Entropy For Parreto (IV), Burr, and Its Order Statistics Distributions

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Abstract. Main aim of this paper is to derive the exact analytical expressions of entropy for Pareto, Burr and related distributions. Entropy for i^{th} order statistics of these distributions corresponding to the random sample size n is introduced. These distributions arise as tractable parametric models in reliability, actuarial science, economics, finance and telecommunications. We showed that all the calculations can be obtained from one main dimensional integral, its expression is obtained through some particular change of variable. Indeed, we consider that this technique for that improper integral has its own importance.

Key Words: Gamma and Beta functions; Polygamma functions; Entropy; Order Statistics; Pareto and Burr models.

INTRODUCTION

The development of the idea of entropy of random variables by Claud Shannon [1]. Provided the beginning of information theory. The applications of entropy originated in the nineteenth century in the field of Statistical Mechanics and Thermodynamics. During the last fifty years or so, a number of research paper and monographs discussing and extending Shannon's original work have appeared. In this paper the exact form of entropy for pareto (IV) and related distributions is determined. Entropy for i^{th} order statistics corresponding to the random size n from these distributions is introduced. The entropy of a random variable X taking its values in R with probability density function $f_X(x)$, is defined by

$$H_X = -\int_R f(x) \ln f(x) dx.$$

Provided that the integral exists. Analytical expression for the entropy of univariate continuous distributions are discussed by Cover and Thomas [3], Verdugo Lazo and Rathie [4]. The entropy expression for pareto (IV) distribution are given in section 2, entropy for i^{th} order statistics of this distribution are given in section 3, and entropy expressions for Pareto type (III), (II), (I) and i^{th} order statistics of these distributions corresponding to the random samples from are give in section 4. The entropy expression with respect to its order statistics are given in section 5.

PARETO(IV) DISTRIBUTION

Pareto type (IV) distribution as discussed in Arnold [2] chapter 3, a hierarchy Pareto distribution is established by starting with the classical Pareto (I) distribution and subsequently introducing additional parameters related to location, scale, shape and inequality (Gini index). Such an approach leads to a very general family of distributions, called the Pareto (IV) family, with the cumulative distribution function

$$F_X(x) = 1 - \left(1 + \left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha}, \quad x > \mu, \tag{1}$$

where $-\infty < \mu < +\infty$ is the location parameter, $\theta > 0$ is the scale parameter, $\gamma > 0$ is the inequality parameter and $\alpha > 0$ is the shape parameter which characterizes the tail of the distribution. We denote this distribution by Pareto (IV) $(\mu, \theta, \gamma, \alpha)$. Parameter γ is called the inequality parameter because of its interpretation in the economics context. That is, if we choose $\alpha = 1$ and $\mu = 0$ in expression (1), the parameter $(\gamma \le 1)$ is precisely the Gini index of inequality. For the Pareto (IV) $(\mu, \theta, \gamma, \alpha)$ distribution, we have the density function

$$f_X(x) = \frac{\alpha\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}}{\theta\gamma\left(1+\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{\alpha+1}}, \quad x > \mu. \tag{2}$$

The log-density is:

$$\ln f_X(x) = \ln\left(\frac{\alpha}{\theta\gamma}\right) + \left(\frac{1}{\gamma}-1\right)\ln\left(\frac{x-\mu}{\theta}\right) - (\alpha+1)\ln\left[1+(\frac{x-\mu}{\theta})^{\frac{1}{\gamma}}\right],$$

and the entropy is:

$$\begin{aligned} H_X &= \int_x f(x)\ln f(x)dx = -\ln\left(\frac{\alpha}{\theta\gamma}\right) \\ &+ \left(1-\frac{1}{\gamma}\right)E\left[\ln\left(\frac{X-\mu}{\theta}\right)\right] + (\alpha+1)E\left[\ln\left(1+\left(\frac{X-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)\right]. \end{aligned} \tag{3}$$

We need to find the expressions $E\left[\ln\left(\frac{X-\mu}{\theta}\right)\right]$ and $E\left[\ln\left(1+\left(\frac{X-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)\right]$.
First we calculate the expectation of

$$E\left[\left(\frac{X-\mu}{\theta}\right)^r\right] = \int_x \left(\frac{x-\mu}{\theta}\right)^r f(x)dx,$$

