Phase space methods in a continuous tensor product of Hilbert spaces

A. Vourdas

Dept of Computing
University of Bradford
Bradford BD7 1DP
UK

Abstract. A continuum of coupled oscillators is considered, described by a continuous tensor product of Hilbert spaces. The mode position $\mathcal{U}_x$ and the mode momentum $\mathcal{U}_p$ are operators which act collectively on all oscillators. They obey equations of motion which are very similar to those of a harmonic oscillator. Q-functionals are used to introduce entropic quantities that describe correlations among the oscillators. If the system is in an entangled state, the formalism can be used to quantify concepts like the location of entanglement; and the speed with which the entanglement propagates.

Key Words: Quantum systems, phase space methods, entropy

INTRODUCTION

In a recent paper [1] we have introduced the concept of mode phase space in a system comprised of a continuum of coupled oscillators. We have defined mode-position and mode-momentum operators $\mathcal{U}_x$ and $\mathcal{U}_p$, which act collectively on all oscillators. The expectation value of $\mathcal{U}_x$ gives the location of a quantum state in the chain of oscillators. The expectation value of $\mathcal{U}_p$ shows how fast the mode position changes with time. The $\mathcal{U}_x, \mathcal{U}_p$ obey a commutation relation which leads to an uncertainty relation between the uncertainties related to these operators Δx and Δp. From a physical point of view, most of the quantum state is in the region $(\langle \mathcal{U}_x \rangle - \Delta x, \langle \mathcal{U}_x \rangle + \Delta x)$ and oscillators outside this region are close to the vacuum state. The propagation of a quantum state in the chain of oscillators occurs with momenta in the interval $(\langle \mathcal{U}_p \rangle - \Delta p, \langle \mathcal{U}_p \rangle + \Delta p)$.

The corresponding mode phase space $\mathcal{U}_x - \mathcal{U}_p$ is different concept from the phase spaces of the individual oscillators. It describes the collective quantum behaviour of all oscillators. Exponentials of $\mathcal{U}_x$ and $\mathcal{U}_p$ perform mode displacements in it, i.e., they translate a quantum state along the chain of oscillators; and they also change its mode momentum. The mode displacements form a Heisenberg-Weyl group[1].

These ideas apply to all quantum states regardless of whether they are entangled or not. But when the states are entangled, they can be used to quantify the concept of entanglement location and propagation. The entanglement of the state is located in the region $(\langle \mathcal{U}_x \rangle - \Delta x, \langle \mathcal{U}_x \rangle + \Delta x)$ and propagates with momenta in the interval $(\langle \mathcal{U}_p \rangle - \Delta p, \langle \mathcal{U}_p \rangle + \Delta p)$.

From a mathematical point of view, the Hilbert space of our system is a **continuous**

tensor product of Hilbert spaces. There are interesting mathematical problems in this case which have been discussed in [2, 3, 4, 5, 6]. In [1] we have used the exponential Hilbert space approach [7] which links the formalism of a single harmonic oscillator to the formalism of a continuum of oscillators.

In the present paper we review and expand further this work. In section II we introduce various operators and discuss their commutators and their physical meaning. In section III we discuss coherent states. In section IV we introduce partial traces, reduced density matrices and entropies. They use multidimensional integrals in a tensor product of a finite number of Hilbert spaces. In our case we have a continuous tensor product of Hilbert spaces and they become functional integrals. Entropies can be defined in various ways and here they are defined in terms of the Q functionals. In section V we discuss the time evolution of these systems. We conclude in section VI with a discussion of our results.

COLLECTIVE POSITION AND MOMENTUM OPERATORS

We introduce operators

$$\mathcal{U}_\phi = \int dx a^\dagger(x)\phi a(x) \tag{1}$$

where ϕ is an operator acting on the Hilbert space of functions of x. We can show that

$$[\mathcal{U}_\phi, \mathcal{U}_\chi] = \mathcal{U}_{[\phi,\chi]} \tag{2}$$

If $\phi_1, ..., \phi_N$ are generators of a Lie algebra then the $\mathcal{U}_{\phi_1}, ..., \mathcal{U}_{\phi_N}$ form the same Lie algebra.

