Information Intrinsic Geometric Flows

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Abstract. Geometric Flow Theory is cross fertilized by diverse elements coming from Pure Mathematic and Mathematical Physic, but its foundation is mainly based on Riemannian Geometry, as explained by M. Berger in a recent panoramic view of this discipline [4], its extension to complex manifolds, the Erich Kähler's Geometry, vaunted for its unabated vitality by J.P. Bourguignon [6], and Minimal Surface Theory [8,9]. This paper would like to initiate seminal studies for applying intrinsic geometric flows in the framework of information geometry theory. More specifically, after having introduced Information metric deduced for Complex Auto-Regressive (CAR) models from Fisher Matrix (Siegel Metric and Hyper-Abelian Metric from Entropic Kähler Potential), we study asymptotic behavior of reflection coefficients of CAR models driven by intrinsic Information geometric Kähler-Ricci and Calabi flows. These Information geometric flows can be used in different contexts to define distance between CAR models interpreted as geodesics of Entropy Manifold. We conclude with potential application of Intrinsic Geometric Flow on Gauss Map to transform Manifold of any dimension by mean of Generalized Weierstrass Formula introduced by Kenmotsu [8] that can represent arbitrary surfaces with non-vanishing mean curvature in terms of the mean curvature function and the Gauss map. One of the advantages of the generalized formulae is that they allow to construct a new class of deformations of surfaces by use of Intrinsic Geometric Flow on Gauss Map. We conclude with the Heat equation interpretation in the framework of Information Geometry.

Keywords: Chentsov Information Geometry, Siegel Metric, Hyper-Abelian Metric, Entropic Kähler Potential, Intrinsic Geometric Flow, Kähler-Ricci Flow, Calabi Flow, Gauss map,
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SIEGEL METRIC FOR COMPLEX AUTOREGRESSIVE MODEL

In Chentsov Information geometry theory, we consider families of parametric density functions $G_\Theta = \{p(./\theta) : \theta \in \Theta\}$ with $\Theta = [\theta_1 \quad \cdots \quad \theta_n]^T$, from which we can define a Riemannian Manifold Structure by meam of Fisher Information matrix :

$$g_{ij}(\theta) = E_\theta\left[\frac{\partial \ln p(./\theta)}{\partial \theta_i}\frac{\partial \ln p(./\theta)}{\partial \theta_j^*}\right] \text{ with metric } ds^2 = \sum_{i,j=1}^{n} g_{ij}(\theta).d\theta_i.d\theta_j^* .$$

We demonstrate easily that this Fisher metric is equivalent to the Siegel metric, introduced by Siegel in the 60's in the framework of Symplectic Geometry, in the case of Complex Multivariate Gaussian Law (Complex Circular Process):

$$p(X/R_n, m_n) = (2\pi)^{-n}|R_n|^{-1} e^{-Tr[\hat{R}_n.R_n^{-1}]} \text{ with } \hat{R}_n = (X - m_n)(X - m_n)^+ \text{ such that } E[\hat{R}_n] = R_n$$

and only consider random process with zero mean $m_n = E[X] = 0$, the Fisher matrix is reduced to $g_{ij}(\theta) = Tr[(R_n.\partial_i R_n^{-1}).(R_n.\partial_j R_n^{-1})]$ and metric $\boxed{ds^2 = Tr[(R_n.dR_n^{-1})^2]}$

In case of Complex Autoregressive models (CAR model), we can exploit the following specific blocks structure of covariance matrices :

$$\mathbf{R}_{\mathbf{n}}^{(1)} = \begin{bmatrix} \alpha_{n-1}^{(1)-1} + A_{n-1}^{(1)+}.R_{n-1}^{(1)}.A_{n-1}^{(1)} & -A_{n-1}^{(1)+}.R_{n-1}^{(1)} \\ -R_{n-1}^{(1)}.A_{n-1}^{(1)} & R_{n-1}^{(1)} \end{bmatrix} = R_n^{(1)1/2}.R_n^{(1)1/2+} \text{ with } \mathbf{R}_{\mathbf{n}}^{(1)1/2} = \begin{bmatrix} \frac{1}{\sqrt{\alpha_{n-1}^{(1)}}} & -A_{n-1}^{(1)+}.R_{n-1}^{(1)1/2} \\ 0 & R_{n-1}^{(1)1/2} \end{bmatrix}$$

