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Koszul Information Geometry & Souriau Lie Group Thermodynamics

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Koszul-Vinberg Characteristic Function

- **François Massieu in 1869 demonstrated that some thermal properties of physical systems could be derived from "characteristic functions".**
- **This idea was developed by Gibbs and Duhem with the notion of potentials in thermodynamics, and introduced by Poincaré in probability.**
- **We will study generalization of this concept by**
  - Jean-Louis Koszul in Mathematics
  - Jean-Marie Souriau in Statistical Physics.
- **The Koszul-Vinberg Characteristic Function (KVCF) on convex cones will be presented as cornerstone of "Information Geometry" theory:**
  - defining Koszul Entropy as Legendre transform of minus the logarithm of KVCF (their gradients defining mutually inverse diffeomorphisms)
  - Fisher Information Metrics as hessian of these dual functions.
- **Koszul proved that these metrics are invariant by all automorphisms of the convex cones.**

Koszul-Vinberg Characteristic Function

- **Jean-Marie Souriau has extended the Characteristic Function in Statistical Physics:**
  - looking for other kinds of invariances through co-adjoint action of a group on its momentum space
  - defining physical observables like energy, heat and momentum as pure geometrical objects.
- **In covariant Souriau model, Gibbs equilibriums states are indexed by a geometric parameter, the Geometric Temperature, with values in the Lie algebra of the dynamical Galileo/Poincaré groups, interpreted as a space-time vector (a vector valued temperature of Planck), giving to the metric tensor a null Lie derivative.**
- **Fisher Information metric appears as the opposite of the derivative of Mean "Moment map" by geometric temperature, equivalent to a Geometric Capacity or Specific Heat.**
- **We will synthetize the analogies between both Koszul and Souriau models, and will reduce their definitions to the exclusive "Inner Product" selection using symmetric bilinear "Cartan-Killing form" (introduced by Elie Cartan in 1894).**

Information Theory

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Claude. SHANNON

Homogeneous Convex Cones G.

E.B. VINBERG
Jean-Louis KOSZUL

Geometric Science of Information

Probability/G. on structures

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Affine G.

Takeshi SASAKI
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KOSZUL-VINBERG METRIC (KOSZUL-VINBERG CHARACTERISTIC FUNCTION)

Information G.

Calyampudi R. RAO
Nikolai N. CHENTSOV

FISHER METRIC

Probability in Metric Space

Maurice R. FRECHET

Probability on Riemannian Manifold

Michel EMERY
Marc ARNAUDON

Hessian G.

Hirohiko SHIMA
Jean-Louis KOSZUL

Sympl. G.

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Y. SHISHIDO

Contact G.

Vladimir ARNOLD

Elie CARTAN
Carl Ludwig SIEGEL

Homogeneous Symmetric Bounded Domains G.

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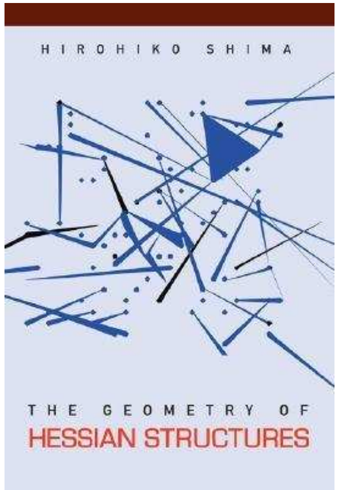
Foundation of Information Geometry: Jean-Louis Koszul Works

Hessian Geometry by J.L. Koszul

- **Hirohiko Shima Book, « Geometry of Hessian Structures », world Scientific Publishing 2007, dedicated to Jean-Louis Koszul**
- **Hirohiko Shima Keynote Talk at GSI'13**
  - http://www.see.asso.fr/file/5104/download/9914
- **Prof. M. Boyom tutorial :**
  - http://repmus.ircam.fr/_media/brillouin/ressources/une-source-de-nouveaux-invariants-de-la-geometrie-de-l-information.pdf


Jean-Louis Koszul



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J.L. Koszul, « Ouverts convexes homogènes des espaces affines », Math. Z. 79, pp. 254-259., 1962
J.L. Koszul, « Variétés localement plates et convexité », Osaka J. Maht. 2, pp. 285-290., 1965
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Projective Legendre Duality and Koszul Characteristic Function

INFORMATION GEOMETRY METRIC

$$g^* = d^2\Psi^* = d^2 S$$

$$g = -d^2 \log \Phi = d^2 \Psi$$

ds^2=d^2ENTROPY

ds^2=-d^2LOG[FOURIER]

LEGENDRE TRANSFORM

FOURIER/LAPLACE TRANSFORM

$$\Psi^*(x^*) = \left\langle x, x^* \right\rangle - \Psi(x)$$

$$\Psi(x) = -\log \Phi(x) = -\log \int_{\Omega^*} e^{-\langle x,y \rangle} dy$$

$$\Psi^* = -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi$$

ENTROPY=
LEGENDRE(- LOG[LAPLACE])

$$p_x(\xi) = e^{-\langle \xi,x \rangle} / \int_{\Omega^*} e^{-\langle \xi,x \rangle} d\xi = e^{-\langle x,\xi \rangle + \Phi(x)}$$

Legendre Transform of minus logarithm of characteristic function (Laplace transform) = ENTROPY !!!

$$x^* = \int_{\Omega^*} \xi . p_x(\xi) d\xi$$

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Koszul-Vinberg Characteristic Function/Metric of convex cone