We note that we do **not** consider operators of the form:

$$\mathcal{W}_\phi = \int dx a^\dagger(x)\phi a^\dagger(x); \qquad \mathcal{V}_\phi = \int dx a(x)\phi a(x) \tag{3}$$

They do not obey Eq(2).

Special cases of the operators (1) are the mode position and momentum operators

$$\mathcal{U}_x = \int_{-\infty}^{\infty} dx x a^\dagger(x) a(x); \qquad \mathcal{U}_p = -i\int_{-\infty}^{\infty} dx a^\dagger(x)\partial_x a(x) \tag{4}$$

Other special cases of (1) are the operators

$$\mathcal{U}_{\mathbf{1}} = n_T = \int_{-\infty}^{\infty} dx a^\dagger(x) a(x); \qquad \mathcal{U}_N = \frac{1}{2}\int_{-\infty}^{\infty} dx a^\dagger(x)(x^2 - \partial_x^2 - 1)a(x) \tag{5}$$

n_T is the total number of photons.

These operators are collective variables, acting on all oscillators. According to Eq(2) they obey the commutation relation

$$[\mathcal{U}_x, \mathcal{U}_p] = i\mathcal{U}_{\mathbf{1}}; \qquad [\mathcal{U}_x, \mathcal{U}_{\mathbf{1}}] = [\mathcal{U}_p, \mathcal{U}_{\mathbf{1}}] = 0 \tag{6}$$

The commutators between the operator $\mathcal{U}_N$ and the operators $\mathcal{U}_x$ and $\mathcal{U}_p$ are:

$$[\mathcal{U}_N, \mathcal{U}_x] = -i\mathcal{U}_p; \qquad [\mathcal{U}_N, \mathcal{U}_p] = i\mathcal{U}_x \tag{7}$$

We have explained in [1] that the commutation relation of Eq.(6) leads to an uncertainty relation. In order to quantify this we consider the operators

$$\mathcal{U}_{x^2} = \int dx x^2 a^\dagger(x) a(x); \qquad \mathcal{U}_{p^2} = -\int dx a^\dagger(x) \partial_x^2 a(x) \tag{8}$$

We define the mode position uncertainty as:

$$\Delta x \quad = \quad \left[\frac{\langle \mathcal{U}_{x^2} \rangle}{\langle n_T \rangle} - \left(\frac{\langle \mathcal{U}_x \rangle}{\langle n_T \rangle} \right)^2 \right]^{1/2} \tag{9}$$

In a similar way we define the Δp. The uncertainty relation states that

$$(\Delta x \Delta p)^2 \geq \frac{1}{4} \tag{10}$$

We stress that this uncertainty relation refers to the whole system and not to a particular oscillator. Physically the expectation value of $\mathcal{U}_x$ gives the location of a quantum state in the chain of oscillators in the sense that oscillators outside the region $(\langle \mathcal{U}_x \rangle - \Delta x, \langle \mathcal{U}_x \rangle + \Delta x)$ are close to the vacuum state. The expectation value of $\mathcal{U}_p$ shows how fast the mode position changes with time. The propagation of the quantum state in the chain of oscillators occurs with momenta in the interval $(\langle \mathcal{U}_p \rangle - \Delta p, \langle \mathcal{U}_p \rangle + \Delta p)$.

COHERENT STATES

Displacement operators are given by

$$D(\{z(x)\}) = \exp\left[\int_{-\infty}^{\infty} dx (z(x) a^\dagger(x) - z^*(x) a(x)) \right] \tag{11}$$

Using them we define coherent states as

$$|\{z(x)\}\rangle_{\rm coh} = D(\{z(x)\})|0\rangle \tag{12}$$

where $|0\rangle$ is the vacuum in H. The total number of photons in these coherent states is

$${}_{\rm coh}\langle \{z(x)\}|n_T|\{z(x)\}\rangle_{\rm coh} = (z(x), z(x)) \tag{13}$$

where

$$(w(x), z(x)) \equiv \int_{-\infty}^{\infty} dx w^*(x) z(x) \tag{14}$$

The overlap of two coherent states is:

$${}_{\rm coh}\langle \{z(x)\}|\{w(x)\}\rangle_{\rm coh} = \exp\left[-\frac{1}{2}(w(x), w(x)) - \frac{1}{2}(z(x), z(x)) + (z(x), w(x)) \right] \tag{15}$$