$$\mathbf{R}_{\mathbf{n}}^{(2)-1} = \begin{bmatrix} \alpha_{n-1}^{(2)} & \alpha_{n-1}^{(2)}.A_{n-1}^{(2)+} \\ \alpha_{n-1}^{(2)}.A_{n-1}^{(2)+} & R_{n-1}^{(2)} + \alpha_{n-1}^{(2)}.A_{n-1}^{(2)}A_{n-1}^{(2)+} \end{bmatrix} \text{ where } \alpha_n^{-1} = \left[1-|\mu_n|^2\right].\alpha_{n-1}^{-1} \text{ and } A_n = \begin{bmatrix} A_{n-1} \\ 0 \end{bmatrix} + \mu_n.\begin{bmatrix} A_{n-1}^{(-)} \\ 1 \end{bmatrix}$$

We can deduce the expression of the Siegel metric previously defined :

$$ds_n^2 = Tr\left[\left(R_n.dR_n^{-1}\right)^2\right] = \left(\frac{d\alpha_{n-1}}{\alpha_{n-1}}\right)^2 + \alpha_{n-1}.dA_{n-1}^+.R_{n-1}.dA_{n-1} + tr\left[\left(R_{n-1}.dR_{n-1}^{-1}\right)^2\right]$$

that leads to : $ds_n^2 = ds_{n-1}^2 + \left(\frac{d\alpha_{n-1}}{\alpha_{n-1}}\right)^2 + \alpha_{n-1}.dA_{n-1}^+.R_{n-1}.dA_{n-1}$

We can then define a new hyperbolic distance between CAR models as Inferior Bound of this metric : $\boxed{ds_n^2 > \sum_{k=0}^{n-1}\left(\frac{d\alpha_k}{\alpha_k}\right)^2 + \sum_{k=1}^{n-1}\frac{|d\mu_k|^2}{1-|\mu_k|^2}}$

INFORMATION METRIC FROM ERICH KÄHLER GEOMETRY

Natural extension of Riemannian Geometry to Complex Manifold has been introduced by a seminal paper of Erich Kähler during 30th 's of last century. We can easily apply this geometric framework for information metric definition. Let a complex Manifold M^n of dimension n, we can associate a Kählerian metric, which can be locally defined by its definite positive Riemannian form : $ds^2 = 2\sum_{i,j=1}^{n} g_{i\bar{j}}.dz^i dz^{\bar{j}}$ with $\left[g_{i\bar{j}}\right]_{i,j}$ an Hermitian definite positive matrix. Kähler assumption assumes that we can define a Kähler potential Φ, such that $g_{i\bar{j}} = \frac{\partial^2\Phi}{\partial z^i \partial z^{\bar{j}}}$. Fundamental relation, given by Erich Kähler, is that Ricci tensor can be expressed by : $R_{i\bar{j}} = -\frac{\partial^2 \log\left(\det g_{k\bar{l}}\right)}{\partial z_i \partial \bar{z}_j}$ with the associated scalar curvature $R = \sum_{k,l=1}^{n} g^{k\bar{l}} R_{k\bar{l}}$. One important geometric flow, in physic & mathematic, is the Kähler-Ricci flow which drive the evolution of the metric by : $\frac{\partial g_{ij}}{\partial t} = -R_{ij} + \frac{1}{n} R g_{ij}$. This flow converges to a Kähler-Einstein metric $R_{i\bar{j}} = k_0 g_{k\bar{l}}$, or equivalently to : $-\frac{\partial^2 \log\left(\det g_{k\bar{l}}\right)}{\partial z_i \partial \bar{z}_j} = k_0 \frac{\partial^2\Phi}{\partial z^i \partial z^{\bar{j}}}$, known as Monge-Ampère equation : $\det(g_{k\bar{l}}) = |\psi|^2 e^{-k_0\Phi}$ where Φ is a Kähler potential and ψ a non specified holomorphic function, but that could be reduced to unity : if $k_0 \neq 0$) by choice of a new Φ potential, or if $k_0 = 0$ by local holomorphic coordinates selection so that volume $\det(g_{k\bar{l}})$ is reduced to 1 (cancellation of Ricci tensor is existence condition of this coordinates system).