- J.L. Koszul and E. Vinberg have introduced an affinely invariant Hessian metric on a sharp convex cone through its characteristic function.
- Ω is a sharp open convex cone in a vector space E of finite dimension on R (a convex cone is sharp if it does not contain any full straight line).
- Ω^* is the dual cone of Ω and is a sharp open convex cone.
- Let $d\xi$ the Lebesgue measure on E^* dual space of E, the following integral:

$$\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \quad \forall x \in \Omega$$

is called the Koszul-Vinberg characteristic function

Koszul-Vinberg Characteristic Function/Metric of convex cone

- **Koszul-Vinberg Metric :** $g = d^2 \log \psi_\Omega$

$$d^2 \log \psi(x) = d^2\left[\log \int \psi_u du\right] = \frac{\int \psi_u d^2 \log \psi_u du}{\int \psi_u du} + \frac{1}{2}\frac{\iint \psi_u \psi_v (d\log\psi_u - d\log\psi_v)^2 dudv}{\iint \psi_u \psi_v dudv}$$

- **We can define a diffeomorphism by:** $x^* = -\alpha_x = -d\log\psi_\Omega(x)$

with $\langle df(x), u\rangle = D_u f(x) = \left.\frac{d}{dt}\right|_{t=0} f(x+tu)$

- **When the cone** Ω **is symmetric, the map** $x^* = -\alpha_x$ **is a bijection and an isometry with a unique fixed point (the manifold is a Riemannian Symmetric Space given by this isometry):**

$(x^*)^* = x$, $\langle x, x^*\rangle = n$ **and** $\psi_\Omega(x)\psi_{\Omega^*}(x^*) = cste$

- x^* **is characterized by** $x^* = \arg\min\{\psi(y) / y \in \Omega^*, \langle x, y\rangle = n\}$

- x^* **is the center of gravity of the cross section** $\{y \in \Omega^*, \langle x, y\rangle = n\}$ **of** Ω^***:**

$$x^* = \int_{\Omega^*} \xi . e^{-\langle \xi, x\rangle} d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi$$

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Koszul Entropy via Legendre Transform

- we can deduce "Koszul Entropy" defined as Legendre Transform of minus logarithm of Koszul-Vinberg characteristic function $\Phi(x) = -\log \psi_\Omega(x)$:

$\Phi^*(x^*) = \langle x, x^* \rangle - \Phi(x)$ with $x^* = D_x \Phi$ and $x = D_{x^*} \Phi^*$ where

- **Demonstration:** we set $\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \quad \forall x \in \Omega$

Using $x^* = \int_{\Omega^*} \xi . e^{-\langle \xi, x \rangle} d\xi / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$

and $\langle -x^*, h \rangle = d_h \log \psi_\Omega(x) = -\int_{\Omega^*} \langle \xi, h \rangle e^{-\langle \xi, x \rangle} d\xi / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$

we can write: $-\langle x^*, x \rangle = \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$

and

$$\Phi^*(x^*) = -\int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi + \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

$$\Phi^*(x^*) = \left[\left(\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \right) . \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \right] / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

Koszul-Vinberg Characteristic Function Legendre Transform

$$\Phi^*(x^*) = \left\langle x, x^* \right\rangle - \Phi(x) = -\int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi + \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

$$\Phi^*(x^*) = \left[\left(\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \right) . \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \right] \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

$$\Phi^*(x^*) = \left[\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right]$$

$$\Phi^*(x^*) = \left[\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi . \left(\int_{\Omega^*} \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right) - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right] \quad \text{with} \quad \int_{\Omega^*} \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi = 1$$

$$\Phi^*(x^*) = \left[-\int_{\Omega^*} \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} . \log \left(\frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} \right) d\xi \right]$$

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Koszul Entropy via Legendre Transform

- **We can then consider this Legendre transform as an entropy, that we could named "Koszul Entropy":**

$$\Phi^* = -\int_{\Omega^*} \frac{e^{-\langle \xi, x\rangle}}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} \log \frac{e^{-\langle \xi, x\rangle}}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} d\xi = -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi$$

With $$p_x(\xi) = e^{-\langle \xi, x\rangle} / \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi = e^{-\langle x,\xi\rangle - \log \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} = e^{-\langle x,\xi\rangle + \Phi(x)}$$

and $$x^* = D_x\Phi = \int_{\Omega^*} \xi . p_x(\xi) d\xi = \int_{\Omega^*} \xi . e^{-\langle x,\xi\rangle + \Phi(x)} d\xi = \int_{\Omega^*} \xi . e^{-\Phi^*(\xi)} d\xi$$

$$\Phi(x) = -\log \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi = -\log \int_{\Omega^*} e^{-[\Phi^*(\xi)+\Phi(x)]} d\xi$$

$$\Phi(x) = \Phi(x) - \log \int_{\Omega^*} e^{-\Phi^*(\xi)} d\xi \Rightarrow \int_{\Omega^*} e^{-\Phi^*(\xi)} d\xi = 1$$

$$\text{Jensen Ineq.}: \Phi^* \text{ conv.} \Rightarrow \Phi^*(E[\xi]) \leq E[\Phi^*(\xi)]$$

$$\text{Legendre Transform}: \Phi^*(x^*) \geq \langle x, x^* \rangle - \Phi(x)$$

$$\Rightarrow \Phi^*(x^*) \geq \int_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = E[\Phi^*(\xi)]$$

$$\log p_x(\xi) = \log e^{-\langle x,\xi\rangle + \Phi(x)} = \log e^{-\Phi^*(\xi)} = -\Phi^*(\xi)$$

$$\Rightarrow -\int_{\Omega^*} \log p_x(\xi) p_x(\xi) d\xi = \int_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = \Phi^*(x^*)$$

$$\text{if and only if } \int_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = \Phi^*\left(\int_{\Omega^*} \xi . p_x(\xi) d\xi \right)$$

$$\text{or } E[\Phi^*(\xi)] = \Phi^*(E[\xi])$$

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Barycenter of Koszul Entropy = Koszul Entropy of Barycenter

$$E\left[\Phi^{*}(\xi)\right]=\Phi^{*}\left(E[\xi]\right)$$

$$\int_{\Omega^*}\Phi^{*}(\xi)p_x(\xi)d\xi=\Phi^{*}\left(\int_{\Omega^*}\xi.p_x(\xi)d\xi\right)$$