By using the change of variable:

$$1+\left(\frac{X-\mu}{\theta}\right)^{\frac{1}{\gamma}} = \frac{1}{1-t} \quad 0 < t < 1,$$

we obtained:

$$E\left[\left(\frac{X-\mu}{\theta}\right)^r\right] = \alpha\frac{\Gamma(r\gamma+1)\Gamma(\alpha-r\gamma)}{\Gamma(\alpha+1)} \quad , \quad \alpha-r\gamma\neq 0,-1,-2,\cdots. \tag{4}$$

Differentiating both sides of (4) with respect to r we obtain:

$$\frac{d}{dr}E\left[\left(\frac{X-\mu}{\theta}\right)^r\right] = E\left[\left(\frac{X-\mu}{\theta}\right)^r \ln\left(\frac{X-\mu}{\theta}\right)\right] = \frac{1}{\Gamma\alpha}\left[\gamma\Gamma'(r\gamma+1)\Gamma(\alpha-r\gamma)-\gamma\Gamma'(\alpha-r\gamma)\Gamma(r\gamma+1)\right]. \tag{5}$$

Using relation (5), at $r=0$ we obtain

$$E\left[\ln\left(\frac{X-\mu}{\theta}\right)\right] = \gamma\left[\Psi(1)-\Psi(\alpha)\right], \tag{6}$$

where Ψ is the digamma function defined by $\Psi(z)=\frac{d}{dz}\ln\Gamma(z)$ where Γ is the gamma function.

Taking derivative with respect to α, from the both sides of the relation

$$1=\int_{\mu}^{+\infty} f(x)dx, \tag{7}$$

leads to

$$E\left[\ln\left(1+\left(\frac{X-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)\right] = \frac{1}{\alpha}. \tag{8}$$

Substitute (6) and(8) in relation (3) we have:

$$H_X = \ln\left(\frac{\gamma\theta}{\alpha}\right) + (\gamma-1)\left[\Psi(1)-\Psi(\alpha)\right] + \frac{\alpha+1}{\alpha}. \tag{9}$$

ENTROPY FOR ORDER STATISTICS

Let $X_1, X_2, \cdots, X_n$ be a random sample of the probability density function (2). Let $Y_1 \leq Y_2 \leq \cdots \leq Y_{n-1} \leq Y_n$ denote the corresponding order statistics; then

$$\begin{aligned} g_{i,n}(y) &= n\binom{n-1}{i-1}\left[F_{i,n}(y)\right]^{i-1}\left[1-F_{i,n}(Y)\right]^{n-i} f_X(y) \\ &= \frac{n\alpha}{\theta\gamma}\binom{n-1}{i-1}\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}\left(1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha(n-i+1)-1} \\ &\quad \left[1-\left(1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha}\right]^{i-1}, y>\mu. \end{aligned} \tag{10}$$

The entropy expression of the $g_{i,n}(y)$ is:

$$
\begin{aligned}
H_{i,n}(Y) &= -\ln\left[\frac{n\alpha}{\theta\gamma}\binom{n-1}{i-1}\right]+\left(1-\frac{1}{\gamma}\right)E\left[\ln\left(\frac{Y-\mu}{\theta}\right)\right] \\
&\quad +[\alpha(n-i+1)+1]\,E\left[\ln\left(1+\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]+ \\
&\quad (1-i)E\left[\ln\left(1-\left(1+\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha}\right]. \qquad (11)
\end{aligned}
$$

We need to calculate the expressions of:

$$
E\left[\ln\left(\frac{Y-\mu}{\theta}\right)\right], \quad E\left[\ln\left(1+\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]
$$

and

$$
E\left[\ln\left(1-\left(1+\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha}\right].
$$

Derivation of these expressions are based on the following strategy: first, we derive an analytical expression for the following expressions.

$$
E\left[\left(\frac{Y-\mu}{\theta}\right)^{r}\right], \quad E\left[\left(1+\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{r}, \quad E\left[\left(1-\left(1+\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha}\right]^{r}.
$$