In case of Complex Auto-Regressive (CAR) models, if we choose as Kähler potential Φ with $g_{i\bar{j}} = \frac{\partial^2 \Phi}{\partial z^i \partial z^{\bar{j}}}$, the Entropy of the process expressed according to reflexion coeffient in the unit Polydisk $\{z / |z_k| < 1 \ \forall k = 1,...n\}$, the Kähler potential is given by $: \Phi = \sum_{k=1}^{n-1} \rho_k \ln\left[1 - |z_k|^2\right] = \ln K_D(z,z)$ and Bergman kernel $K_D(z,z) = \prod_{k=1}^{n-1} \left(1 - |z_k|^2\right)^{\rho_k}$. Very surprisingly, this case was the first example of potential studied by Erich Kähler in his seminal paper, named by Erich Kähler Hyper-Abelian case. This choice of Kähler potential as Entropy of CAR model, can be justified by remarking that Entropy Hessian along one direction in the tangent plane of parametric manifold is a definite positive form that can be considered as a Kählerian differential metric: $g_{ij}^{(H)}(\theta) = -\frac{\partial^2 H(P_\theta)}{\partial\theta_i . \partial\theta_j} \Rightarrow ds_H^2 = \sum_{i,j=1}^{n} g_{ij}^H(\theta).d\theta_i.d\theta_j$.

In case of Complex Autoregressive models, with as previously Whishart density, Entropy is given by :

$$H_n = -\int P(X_n / m_n, R_n) \ln\left[P(X_n / m_n, R_n)\right] dX_n = \ln|R_n| + n \ln(\pi.e) \text{ with } X_n = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$$

If we use the blocks structure of covariance matrix in case of CAR models, we obtain the Entropy expressed according to reflection coefficients :

$$\boxed{H_n = \sum_{k=1}^{n-1} (k-n).\ln\left[1 - |\mu_k|^2\right] + n.\ln\left[\pi.e.\alpha_0^{-1}\right] \text{ with } \alpha_0^{-1} = P_0 = \frac{1}{n}\sum_{k=1}^{n} |x_k|^2}$$

The Kähler metric is given by Hessien of Entropy, where Entropy is considered as Kähler potential $g_{ij}(\theta) = -\frac{\partial^2 \Phi(p)}{\partial\theta_i \partial\theta_j^*}$ with $\Phi(p) = \int p(x/\theta).\ln p(x/\theta).dx$

Let $\theta^{(n)} = \begin{bmatrix} P_0 & \mu_1 & & \mu_{n-1} \end{bmatrix}^T$, we have then $g_{11} = nP_0^{-2}$, $g_{ij} = \frac{(n-i).\delta_{ij}}{\left(1 - |\mu_i|^2\right)^2}$ and metric of this Hyper-Abelian Case : $\boxed{ds_n^2 = n.\left(\frac{dP_0}{P_0}\right)^2 + \sum_{i=1}^{n-1} (n-i) \frac{|d\mu_i|^2}{\left(1 - |\mu_i|^2\right)^2}}$

INFORMATION RICCI & CALABI FLOWS FOR COMPLEX AR

Historical Root of Ricci flow can be found in Hilbert work on General Relativity. The "Hilbert Action" S is defined as the integral of scalar Curvature *R* on the Manifold M^n : $S(g) = \int_{M^n} R.\sqrt{\det(g)}.d^n x = \int_{M^n} R.d\eta$ with volume $V(g) = \int_{M^n} d\eta$ and

$R=\sum_{\mu}\sum_{\nu} g^{\mu\nu}.R_{\mu\nu}$. Fundamental theorem of Hilbert said that for $n\geq 3$, $S(g)$ is minimal with $V(g)=cste$ if $R(g)$ is constant and g is an Einstein metric: $R_{ij}=\frac{1}{n}R.g_{ij}$. So the more natural geometric flow that converges to Einstein metric is given by : $\frac{\partial g_{ij}}{\partial t}=2\left[-R_{ij}+\frac{1}{n}Rg_{ij}\right]$, but unfortunately this flow exhibits some convergence problems in finite time. That the main raison why R. Hamilton has introduced the following normalized Ricci flow :

$\frac{\partial g_{ij}}{\partial t}=2\left[-R_{ij}+\frac{1}{n}r.g_{ij}\right]$ **with** $r=\left(\int_{M^n} Rd\eta\right)/\left(\int_{M^n} d\eta\right)$. This flow can be interpreted as Fourrier Heat Operator acting on metric g , $\frac{\partial g_{ij}}{\partial t}=\Delta_g g_{ij}$, by using isothermal coordinates introduced by G. Darmois. More specifically A. Lichnerowicz has proved that in isothermal coordinates, we have : $-2.R_{ij}=\sum_k\sum_l g^{kl}.\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}+Q_{ij}\left(g^{-1},\partial g\right)$.