$$p_x(\xi)=e^{-\langle\xi,x\rangle}/\int_{\Omega^*}e^{-\langle\xi,x\rangle}d\xi=e^{-\langle x,\xi\rangle-\log\int_{\Omega^*}e^{-\langle\xi,x\rangle}d\xi}=e^{-\langle x,\xi\rangle+\Phi(x)}$$

$$x^{*}=D_x\Phi=\int_{\Omega^*}\xi.p_x(\xi)d\xi=\int_{\Omega^*}\xi.e^{-\langle x,\xi\rangle+\Phi(x)}d\xi=\int_{\Omega^*}\xi.e^{-\Phi^{*}(\xi)}d\xi$$

$$\Phi^{*}(x^{*})=\underset{x}{Sup}\left[\langle x,x^{*}\rangle-\Phi(x)\right]$$

Koszul metric & Fisher Metric

- To make the link with Fisher metric given by matrix $I(x)$, we can observe that the second derivative of $\log p_x(\xi)$ is given by:

$$\log p_x(\xi) = -\Phi^*(\xi) = \Phi(x) - \langle x, \xi \rangle$$

$$\frac{\partial^2 \log p_x(\xi)}{\partial x^2} = \frac{\partial^2 \left[\Phi(x) - \langle x, \xi \rangle\right]}{\partial x^2} = \frac{\partial^2 \Phi(x)}{\partial x^2}$$

$$\Rightarrow I(x) = -E_\xi\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right] = -\frac{\partial^2 \Phi(x)}{\partial x^2} = \frac{\partial^2 \log \psi_\Omega(x)}{\partial x^2}$$

- We could then deduce the close interrelation between Fisher metric and hessian of Koszul-Vinberg characteristic logarithm.

$$I(x) = -E_\xi\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right] = \frac{\partial^2 \log \psi_\Omega(x)}{\partial x^2}$$

FISHER METRIC (Information Geometry) =
KOSZUL HESSIAN METRIC (Hessian Geometry)

Koszul Metric and Fisher Metric as Variance

- We can also observed that the Fisher metric or hessian of KVCF logarithm is related to the variance of ξ :

$$\log \Psi_{\Omega}(x) = \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \Rightarrow \frac{\partial \log \Psi_{\Omega}(x)}{\partial x} = -\frac{1}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} \int_{\Omega^*} \xi . e^{-\langle \xi, x \rangle} d\xi$$

$$\frac{\partial^2 \log \Psi_{\Omega}(x)}{\partial x^2} = -\frac{1}{\left(\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \right)^2} \left[-\int_{\Omega^*} \xi^2 . e^{-\langle \xi, x \rangle} d\xi . \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi + \left(\int_{\Omega^*} \xi . e^{-\langle \xi, x \rangle} d\xi \right)^2 \right]$$

$$\frac{\partial^2 \log \Psi_{\Omega}(x)}{\partial x^2} = \int_{\Omega^*} \xi^2 . \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi - \left(\int_{\Omega^*} \xi . \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right)^2 = \int_{\Omega^*} \xi^2 . p_x(\xi) d\xi - \left(\int_{\Omega^*} \xi . p_x(\xi) d\xi \right)^2$$

$$I(x) = -E_{\xi}\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2} \right] = \frac{\partial^2 \log \psi_{\Omega}(x)}{\partial x^2} = E_{\xi}\left[\xi^2 \right] - E_{\xi}\left[\xi \right]^2 = Var(\xi)$$

New Definition of Maximum Entropy

- **We have then observed that Koszul Entropy provides density of Maximum Entropy:**

$$p_{\bar{\xi}}(\xi) = \frac{e^{-\left\langle \xi, \Theta^{-1}\left(\bar{\xi}\right)\right\rangle}}{\int_{\Omega^*} e^{-\left\langle \xi, \Theta^{-1}\left(\bar{\xi}\right)\right\rangle} d\xi}$$ **with** $$x = \Theta^{-1}(\bar{\xi})$$ **and** $$\bar{\xi} = \Theta(x) = \frac{d\Phi(x)}{dx}$$

where $$\bar{\xi} = \int_{\Omega^*} \xi . p_{\bar{\xi}}(\xi) d\xi$$ **and** $$\Phi(x) = -\log \int_{\Omega^*} e^{-\langle x, \xi \rangle} d\xi$$

Koszul Density: Application for SPD matrices

- We can then named this new density as "**Koszul Density**":

$$p_x(\xi) = e^{-\langle \xi, x\rangle} \Big/ \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi = e^{-\langle x,\xi\rangle - \log \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} = e^{-\langle x,\xi\rangle + \Phi(x)}$$

With $x^* = D_x \Phi = \int_{\Omega^*} \xi . p_x(\xi) d\xi = \int_{\Omega^*} \xi . e^{-\langle x,\xi\rangle + \Phi(x)} d\xi = \int_{\Omega^*} \xi . e^{-\Phi^*(\xi)} d\xi$

$$\begin{cases} \langle x, y\rangle = Tr(xy), \forall x, y \in Sym_n(R) \\ \psi_\Omega(x) = \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi \underset{\substack{\langle x,y\rangle = Tr(xy) \\ \Omega^* = \Omega \text{ self-dual}}}{=} \det x^{-\frac{n+1}{2}} \psi(I_n) \\ x^* = \bar{\xi} = -d \log \psi_\Omega = \frac{n+1}{2} d \log \det x = \frac{n+1}{2} x^{-1} \end{cases}$$

$$\Rightarrow \quad p_x(\xi) = e^{-Tr(x\xi) + \frac{n+1}{2} \log \det x} = \left[\det\left(\alpha \bar{\xi}^{-1}\right)\right]^{\alpha} e^{-Tr\left(\alpha \bar{\xi}^{-1} \xi\right)} \text{ with } \bar{\xi} = \int_{\Omega^*} \xi . p_x(\xi) . d\xi$$