The resolution of the identity in terms of these coherent states, is given by the functional integral:

$$\int \mathcal{D}^2[z(x)]\ |\{z(x)\}\rangle_{\rm coh}\ {}_{\rm coh}\langle\{z(x)\}| = \mathbf{1}; \qquad \mathcal{D}^2[z(x)] = \prod_{x\in R} \frac{d^2 z(x)}{\pi} \tag{16}$$

We consider a state described with the density matrix ρ. The corresponding Q-functional is

$$Q[\{z(x)\}] =_{\rm coh} \langle\{z(x)\}|\rho|\{z(x)\}\rangle_{\rm coh} \tag{17}$$

and obeys the relation

$$\int \mathcal{D}^2[z(x)]\ Q[\{z(x)\}] = 1 \tag{18}$$

In the case of a pure state $|f\rangle$

$$\begin{aligned} |f\rangle &= \int \mathcal{D}^2[z(x)]\ |\{z(x)\}\rangle_{\rm coh}\ f[\{z(x)\}] \\ f[\{z(x)\}] &\equiv\ {}_{\rm coh}\langle\{z(x)\}|f\rangle \end{aligned} \tag{19}$$

we get

$$Q[\{z(x)\}] = |f[\{z(x)\}]|^2. \tag{20}$$

As an example we consider the coherent states $|\{w(x)\}\rangle_{\rm coh}$. Using Eq.(15) we find that the corresponding Q-functional is

$$Q[\{z(x)\}] = \exp\left[-(w(x),w(x)) - (z(x),z(x)) + 2\Re(z(x),w(x))\right] \tag{21}$$

For the vacuum, $w(x) = 0$ and the corresponding Q-functional is

$$Q[\{z(x)\}] = \exp\left[-(z(x),z(x))\right] \tag{22}$$

PARTIAL TRACES

We consider an interval $I \subset R$. Examples are $I = (-\infty, 1)$ or $I = (1,2)\bigcup(3,4)$, etc. We introduce the 'reduced Hilbert space'

$$H(I) = \bigotimes_{x\in I} \mathcal{H}(x) \tag{23}$$

The term reduced indicates that x takes values in a subset of R. Many of the above relations for states, in H, are also valid for states in H(I).

When x is restricted to an interval I, we use a notation which indicates the interval explicitly. For example, we denote the coherent states in $H(I)$ as $|\{z(x); x\in I\}\rangle_{\rm coh}$. It is easily seen that the partial trace of the density matrix $|\{z(x)\}\rangle_{\rm coh\ coh}\langle\{z(x)\}|$ with respect to the modes labeled with $x \in R-I$ is

$$\mathrm{Tr}_{R-I}\left[|\{z(x)\}\rangle_{\rm coh\ coh}\langle\{z(x)\}|\right] = |\{z(x); x \in I\}\rangle_{\rm coh\ coh}\langle\{z(x); x \in I\}| \tag{24}$$

The overlap of these coherent states is given by:

$$\begin{aligned} {}_{\rm coh}\langle\{z(x); x \in I\}|\{w(x); x \in I\}\rangle_{\rm coh} &= \exp[-\frac{1}{2}(w(x), w(x))_I \\ &- \frac{1}{2}(z(x), z(x))_I + (z(x), w(x))_I] \end{aligned} \tag{25}$$

where

$$(z(x), w(x))_I = \int_{x \in I} dx z^*(x) w(x) \tag{26}$$

The resolution of the identity within $H(I)$ is

$$\begin{aligned} &\int \mathcal{D}^2[z(x); x \in I]\ |\{z(x); x \in I\}\rangle_{\rm coh\ coh}\langle\{z(x); x \in I\}| = \mathbf{1}_I \\ &\mathcal{D}^2[z(x); x \in I] = \prod_{x \in I} \frac{d^2 z(x)}{\pi} \end{aligned} \tag{27}$$