Recently, Ionnis Bakas has established connection between 2 distinct classes of geometric deformations, Ricci and Calabi Flows, respectively of 2nd and 4th order. Calabi flow preserves the Kähler class and minimize the quadratic curvature for extremal metrics. All geometric flows share some common qualitative features with the linear heat flow equation. The Ricci and Calabi flows correspond to intrinsic deformations. Let M denote a complex n-dimensional manifold, that admits a Kähler metric g with : $ds^2=2\sum_{i,j} g_{i\bar{j}}dz_i dz_j^*$. The Calabi flow, defined for Kähler Manifolds, is given by : $\frac{\partial g_{i\bar{j}}}{\partial t}=\frac{\partial^2 R}{\partial z_i \partial z_j^*}$. If we use that $g_{i\bar{j}}=\frac{\partial^2 \phi\left(z,z^*\right)}{\partial z_i \partial z_j^*}$, then it is equivalent to $\frac{\partial \phi}{\partial t}=R-\overline{R}$. It preserves the volume and the Kähler class of the metric. Critical points of the flow are called extremal metrics. The following functional $S(g)=\int_M R^2(g)dV(g)$ decreases monotonically along the Calabi Flow and the minimum of the functional $S(g)$ is given by Schwarz's inequality :

$S(g)=\left(\int_M R(g)dV(g)\right)^2/\int_M dV(g)$. Bakas [1] has recently proved that there is a relation between the Ricci and Calabi Flows on Kähler manifolds of arbitrary dimension that manifests by squaring the time evolution operator. The proof is given by taking time derivative of the Ricci Flow :

$$\frac{\partial g_{i\bar{j}}}{\partial t} = -R_{i\bar{j}} \Rightarrow \begin{cases} \frac{\partial^2 g_{i\bar{j}}}{\partial t^2} = -\frac{\partial R_{i\bar{j}}}{\partial t} \\ R_{i\bar{j}} = -\frac{\partial^2 \log(\det(g))}{\partial z_i \partial z_j^*} \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 g_{i\bar{j}}}{\partial t^2} = \frac{\partial^2}{\partial z_i \partial z_j^*}\left(\frac{\partial \log(\det(g))}{\partial t}\right) \\ \frac{\partial \log(\det(g))}{\partial t} = -\sum_{i,j} g^{i\bar{j}} R_{i\bar{j}} = -R \end{cases} \Rightarrow \frac{\partial^2 g_{i\bar{j}}}{\partial t^2} = -\frac{\partial^2 R}{\partial z_i \partial z_j^*}$$

If the second derivative of the metric with respect to the Ricci time is identified with minus its first derivative to the Calabi time, the two flows are the same : $\frac{\partial^2}{\partial t_R^2} = -\frac{\partial t}{\partial t_C}$

Considering the two dimensions case, and a system of conformally flat (Kähler) coordinates : $ds^2 = 2e^{\Phi(z,z^*,t)} dz.dz^*$. The only non-vanishing components of the Ricci curvature tensor is : $R_{zz^*} = -\frac{\partial^2 \Phi}{\partial z \partial z^*}$ because $R_{zz^*} = -\frac{\partial^2 \log(\det(g))}{\partial z \partial z^*}$ **and** $g_{zz^*} = e^{\Phi}$

For the Ricci flow :

$$\frac{\partial g_{i\bar{j}}}{\partial t} = -R_{i\bar{j}} \Rightarrow \frac{\partial e^{\Phi}}{\partial t} = \frac{\partial^2 \Phi}{\partial z \partial z^*} \Rightarrow e^{\Phi} \frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial z \partial z^*} \Rightarrow \frac{\partial \Phi}{\partial t} = \Delta \Phi \quad \textbf{with} \quad \Delta . = e^{-\Phi} \frac{\partial^2 .}{\partial z \partial z^*}$$