Covariant Definition of Thermodynamic Equilibriums

- **Jean-Marie Souriau , student of Elie Cartan at ENS Ulm in 1946, has**
  - given a **covariant definition of thermodynamic equilibriums**
  - formulated statistical mechanics and thermodynamics in the framework of **Symplectic Geometry**

by use of symplectic moments and distribution-tensor concepts,

giving a geometric status for:

  - Temperature
  - Heat
  - Entropy

- **This work has been extended by C. Vallée & G. de Saxcé, P. Iglésias and F. Dubois.**

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Covariant Definition of Thermodynamic Equilibriums

- **The first general definition of the "moment map" (constant of the motion for dynamical systems) was introduced by Jean-Marie Souriau during 1970's**
  - with geometric generalization such earlier notions as the Hamiltonian and the invariant theorem of Emmy Noether describing the connection between symmetries and invariants (it is the moment map for a one-dimensional Lie group of symmetries).
- **In symplectic geometry the analog of Noether's theorem is the statement that the moment map of a Hamiltonian action which preserves a given time evolution is itself conserved by this time evolution.**
- **The conservation of the moment of a Hamilotnian action was called by Souriau the "Symplectic or Geometric Noether theorem"**
  - considering phases space as symplectic manifold, cotangent fiber of configuration space with canonical symplectic form, if Hamiltonian has Lie algebra, **moment map is constant along system integral curves**.
  - **Noether theorem is obtained by considering independently each component of moment map**

THALES

Souriau Covariant Model

- Let M be a differentiable manifold with a continuous positive density $d\omega$ and let E a finite vector space and $U(\xi)$ a continuous function defined on M with values in E. A continuous positive function $p(\xi)$ solution of this problem with respect to calculus of variations:

$$\underset{p(\xi)}{ArgMin}\left[s = -\int_M p(\xi)\log p(\xi)d\omega \right] \text{ such that } \begin{cases} \int_M p(\xi)d\omega = 1 \\ \int_M U(\xi)p(\xi)d\omega = Q \end{cases}$$

- is given by:

$$p(\xi) = e^{\Phi(\beta)-\beta.U(\xi)} \text{ and } Q = \frac{\int_M U(\xi)e^{-\beta.U(\xi)}d\omega}{\int_M e^{-\beta.U(\xi)}d\omega}$$

and $\Phi(\beta) = -\log \int_M e^{-\beta.U(\xi)}d\omega$

- Entropy $s = -\int_M p(\xi)\log p(\xi)d\omega$ can be stationary only if there exist a scalar Φ and an element β belonging to the dual of E.
- Entropy appears naturally as Legendre transform of Φ :

$$s(Q) = \beta.Q - \Phi(\beta)$$

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Souriau Covariant Model

- This value $s(Q)=\beta.Q-\Phi(\beta)$ is a strict minimum of s, and the equation:

$$Q=\frac{\int_M U(\xi)e^{-\beta.U(\xi)}d\omega}{\int_M e^{-\beta.U(\xi)}d\omega}$$

has a maximum of one solution for each value of Q.

- The function $\Phi(\beta)$ is differentiable and we can write $d\Phi=d\beta.Q$ and identifying *E* with its dual: $Q=\frac{\partial\Phi}{\partial\beta}$

- Uniform convergence of $\int_M U(\xi)\otimes U(\xi)e^{-\beta.U(\xi)}d\omega$ proves that $-\frac{\partial^2\Phi}{\partial\beta^2}>0$ and that $-\Phi(\beta)$ is convex.

- Then, $Q(\beta)$ and $\beta(Q)$ are mutually inverse and differentiable, where $ds=\beta.dQ$.
- Identifying E with its bidual: $\beta=\frac{\partial s}{\partial Q}$

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Souriau-Gibbs Canonical Ensemble

- **In statistical mechanics, a canonical ensemble is the statistical ensemble that is used to represent the possible states of a mechanical system that is being maintained in thermodynamic equilibrium.**
- **Souriau has defined this Gibbs canonical ensemble on Symplectic manifold *M* for a Lie group action on *M***
- **The seminal idea of Lagrange was to consider that a statistical state is simply a probability measure on the manifold of motions**
- **In Jean-Marie Souriau approach, one movement of a dynamical system (classical state) is a point on manifold of movements.**
- **For statistical mechanics, the movement variable is replaced by a random variable where a statistical state is probability law on this manifold.**

Souriau-Gibbs Canonical Ensemble

- **Symplectic manifolds have a completely continuous measure, invariant by diffeomorphisms: the Liouville measure** λ
- **All statistical states will be the product of Liouville measure by the scalar function given by the generalized partition function** $e^{\Phi-\beta.U}$ **defined by the generalized energy** U **(the moment that is defined in dual of Lie Algebra of this dynamical group) and the geometric temperature** β **, where** Φ **is a normalizing constant such the mass of probability is equal to 1,** $\Phi = -\log \int_M e^{-\beta.U} d\omega$
- **Jean-Marie Souriau generalizes the Gibbs equilibrium state to all Symplectic manifolds that have a dynamical group.**
- **To ensure that all integrals could converge, the canonical Gibbs ensemble is the largest open proper subset (in Lie algebra) where these integrals are convergent. This canonical Gibbs ensemble is convex.**
  - the mean value of the energy $Q = \frac{\partial \Phi}{\partial \beta}$
  - a generalization of heat capacity $K = -\frac{\partial Q}{\partial \beta}$
  - Entropy by Legendre transform $s = \beta.Q - \Phi$

