{A}(X)$ of (finitely or infinitely many) particles in a space X, e.g. in the Euclidean 3-space by

$$U \mapsto ent_U(\mathcal{A}) = ent(\mathcal{A}_{|U}),\ U \subset X,$$

that assigns the *entropies* of *the U-reductions $\mathcal{A}_{|U}$ of* $\mathcal{A}$, to all bounded open subsets $U \subset X$. In the physicists' parlance, this entropy is

"*the logarithm of the number of the states of* $\mathcal{E}$

that are effectively observable from U",

We want to replace "*effectively observable number of states*" by

"*the number of effective degrees of freedom of ensembles of moving*

balls".

• *Classical (Non-parametric) Sphere Packings.*

• *Homotopy and Cohomotopy Energy Spectra.*

• *Homotopy Dimension, Cell Numbers and Cohomology Valued Measures.*

• *Infinite Packings and Equivariant Topology of Infinite Dimensional*

Spaces Acted upon by Non-compact Groups.

• *Bi-Parametric Pairing between Spaces of Packings and Spaces of Cycles.*

• *Non-spherical Packings, Spaces of Partitions and Bounds on Waists.*

• *Symplecting Packings.*

Graded Ranks, Poincare Polynomials and Ideal Valued Measures.

The images as well as kernels of (co)homology homomorphisms that are induced by continuous maps are *graded* Abelian groups and their ranks are properly represented not by individual numbers but by *Poincaré polynomials.*

The set function $U \mapsto \mathrm{Poincaré}_U$ that assigns Poincaré polynomials to subsets $U \subset A$, (e.g. $U = A_r$) has some measure-like properties that become more pronounced for the set function

$$A \supset U \mapsto \mu(U) \subset H^*(A;\Pi),$$

$$\mu(U) = Ker\left(H^*(A;\Pi) \to H^*(A \setminus U;\Pi)\right),$$

where Π is an Abelian (homology coefficient) group, e.g. a field $\mathbb{F}$.

$\mu(U)$ *is additive for the sum-of-subsets in* $H^*(A;\Pi)$ *and super-multiplicative for the the* $\smile$*-product of ideals in the case* Π *is a commutative ring*:

$$\mu(U_1 \cup U_2) = \mu(U_i) + \mu(U_2)$$

for *disjoint* open subsets U_1 and U_2 in A, and

$$\mu(U_1 \cap U_2) \supset \mu(U_1) \smile \mu(U_2)$$

for all open $U_1, U_2 \subset A$