For the Calabi flow :

$$\frac{\partial g_{i\bar{j}}}{\partial t} = \frac{\partial^2 R}{\partial z \partial z^*} \quad \textbf{and} \quad R = \sum_{i,j} g^{i\bar{j}} R_{i\bar{j}} = -e^{-\Phi} \frac{\partial^2 \Phi}{\partial z \partial z^*} = -\Delta \Phi \Rightarrow \frac{\partial e^{\Phi}}{\partial t} = -\frac{\partial^2 \Delta \Phi}{\partial z \partial z^*} \Rightarrow \frac{\partial \Phi}{\partial t} - \Delta \Delta \Phi$$

If we use definition of metric g as previously for CAR models, as derived from Entropic Kähler potential in case of a Complex Autoregressive models, then we can express Ricci tensor : $\boxed{R_{k\bar{l}} = -2\delta_{kl}\left(1 - |\mu_k|^2\right)^{-2} \quad \textbf{for} \quad k = 2,...,n-1 \ \textbf{and} \ R_{11} = -2P_0^{-2}}$

Identically, we can deduce scalar curvature $R = \sum_{k,l} g^{k\bar{l}} . R_{k\bar{l}}$ of CAR models :

$\boxed{R = -2.\left[n^{-1} + \sum_{j=1}^{n-1}(n-j)^{-1}\right] = -2.\left[\sum_{j=0}^{n-1}(n-j)^{-1}\right]}$ (this curvature diverges when n tends to infinity).We can observe that we have an Einstein metric but more generally defined as : $\boxed{\left[R_{i\bar{j}}\right] = B^{(n)}\left[g_{i\bar{j}}\right] \ \textbf{with} \ R = Tr\left[B^{(n)}\right]}$ where $B^{(n)} = -2diag\left\{..,(n-i)^{-1},..\right\}$

If we study Kähler-Ricci flow acting on reflection coefficients, we have :

$\frac{\partial \ln\left(1 - |\mu_i|^2\right)}{\partial t} = -\frac{1}{(n-i)}$ and $\frac{\partial \ln P_0}{\partial t} = \frac{1}{n}$. From which, we obtain the behaviour of refelction coefficient in asymptotic case :

$|\mu_i(t)|^2 = 1 - \left(1 - |\mu_i(0)|^2\right).e^{-\frac{t}{(n-i)}} \xrightarrow[t \to \infty]{} |\mu_i(t)|^2 = 1$ that converges to unit circle.

If We introduce Calabi flow for CAR model, solution is defined as steady state of the PDE equation : $\frac{\partial \psi}{\partial t} = R_{\psi} - \overline{R}$ with ψ Kähler potential associated to the Kähler metric $g_{i\bar{j}} = \frac{\partial^2 \psi}{\partial z^i \partial z^{\bar{j}}}$, R_{ψ} scalar curvature and $\overline{R}$ its mean value on the Manifold. This can be used to defined shortest path between two parametric models. If we consider the path $\psi(t)\ (0 \le t < 1)$ and if we assume the existence of Kähler potential $\{\phi(s,t) : 0 \le t < 1\}$ driven by Calabi Flow : $\frac{\partial \phi}{\partial t} = R(\phi) - \overline{R}$ **and** $\phi(0,t) = \psi(t)$.

Length of path $L(s)$ is then given by : $L(s) = \int_0^1 \left(\int_V \left(\frac{\partial \phi(s,t)}{\partial t} \right)^2 d\eta_{\phi(s,t)} \right)^{\frac{1}{2}} dt$.

If we apply same approach as previously for CAR models, Calabi flow will act on Entropy $-H(p)$ defined as Kähler potential :

$$-\sum_{k=1}^{n-1} (k-n). \frac{\partial \ln\left[1 - |\mu_k|^2\right]}{\partial t} - n. \frac{\partial \ln[\pi.e.P_0]}{\partial t} = -2.\left[\sum_{k=1}^{n} \frac{1}{n-k} + \frac{1}{n} \right]$$

We then deduce the asymptotic behaviour of PARCOR coefficients submitted to Calabi Flow : $\frac{\partial \ln\left(1 - |\mu_k|^2\right)}{\partial t} = -\frac{2}{(n-k)^2}$ **with** $\frac{\partial \ln P_0}{\partial t} = \frac{2}{n^2}$. We can observe, as previsouly, that Calabi flow will drive PARCOR coefficients evolutions to unit circle.