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- **For the group of time translation, this is the classical thermodynamic**
- **Souriau has observed that if we apply this theory for non-commutative group (Galileo or Poincaré groups):**
  - the symmetry has been broken
  - Classical Gibbs equilibrium states are no longer invariant by this group
- **This symmetry breaking provides new equations, discovered by Jean-Marie Souriau.**
- **For each temperature β , Jean-Marie Souriau has introduced a tensor f_β , equal to the sum of cocycle f and Heat coboundary (with [.,.] Lie bracket):**

$$f_\beta(Z_1,Z_2)=f(Z_1,Z_2)+Q.ad_{Z_1}(Z_2) \quad \text{with} \quad ad_{Z_1}(Z_2)=[Z_1,Z_2]$$

- **This tensor f_β has the following properties:**
  - f_β is a symplectic cocycle
  - $\beta \in Ker\ f_\beta$
  - The following symmetric tensor g_β , defined on all values of $ad_\beta(.)$ is positive definite:

$$g_\beta([\beta,Z_1],[\beta,Z_2])=f_\beta(Z_1,[\beta,Z_2])$$

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Souriau Lie Group Thermodynamic

$$f_{\beta}(Z_1,Z_2)= f(Z_1,Z_2)+Q.ad_{Z_1}(Z_2) \quad \text{with} \quad ad_{Z_1}(Z_2)=[Z_1,Z_2]$$

$$\beta \in Ker\ f_{\beta} \qquad g_{\beta}([\beta,Z_1],[\beta,Z_2])= f_{\beta}(Z_1,[\beta,Z_2])$$

- **Souriau equations are universal, because they are not dependent of the symplectic manifold but only of:**
  - the dynamical group ***G***
  - its symplectic cocycle f
  - the temperature β
  - the heat Q
- **Souriau called this model "Lie Groups Thermodynamics":**
  - "Peut-être cette thermodynamique des groups de Lie a-t-elle un intérêt mathématique".
- **For dynamic Galileo group (rotation and translation) with only one axe of rotation:**
  - this thermodynamic theory is the theory of centrifuge where the temperature vector dimension is equal to 2 (sub-group of invariance of size 2)
  - these 2 dimensions for vector-valued temperature are "thermic conduction" and "viscosity", unifying "heat conduction" and "viscosity".

Fundamental Souriau Theorem

- Let Ω be the largest open proper subset of $\mathfrak{g}$, Lie algebra of G, such that $\int_M e^{-\beta.U(\xi)} d\omega$ and $\int_M \xi . e^{-\beta.U(\xi)} d\omega$ are convergent integrals
- this set Ω is convex and is invariant under every transformation $\overline{a}_{\mathfrak{g}}$, where $a \mapsto \overline{a}_{\mathfrak{g}}$ is the adjoint representation of G, with:
  - $\beta \rightarrow \overline{a}_{\mathfrak{g}}(\beta)$
  - $\Phi \rightarrow \Phi - \theta\left(a^{-1}\right)\beta = \Phi + \theta(a).\overline{a}_{\mathfrak{g}}(\beta)$
  - $s \rightarrow s$
  - $Q \rightarrow \overline{a}_{\mathfrak{g}^*}(Q) + \theta(a) = \overline{a}_{\mathfrak{g}^*_\theta}(Q)$
  - $\varsigma \rightarrow \overline{a}_M^+(\varsigma)$
- where θ is the cocycle associated with the group G and the moment, and $\overline{a}_M^+(\varsigma)$ is the image under $\overline{a}_M$ of the probability measure ς.
- Rmq: Φ is changed but with linear dependence to β, then Fisher metric is unchanged by dynamical group:

$$I\left(\overline{a}_{\mathfrak{g}}(\beta)\right) = -\frac{\partial^2\left[\Phi - \theta\left(a^{-1}\right)\beta\right]}{\partial\beta^2} = -\frac{\partial^2\Phi}{\partial\beta^2} = I(\beta)$$

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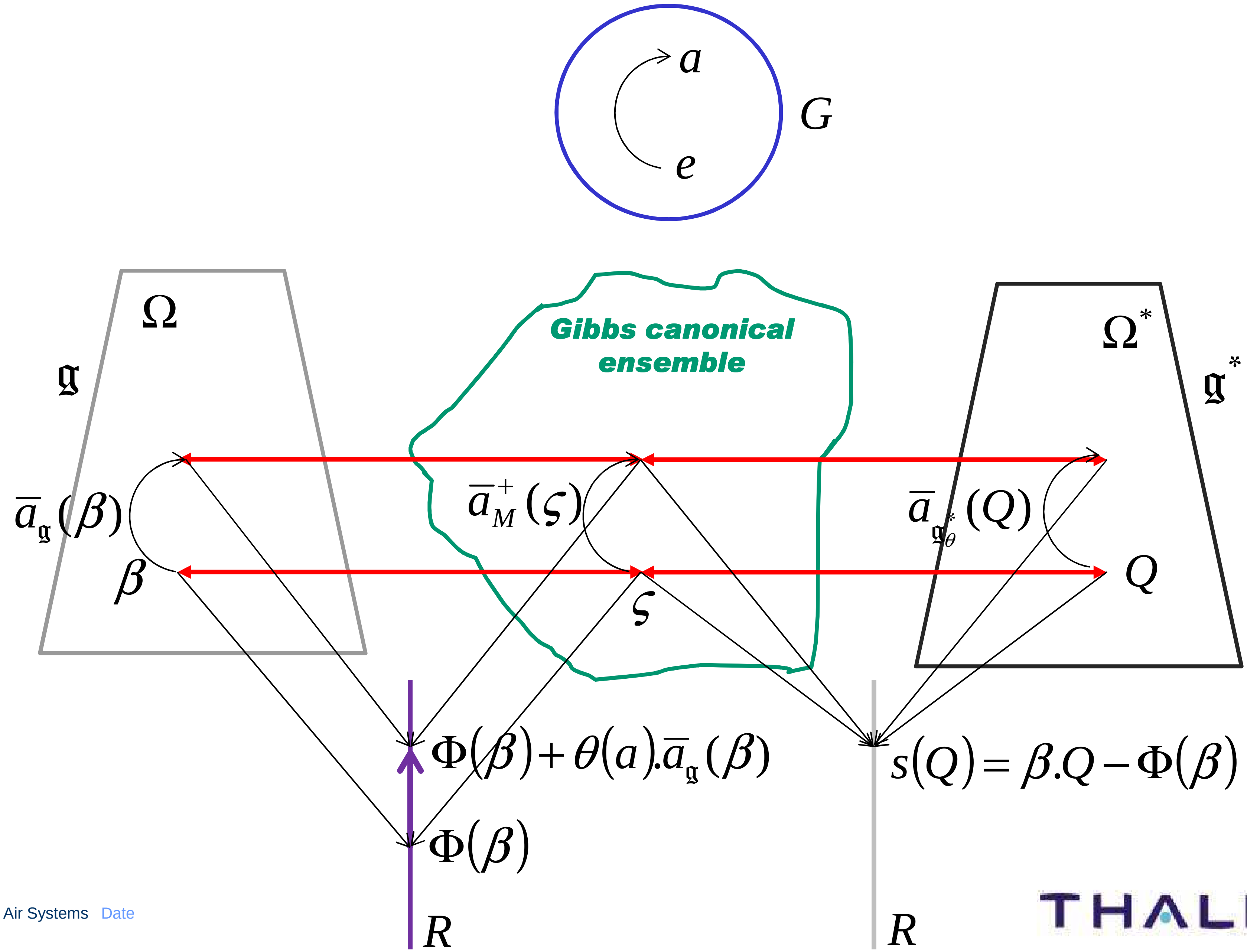
Fundamental Souriau Theorem