Now:

$$
\begin{aligned}
E\left[\left(\frac{Y-\mu}{\theta}\right)^{r}\right] &= \int_{\mu}^{\infty}\left(\frac{y-\mu}{\theta}\right)^{r} g_{i,n}(y)dy = \\
&\quad \frac{n\alpha}{\theta\gamma}\binom{n-1}{i-1}\int_{\mu}^{\infty}\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}+r-1}\left(1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha(n-i+1)-1} \\
&\quad \left[1-\left(1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha}\right]^{i-1} dy. \qquad (12)
\end{aligned}
$$

By change of variable $1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}=t, \quad 0<t<1$,
we have:

$$
\begin{aligned}
E\left[\left(\frac{Y-\mu}{\theta}\right)^{r}\right] &= n\alpha\binom{n-1}{i-1}\sum_{l=0}^{i-1}(-1)^{l}\binom{i-l}{l} \\
&\quad \frac{\Gamma(r\gamma+1)\Gamma(-r\gamma+\alpha(n-i+1+l))}{\Gamma(1+\alpha(n-i+1+l))}. \qquad (13)
\end{aligned}
$$

Differentiating both sides of (13) with respect to r and then at $r=0$ we obtain:

$$E\left[\ln\left(\frac{Y-\mu}{\theta}\right)\right] = n\alpha\gamma\binom{n-1}{i-1} \sum_{l=0}^{i-1}(-1)^l\binom{i-1}{l}\left[\frac{\Psi(1)-\Psi(\alpha(n-i+1+l))}{\alpha(n-i+1+l)}\right]. \tag{14}$$

If denote:

$$\phi(r) = E\left[\left(1+\left(\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^r\right] = \frac{n\alpha}{\theta\gamma}\binom{n-1}{i-1}\int_\mu^\infty\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}-1} \left[\left(1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{r-\alpha(n-i+1)-1}\right]\left[1-\left(1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha}\right]^{i-1} dy. \tag{15}$$

Using the change of variable:

$$\left(1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha} = t, \quad 0<t<1,$$

we have:

$$\phi(r) = n\binom{n-1}{i-1}\int_0^1 t^{n-i-\frac{r}{\gamma}}(1-t)^{i-1}dt = \frac{\Gamma(n+1)\Gamma\left(n-i-\frac{r}{\alpha}+1\right)}{\Gamma(n-i+1)\Gamma\left(n-\frac{r}{\alpha}+1\right)}. \tag{16}$$

$$\frac{d\phi(r)}{dr}\|_{r=0} = E\left[\ln\left(1+\left(\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)\right] = \frac{\Psi(n+1)-\Psi(n-i+1)}{\alpha}. \tag{17}$$

It can be shown that:

$$k(r) = E\left[\ln\left(1-\left(1+\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha}\right]^r = \frac{\Gamma(n+1)\Gamma(r+i)}{\Gamma(i)\Gamma(r+n+1)}, \tag{18}$$

and

$$\frac{dk(r)}{dr}\|_{r=0} = E\left[\ln\left(1-\left(1+\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha}\right] = \Psi(i)-\Psi(n+1). \tag{19}$$

Substitute (14) , (17) and (19) in relation (11) we have:

$$\begin{aligned} H_{i,n}(Y) &= -\ln\left[\frac{n\alpha}{\theta\gamma}\binom{n-1}{i-1}\right]+(\gamma-1)n\binom{n-1}{i-1} \\ & \sum_{l=0}^{i-1}(-1)^l\binom{i-1}{l}\left[\frac{\Psi(1)-\Psi(\alpha(n-i+1+l))}{n-i+1+l}+[\alpha(n-i+1)+1]\right. \\ & \left.\left[\frac{\Psi(n+1)-\Psi(n-i+1)}{\alpha}\right]+(i-1)\left[\Psi(n+1)-\Psi(i)\right]. \right. \end{aligned} \tag{20}$$

In particular case if:
1) $i=n=1 \Longrightarrow H_{1,1}(Y)=H_X$
2) $i=n \Longrightarrow$ we obtain the useful result as:

$$\begin{aligned} H_{n,n}(Y) &= -\ln\left(\frac{n\alpha}{\theta\gamma}\right)+n(\gamma-1)\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l} \\ & \left[\frac{\Psi(1)-\Psi(\alpha(1+l))}{1+l}\right]+\left(\frac{n-1}{n}\right)+\left(\frac{\alpha+1}{\alpha}\right)\left[\Psi(n+1)-\Psi(1)\right]. \end{aligned}$$