DEFORMATION BY GAUSS MAP & WEIERSTRASS FORMULA

Kenmotsu & Konopelchenko have generalized Weierstrass representation for surfaces in multidimensional Riemann spaces. Theory of immersion and deformations of surfaces has been important part of the classical differential geometry. Various methods to describe immersions and different types of deformations have been considered. The classical Weierstrass formulae for minimal surfaces immersed in the three-dimensional Euclidean space R^3 is the best known example of such an approach. Only recently, the Weierstrass formulae have been generalized to the case of generic surfaces in R^3. using the two last years the generalized Weierstrass formulae have been used intensively to study both global properties of surfaces in R^3 and their integrable deformations (e.g. : modified Veselov-Novikov equation). An analog of the Weierstrass formulae for surfaces of prescribed (non zero) mean curvature have been proposed by Kenmotsu in 1979. The Kenmotsu representation is given by : $X^i(z,z^*) = \mathrm{Re}\left[\int^z \psi.\phi^i dz' \right]$ **with** $\vec{\phi} = \left[1 - f^2 \quad i\left(1 + f^2\right) \quad 2f \right]^T$ and the functions f and ψ obey the following compatibility condition : $\frac{\partial \log \psi}{\partial z^*} = -\frac{2f^* \frac{\partial f}{\partial z^*}}{1 + |f|^2}$ and the scalar mean curvature (if we note $\vec{H} = h\vec{N}$) : $h = -\frac{\partial f^*}{\partial z} / \left(\psi \left(1 + |f|^2 \right)^2 \right)$ with the constraint that : $\frac{\partial \log h}{\partial z^*} = \left(1 / \frac{\partial f^*}{\partial z} \right) \frac{\partial^2 f^*}{\partial z \partial z^*} - 2f \frac{\partial f^*}{\partial z^*} / \left(1 + |f|^2 \right)^2$. It was proved by Kenmotsu that any surface in R^3 can be represented in such a form :

$$\vec{X}(z,z^*) = \mathbf{Re}\left[\left[-\frac{\partial f^*}{\partial z}/\left(h.\left(1+|f|^2\right)^2\right)\right]\left(\int 1-f^2dz \quad \int i\left(1+f^2\right)dz \quad \int 2fdz\right)^T\right]$$

with f the stereographic projection of the Gauss map $\vec{N} = \frac{1}{1+|f|^2}\left(2\,\mathrm{Re}(f) \quad 2\,\mathrm{Im}(f) \quad |f|^2-1\right)^T$ and with the mean curvature h .

The most surprising result, as explained by Hoffman [9], is that a general surface in R^n is essentially determined by its Gauss map. Such a result was unexpected because this result is false for minimal surface. Minimal surfaces in R^n come with a one-parameter family of associated surfaces, all having the same Gauss map, but generally distinct. Even more is true in R^3, where essentially any two minimal surfaces have the same Gauss map locally, after a rotation. This representation of surfaces deals basically with the Gauss map for generic surface in R^3, as developed by Hoffman[9]. One of the advantages of the generalized Weierstrass formulae is that they allow to construct a new class of deformations of surfaces by use of Geometric Flow on Gauss Map.

MEANING OF FOURIER HEAT EQUATION BY INFORMATION GEOMETRY

As we have seen previously, heat equation is the main equation of evolution, Kähler flow being interpreted as Heat flow action on differential metric $\frac{\partial g_{ij}}{\partial t} = \Delta_g g_{ij}$. If we focus on geometric interpretation of heat equation, this analogy could be used to make an amother link with Information geometry by mean of Cramer-Rao inequality. If we consider the classical definition of Laplace operator in Euclidean space :