Souriau Lie Group definition of Fisher Metric

- Let f be the derivative of θ (symplectic cocycle of G) at the identity element and let us define:

$$\forall \beta \in \Omega, \quad f_\beta(Z_1, Z_2) = f(Z_1, Z_2) + Q.ad_{Z_1}(Z_2) \quad \text{with} \quad ad_{Z_1}(Z_2) = [Z_1, Z_2]$$

Then

- f_β is a symplectic cocycle of $\mathfrak{g}$,that is independent of the moment of G
- $f_\beta(\beta, \beta) = 0 \quad , \quad \forall \beta \in \Omega$
- There exists a symmetric tensor g_β defined on the image of $ad_\beta(.) = [., \beta]$ such that:

$$g_\beta([\beta, Z_1], Z_2) = f_\beta(Z_1, Z_2) \quad , \quad \forall Z_1 \in \mathfrak{g}, \ \forall Z_2 \in \mathrm{Im}(ad_\beta(.))$$

and

$$g_\beta(Z_1, Z_2) \geq 0 \quad , \quad \forall Z_1, Z_2 \in \mathrm{Im}(ad_\beta(.))$$

that gives the structure of a positive Euclidean space

Koszul Information Geometry, Souriau Lie Group Thermodynamics

	Koszul Information Geometry Model	*Souriau Lie Groups Thermodynamics Model*
Characteristic function	$\Phi(x) = -\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \quad \forall x \in \Omega$	$\Phi(\beta) = -\log \int_M e^{-\beta.U(\xi)} d\omega \quad \forall \beta \in \mathfrak{g}$
Entropy	$\Phi^*(x^*) = -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi$	$s = -\int_M p(\xi) \log p(\xi) d\omega$
Legendre Transform	$\Phi^*(x^*) = \langle x, x^* \rangle - \Phi(x)$	$s(Q) = \beta.Q - \Phi(\beta)$
Density of probability	$p_x(\xi) = e^{-\langle x, \xi \rangle + \Phi(x)}$ $p_x(\xi) = \dfrac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi}$	$p_\beta(\xi) = e^{-\beta.U(\xi) + \Phi(\beta)}$ $p_\beta(\xi) = \dfrac{e^{-\beta.U(\xi)}}{\int_M e^{-\beta.U(\xi)} d\omega}$
Dual Coordinate Systems	$x \in \Omega$ and $x^* \in \Omega^*$ $x^* = \int_{\Omega^*} \xi.p_x(\xi) d\xi = \dfrac{\int_{\Omega^*} \xi.e^{-\langle \xi, x \rangle} d\xi}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi}$	$\beta \in \mathfrak{g}$ and $Q \in \mathfrak{g}^*$ $Q = \int_M U(\xi).p_\beta(\xi) d\omega = \dfrac{\int_M U(\xi) e^{-\beta.U(\xi)} d\omega}{\int_M e^{-\beta.U(\xi)} d\omega}$ β : Souriau Geometric Temperature U : Souriau Moment map Q : Mean of Souriau Moment Map or Geometric heat
Dual Coordinate Systems	$x^* = \dfrac{\partial \Phi(x)}{\partial x}$ and $x = \dfrac{\partial \Phi^*(x^*)}{\partial x^*}$	$Q = \dfrac{\partial \Phi}{\partial \beta}$ and $\beta = \dfrac{\partial s}{\partial Q}$
Hessian Metric	$ds^2 = -d^2 \Phi(x)$	$ds^2 = -d^2 \Phi(\beta)$
Fisher metric	$I(x) = -E_\xi \left[\dfrac{\partial^2 \log p_x(\xi)}{\partial x^2} \right]$ $I(x) = -\dfrac{\partial^2 \Phi(x)}{\partial x^2} = \dfrac{\partial^2 \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi}{\partial x^2}$	$I(\beta) = -E_\xi \left[\dfrac{\partial^2 \log p_\beta(\xi)}{\partial \beta^2} \right]$ $I(\beta) = -\dfrac{\partial^2 \Phi(\beta)}{\partial \beta^2} = \dfrac{\partial^2 \log \int_M e^{-\beta.U(\xi)} d\omega}{\partial \beta^2}$ $I(\beta) = -\dfrac{\partial^2 \Phi(\beta)}{\partial \beta^2} = -\dfrac{\partial Q}{\partial \beta}$ $K = -\dfrac{\partial Q}{\partial \beta}$: Souriau Geometric Capacity