We consider a state described with the density matrix ρ. We call $\rho(I)$ the reduced density matrix which is the partial trace of ρ with respect to the modes labeled with $x \in R-I$

$$\rho(I) = \mathrm{Tr}_{R-I}\rho \tag{28}$$

The corresponding Q-functional is

$$\begin{aligned} Q[\{z(x); x \in I\}] &\equiv {}_{\rm coh}\langle\{z(x); x \in I\}|\rho(I)|\{z(x); x \in I\}\rangle_{\rm coh} \\ &= \int \mathcal{D}^2[z(x); x \in R-I]\ Q[\{z(x)\}] \end{aligned} \tag{29}$$

Using Eq.(18) we show that it obeys the relation

$$\int \mathcal{D}^2[z(x); x \in I]\ Q[\{z(x); x \in I\}] = 1 \tag{30}$$

We define the differential entropy corresponding to $Q[\{z(x); x \in I\}]$ as

$$S(I) = -\int \mathcal{D}^2[z(x); x \in I]\ Q[\{z(x); x \in I\}] \ln Q[\{z(x); x \in I\}] \tag{31}$$

Let I_1, I_2 be two non-overlaping intervals. Then

$$J(I_1, I_2) \equiv S(I_1) + S(I_2) - S(I_1 \cup I_2) \geq 0 \tag{32}$$

We note that eqs(27),(29),(30),(31) involve functional integrals. In practical calculations functional integrals can be calculated analytically in the special case of Gaussian integrals. In the case of ‘almost Gaussian’ integrals we can perform perturbative techniques. In more general cases we can use ‘lattice techniques’ where we approximate a functional integral with a finite-dimensional integral which we calculate numerically.

Factorizable states

A state desrcibed with the density matrix ρ is factorizable if **for every** non-overlaping intervals I_1, I_2

$$\rho(I_1 \cup I_2) = \rho(I_1) \otimes \rho(I_2) \tag{33}$$

A direct consequence of this is that for factorizable states

$$r(I_1, I_2) \equiv \frac{Q[\{z(x); x \in I_1 \cup I_2\}]}{Q[\{z(x); x \in I_1\}]\, Q[\{z(x); x \in I_2\}]} = 1 \tag{34}$$

and

$$J(I_1, I_2) = 0. \tag{35}$$

For general states, the ratio $r(I_1, I_2)$ is different than one and the $J(I_1, I_2)$ is a positive number. Both the $r(I_1, I_2)$ and the $J(I_1, I_2)$ can be used as measures of (classical and quantum) correlations between the oscillators in the intervals I_1 and I_2.

TIME EVOLUTION

We consider the Hamiltonian

$$h = \mathcal{U}_N \tag{36}$$

where $\mathcal{U}_N$ has been given in Eq.(**??**). Using Eq.(2) we show that

$$\begin{aligned} \partial_t \mathcal{U}_x &= i[h, \mathcal{U}_x] = \mathcal{U}_p \\ \partial_t \mathcal{U}_p &= i[h, \mathcal{U}_p] = -\mathcal{U}_x \end{aligned} \tag{37}$$

They are similar to the equations of motion of a harmonic oscillator.

For coherent states we show that

$$\exp[ith]\, |\{z(x,0)\}\rangle_{\text{coh}} = |\{z(x,t)\}\rangle_{\text{coh}} \tag{38}$$

where

$$z(x,t) = \int z(y,0)\mathcal{K}(y,x;t)dy; \qquad \mathcal{K}(y,x;t) = \sum_N h_N(y)h_N(x)e^{iNt} \tag{39}$$

Equivalent to Eq.(39) is the fact that $z(x,t)$ obeys the Schrödinger equation

$$\frac{1}{2}\left(-\partial_x^2 + x^2\right) z(x,t) = i\partial_t z(x,t) \tag{40}$$

Consequently

$$\partial_t |z(x,t)|^2 = \partial_x J_x; \qquad J_x = \frac{i}{2}\left[z^*(x,t)\partial_x z(x,t) - z(x,t)\partial_x z^*(x,t)\right] \tag{41}$$

$|z(x,t)|^2$ is the density of photons and its integral over x gives the total number of photons as we have seen in Eq.(13). J_x is current of photons between the various oscillators and Eq(41) is the conservation relation. We stress that this is valid for the operators of Eq.(1) that we use in this paper; it would not be valid for the operators of Eq.(3).