$\frac{\partial^2\theta}{\partial x^2} = \underset{\nabla x\mapsto 0}{Lim}\frac{[\theta(x+\nabla x)-\theta(x)]-[\theta(x)-\theta(x-\nabla x)]}{(\nabla x)^2}$, we can observe that is exactly the difference between the value and its spatial mean: $\frac{\partial^2\theta}{\partial x^2} \approx \frac{2}{(\nabla x)^2}\left[\hat{\theta}(x)-\theta(x)\right]$ with $\hat{\theta}(x) = \frac{\theta(x+\nabla x)+\theta(x-\nabla x)}{2}$. This remark could be extended in any dimension :

$$\Delta\theta = div(\nabla\theta) = \frac{\partial^2\theta}{\partial x_1^2}+...+\frac{\partial^2\theta}{\partial x_n^2} \approx \frac{2^n}{(\nabla x)^2}\left[\hat{\theta}(x)-\theta(x)\right]$$
$$\textbf{with} \quad \hat{\theta}(x) = \frac{\theta(x+\nabla x_1)+\theta(x-\nabla x_1)+...+\theta(x+\nabla x_n)+\theta(x-\nabla x_n)}{2^n}$$

But also for non euclidean space, by extending classical notion of spatial mean in isothermal coordinate :

$\Delta_g\theta = \sum_{i,j}\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ij}\frac{\partial\theta}{\partial x_j}\right)$ is equal in isothermal coordinates :

$$\Delta_g\theta = \sum_{\lambda,\mu} g^{\lambda\mu}\partial^2_{\lambda\mu}\theta = \frac{2^{n^2}\sum_{\lambda,\mu} g^{\lambda\mu}}{(\nabla x)^2}\left[\hat{\theta}(x)-\theta(x)\right] \quad \textbf{with} \quad \hat{\theta}(x) = \sum_{\lambda,\mu}\alpha_{\lambda\mu}\left(\frac{\theta(x+\nabla x_\lambda)+\theta(x-\nabla x_\mu)}{2}\right)$$

$$\textbf{where} \quad \alpha_{\lambda\mu} = \frac{g^{\lambda\mu}}{\sum_{\lambda,\mu} g^{\lambda\mu}} \quad \textbf{and} \quad \nabla x = \nabla x_1 = \nabla x_2 = ... = \nabla x_n$$

Generally, we can then write heat equation $\boxed{\frac{\partial\theta}{\partial t} = \Delta\theta = \rho\left[\hat{\theta}(x)-\theta(x)\right]}$ and $\frac{d\theta}{dt}\frac{d\theta^+}{dt} = \rho^2\left(\theta-\hat{\theta}\right)\left(\theta-\hat{\theta}\right)^+$. If we use Cramer-Rao Inequalitty $R_\theta \geq I(\theta)^{-1}$, then $E\left[d\theta.d\theta^+\right] = \rho^2 R_\theta dt^2 \geq \rho^2 I(\theta)^{-1} dt^2$. That could be developed by taking trace of the expression : $E\left[Tr\left(I(\theta)d\theta.d\theta^+\right)\right] \geq \rho^2 dt^2 Tr(I_n)$. By rewriting last expression as $E\left[d\theta^+ I(\theta)d\theta\right] \geq n\rho^2 dt^2$, we can observe that appears the information metric :

$$\boxed{E\left[ds_I^2\right] \geq n\rho^2 dt^2}$$

But as we have classically $E\left[d\left(\theta-\hat{\theta}\right)d\left(\theta-\hat{\theta}\right)^+\right] = E\left(d\theta.d\theta^+\right) - d\hat{\theta}.d\hat{\theta}^+$, we can extend previous relation to :

$$\boxed{\begin{array}{l} E\left[d\left(\theta-\hat{\theta}\right)^+.I(\theta).d\left(\theta-\hat{\theta}\right)\right] + d\hat{\theta}^+.I(\theta).d\hat{\theta} \geq n\rho^2 dt^2 \underset{\hat{\theta}\mapsto\theta}{\rightarrow} d\theta^+.I(\theta).d\theta = ds_I^2 = n\rho^2 dt^2 \\ \frac{d\theta^+}{dt}.I(\theta).\frac{d\theta}{dt} = n\rho^2 \end{array}}$$

For a geodesic curve $\theta = \theta(t)$, its tangent vector $\dot{\theta} = \frac{d\theta}{dt}(t)$ is of constant length with respect to the metric ds_I, thus : $\boxed{\sum_{i,j} g_{ij}\frac{d\theta_i}{dt}\frac{d\theta_j}{dt} = n\rho^2}$. The constant may be chosen to be of value 1 when the curve parameter t is chosen to be the arc-length parameter "s".

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