Geometric heat Capacity / Specific heat

- We observe that the Information Geometry metric could be considered as a generalization of "Heat Capacity". Souriau called it the "Geometric Capacity". This geometric capacity is related to calorific capacity.

$$Q = \frac{\partial \Phi}{\partial \beta} \qquad I(\beta) = -\frac{\partial^2 \Phi(\beta)}{\partial \beta^2} = -\frac{\partial Q}{\partial \beta}$$

$$\beta = \frac{1}{kT} \qquad K = -\frac{\partial Q}{\partial \beta} = -\frac{\partial Q}{\partial T}\left(\frac{\partial \frac{1}{kT}}{\partial T}\right) = \frac{1}{kT^2}\frac{\partial Q}{\partial T}$$

- Q is related to the mean, and K is related to the variance of U

$$Q = \frac{\partial \Phi}{\partial \beta} = \int_M U(\xi).p_\beta(\xi)d\omega = E_\xi[U]$$

$$I(\beta) = -\frac{\partial Q}{\partial \beta} = E_\xi[U^2] - E_\xi[U]^2 = \int_M U(\xi)^2.p_\beta(\xi)d\omega - \left(\int_M U(\xi).p_\beta(\xi)d\omega\right)^2$$

Koszul Information Geometry, Souriau Lie Group Thermodynamics

	Koszul Information Geometry Model	***Souriau Lie Groups Thermodynamics Model***
Convex Cone	$x \in \Omega$ Ω *convex cone*	$\beta \in \Omega$ Ω *convex cone: largest open subset of* $\mathfrak{g}$, *Lie algebra of G, such that* $\int_M e^{-\beta.U(\xi)} d\omega$ *and* $\int_M \xi.e^{-\beta.U(\xi)} d\omega$ *are convergent integrals*
Transformation	$x \rightarrow gx$ with $g \in Aut(\Omega)$	$\beta \rightarrow \bar{a}_{\mathfrak{g}}(\beta)$
Transformation of Potential (non invariant)	$\Phi_\Omega(x) \rightarrow \Phi_\Omega(gx) = \Phi_\Omega(x) + \log(\lvert \det g \rvert)$	$\Phi(\beta) \rightarrow \Phi(\bar{a}_{\mathfrak{g}}(\beta)) = \Phi(\beta) - \theta(a^{-1})\beta$
Transformation of Entropy (invariant)	$\Phi^*_{\Omega^*}(x^*) \rightarrow \Phi^*_{\Omega^*}\left(\frac{\partial \Phi_\Omega(gx)}{\partial x}\right) = \Phi^*_{\Omega^*}(x^*)$ with $x^* = \frac{\partial \Phi_\Omega(x)}{\partial x}$	$s(Q) \rightarrow s'(Q') = \beta'.Q' - \Phi' = \beta.Q - \Phi = s(Q)$ .with $\beta' = \bar{a}_{\mathfrak{g}}(\beta)$ $Q' = \frac{\partial \Phi'}{\partial \beta'} = \frac{\partial(\Phi + \theta(a)\bar{a}_{\mathfrak{g}}(\beta))}{\partial \bar{a}_{\mathfrak{g}}(\beta)} = \bar{a}_{\mathfrak{g}^*}(Q) + \theta(a)$ $\Phi' = \Phi(\beta') = \Phi(\bar{a}_{\mathfrak{g}}(\beta)) = \Phi(\beta) - \theta(a^{-1})\beta$
Information Geometry Metric (invariant)	$I(gx) = -\frac{\partial^2[\Phi_\Omega(x) + \log(\lvert \det g \rvert)]}{\partial x^2} = -\frac{\partial^2 \Phi_\Omega(x)}{\partial x^2} = I(x)$	$I(\bar{a}_{\mathfrak{g}}(\beta)) = -\frac{\partial^2[\Phi(\beta) - \theta(a^{-1})\beta]}{\partial \beta^2} = -\frac{\partial^2 \Phi(\beta)}{\partial \beta^2} = I(\beta)$

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Invariance of Fisher Metric

- **In both Koszul and Souriau models, the Information Geometry Metric and the Entropy are invariant respectively to:**
  - the automosphisms g of the convex cone Ω
  - to $\overline{a}_{\mathfrak{g}}$ adjoint representation of Dynamical group G acting on Ω, the convex cone considered as largest open subset of $\mathfrak{g}$, Lie algebra of G, such that

$\int_M e^{-\beta.U(\xi)} d\omega$ and $\int_M \xi . e^{-\beta.U(\xi)} d\omega$ are convergent integrals.

$$x \to gx \quad \text{with} \quad g \in Aut(\Omega)$$

$$I(gx) = -\frac{\partial^2 \left[\Phi_\Omega(x) + \log(\left|\det g\right|)\right]}{\partial x^2} = -\frac{\partial^2 \Phi_\Omega(x)}{\partial x^2} = I(x)$$

$$\beta \to \overline{a}_{\mathfrak{g}}(\beta)$$

$$I(\overline{a}_{\mathfrak{g}}(\beta)) = -\frac{\partial^2 \left[\Phi(\beta) - \theta(a^{-1})\beta\right]}{\partial \beta^2} = -\frac{\partial^2 \Phi(\beta)}{\partial \beta^2} = I(\beta)$$

THALES

Cartan-Killing Form and Invariant Inner Product

A natural G-invariant inner product could be introduced by Cartan-Killing form:

- **Cartan Generating Inner Product: The following Inner product defined by Cartan-Killing form is invariant by automorphisms of the algebra**

  $\langle x, y \rangle = -B(x, \theta(y))$ **with** $B(x, y) = Tr(ad_x ad_y)$

  where $\theta \in g$ is a Cartan involution (An involution on g is a Lie algebra automorphism θ of g whose square is equal to the identity).