Example

All of the above concepts apply to all states; regardless of whether they are entangled or not. However if the states are entangled, they can be used to quantify the location and propagation of entanglement. In order to exemplify this we consider a system described with the Hamiltonian h of Eq.(36), which at $t=0$ is in the following entangled state which is a superposition of two coherent states:

$$\begin{aligned} |s(0)\rangle &= \mathcal{N}\left[|\{\zeta_1 z_{\rm gau}(x;A)\}\rangle_{\rm coh} + |\{\zeta_2 z_{\rm gau}(x;A)\}\rangle_{\rm coh}\right] \\ z_{\rm gau}(x;A) &= \pi^{-1/4}\exp\left[-\frac{1}{2}x^2 + 2^{1/2}Ax - AA_R\right] \end{aligned} \tag{42}$$

Here we consider coherent states with amplitude which in the x-representation, has Gaussian distribution among the various modes. The complex factors ζ_1, ζ_2 have been inserted so that the average number of photons in these coherent states is $|\zeta_1|^2$ and $|\zeta_2|^2$. The normalization coefficient $\mathcal{N}$ is given by

$$\mathcal{N} = \left[2 + 2e^{-\frac{1}{2}|\zeta_1-\zeta_2|^2}\cos\phi\right]^{-1/2}; \qquad \phi = \Im(\zeta_1^*\zeta_2) \tag{43}$$

The state $|s(0)\rangle$ evolves in time as follows:

$$|s(t)\rangle = \mathcal{N}\left[|\{\zeta_1 z_{\rm gau}(x;Ae^{it})\}\rangle_{\rm coh} + |\{\zeta_2 z_{\rm gau}(x;Ae^{it})\}\rangle_{\rm coh}\right] \tag{44}$$

In this example the expectation values $\langle \mathcal{U}_x\rangle$ and $\langle \mathcal{U}_p\rangle$ and the corresponding uncertainties are

$$\begin{aligned} &\langle \mathcal{U}_x\rangle = 2^{1/2}\langle n_T\rangle |A|\cos(\theta_A + t); \qquad \langle \mathcal{U}_p\rangle = 2^{1/2}\langle n_T\rangle |A|\sin(\theta_A + t) \\ &\Delta x = \Delta p = 2^{-1/2} \end{aligned} \tag{45}$$

where

$$\langle n_T\rangle = \frac{|\zeta_1 + \zeta_2|^2}{2 + 2e^{-\frac{1}{2}|\zeta_1-\zeta_2|^2}\cos\phi} \tag{46}$$

The entanglement is located mostly within the interval $(\langle \mathcal{U}_x\rangle - \Delta x, \langle \mathcal{U}_x\rangle + \Delta x)$ which performs an oscillatory motion in time.

DISCUSSION

We have considered a continuum of coupled oscillators with Hilbert space which is the continuous tensor product of Hilbert spaces. We have introduced the mode position and mode momentum operators $\mathcal{U}_x$ and $\mathcal{U}_p$ which act collectively on all oscillators. Their expectation values describe the average position of a quantum state in the line of oscillators; and also the average momentum with which it propagates. For the Hamiltonian of Eq.(36), we have shown in Eq.(37) that the equations of motion are similar to those of a harmonic oscillator.

It is interesting to understand correlations and entanglement between the various oscillators in the present context. We have defined the entropic quantity $J(I_1, I_2)$ of Eq.(32) which is equal to zero for factorizable states. Non-zero values of $J(I_1, I_2)$ indicate (classical and quantum) correlations between the oscillators in I_1 and the oscillators in I_2. Further work is required in order to distinguish between classical and quantum correlations in the present context.

The $\mathcal{U}_x$ and $\mathcal{U}_p$ can be used to quantify the location of entanglement and the speed with which entanglement propagates. For example, in the case considered in Eq.(44) the entanglement is located mostly within the interval $(\langle \mathcal{U}_x \rangle - \Delta x, \langle \mathcal{U}_x \rangle + \Delta x)$ defined in Eq.(45) and oscillates in time. We have not defined any measures of entanglement and further work is required in this direction.

The work can be used for the study of collective quantum phenomena in systems comprised of an infinite number of oscillators.

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