- **The Cartan-Killing form is invariant under automorphisms $\sigma \in Aut(g)$ of the algebra g :**

  $B(\sigma(x), \sigma(y)) = B(x, y)$

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From Cartan-Killing Form to Koszul Information Metric

$$B(x,y) = Tr\left(ad_x ad_y\right)$$

Cartan – Killing Form

$$\langle x,y\rangle = -B\left(x,\theta(y)\right)$$

with $\theta \in g$, Cartan Involution

Koszul Characteristic Function

$$\Phi(x) = -\log \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi \quad \forall x \in \Omega$$

Koszul Entropy

$$\Phi^*(x^*) = \langle x, x^* \rangle - \Phi(x)$$

$$\Phi^*(x^*) = -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi$$

with $x^* = \int_{\Omega^*} \xi . p_x(\xi) d\xi$

Koszul Density

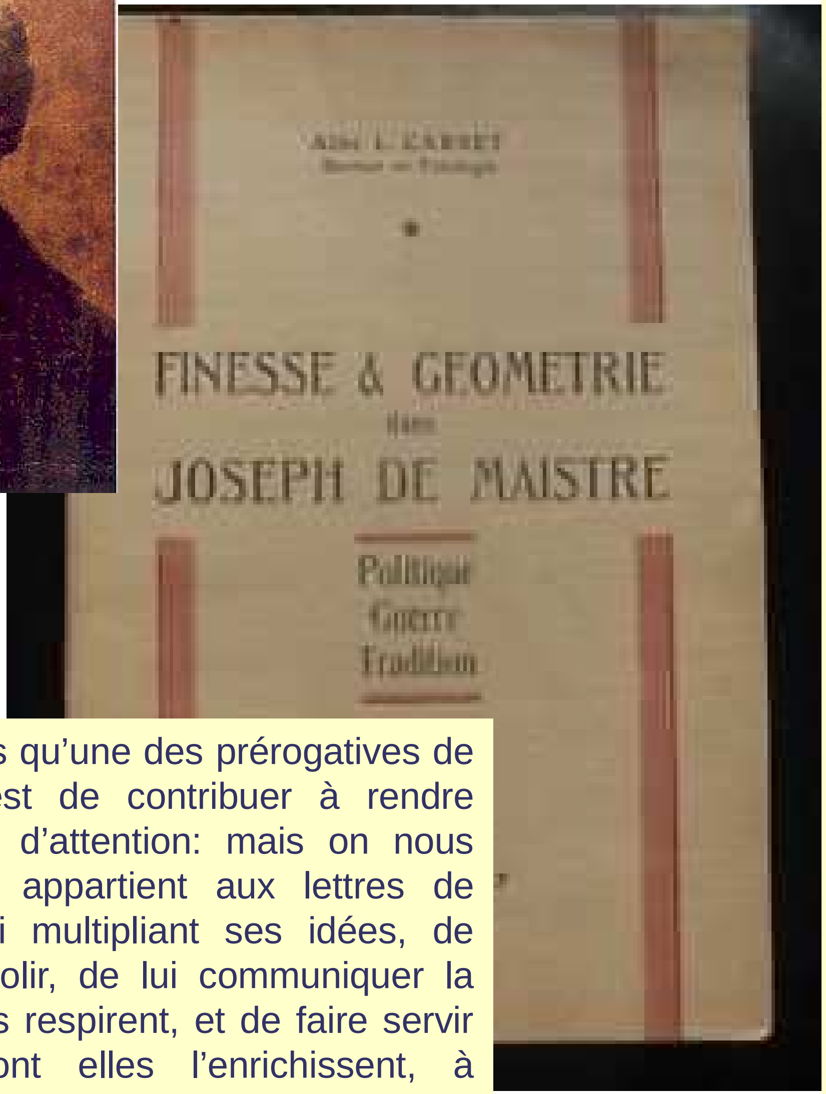
$$p_x(\xi) = \frac{e^{-\langle \xi, x\rangle}}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi}$$

Koszul Metric

$$I(x) = -E_\xi\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right]$$

$$I(x) == -\frac{\partial^2 \Phi(x)}{\partial x^2} = \frac{\partial^2 \log \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi}{\partial x^2}$$

Thank you for your attention



Nous avouerons qu'une des prérogatives de la géométrie est de contribuer à rendre l'esprit capable d'attention: mais on nous accordera qu'il appartient aux lettres de l'étendre en lui multipliant ses idées, de l'orner, de le polir, de lui communiquer la douceur qu'elles respirent, et de faire servir les trésors dont elles l'enrichissent, à l'agrément de la société.

Joseph de Maistre

Si on ajoute que la critique qui accoutume l'esprit, surtout en matière de faits, à recevoir de simples probabilités pour des preuves, est, par cet endroit, moins propre à le former, que ne le doit être la géométrie qui lui fait contracter l'habitude de n'acquiescer qu'à l'évidence; nous répliquerons qu'à la rigueur on pourrait conclure de cette différence même, que la critique donne, au contraire, plus d'exercice à l'esprit que la géométrie: parce que l'évidence, qui est une et absolue, le fixe au premier aspect sans lui laisser ni la liberté de douter, ni le mérite de choisir; au lieu que les probabilités étant susceptibles du plus et du moins, il faut, pour se mettre en état de prendre un parti, les comparer ensemble, les discuter et les peser. Un genre d'étude qui rompt, pour ainsi dire, l'esprit à cette opération, est certainement d'un usage plus étendu que celui où tout est soumis à l'évidence; parce que les occasions de se déterminer sur des vraisemblances ou probabilités, sont plus fréquentes que celles qui exigent qu'on procède par démonstrations: pourquoi ne dirions –nous pas que souvent elles tiennent aussi à des objets beaucoup plus importants ?

Joseph de Maistre