3) $i=1 \Longrightarrow$

$$H_{1,n}(Y)=-\ln\left[\frac{n\alpha}{\theta\gamma}\right]+(\gamma-1)\left[\Psi(1)-\Psi(n\alpha)\right]=H_X(n\alpha).$$

4) The entropy expression for median where n be odd is:

$$n=2m+1, i=m+1 \Longrightarrow$$

$$\begin{aligned} H_{m+1,n}(Y) &= -\ln\left[\frac{n\alpha}{\theta\gamma}\binom{n-1}{m}\right]+(\gamma-1)n\binom{n-1}{m} \\ & \sum_{l=0}^{m}(-1)^l\binom{m}{l}\left[\frac{\Psi(1)-\Psi(\alpha(n-m+l))}{n-m+l}\right]+ \\ & \left(\frac{\alpha(n-m)+1}{\alpha}\right)\left[\Psi(n+1)-\Psi(n-m)\right]+ \\ & m\left[\Psi(n+1)-\Psi(m+1)\right]. \end{aligned}$$

PARETO (III) DISTRIBUTION

By setting $\alpha=1$ in relation (1) we obtain the Pareto cumulative distribution of type (III), and the probability density function of Pareto type (III) is obtained by setting $\alpha=1$ in relation (2) which follows respectively as:

$$F_X(x)=1-\left(1+\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-1}, \quad x>\mu,$$

$$f_X(x)=\frac{1}{\gamma\theta}\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}\left(1-\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-2}, \quad x>\mu.$$

The probability density function of the i^{th} order statistics of pareto type (III) distribution corresponding to the random sample size of n from is:

$$\begin{aligned} g_{i,n}(y) &= \frac{n}{\gamma\theta}\binom{n-1}{i-1}\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}\left(1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-(n-i+2)} \\ & \left[1-\left(1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-1}\right]^{i-1}, \quad y>\mu, \end{aligned}$$

and its entropy expression is:

$$\begin{aligned} H_{i,n}(Y) &= -\ln\left[\frac{n}{\gamma\theta}\binom{n-1}{i-1}\right]+(\gamma-1)n\binom{n-1}{i-1} \\ & \sum_{l=0}^{i-1}(-1)^l\binom{i-1}{l}\left[\frac{\Psi(1)-\Psi(n-i+l+1)}{n-i+l+1}\right]+ \\ & (n-i+2)\left[\Psi(n+1)-\Psi(n-i+1)\right]+(i-1)\left[\Psi(n+1)-\Psi(i)\right]. \end{aligned}$$

In particular case if:
1) $i=n=1 \Longrightarrow H_{1,1}(Y)=H_X=\ln(\gamma\theta)+2= H(Pareto(III))$
2) $i=1 \Longrightarrow$ we obtain the useful result as:

$$H_{1,1}(Y)=-\ln\left(\frac{n}{\gamma\theta}\right)+(\gamma-1)\left[\Psi(1)-\Psi(n)\right]=H_X(\alpha=n).$$

3) $i=n \Longrightarrow$

$$\begin{aligned} H_{n,n}(Y) &= -\ln\left(\frac{n}{\gamma\theta}\right)+n(\gamma-1)\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l} \\ & \left[\frac{\Psi(1)-\Psi(l+1)}{l+1}\right]+2\left[\Psi(n+1)-\Psi(1)\right]+\left(\frac{n-1}{n}\right). \end{aligned}$$

PARETO (II) AND (I) DISTRIBUTIONS

The best known Pareto distributions are type (II) and (I). A Pareto distribution of type (II) is obtained by setting $\gamma=1$ in relation (1).
The entropy of this distribution is:

$$H_X=\ln(\frac{\theta}{\alpha})+\frac{\alpha+1}{\alpha}.$$

The corresponding entropy expression for the i^{th} order statistics of this distribution corresponding is obtained by setting $\gamma=1$ in relation (20):

$$\begin{aligned} H_{i,n}(Y) &= -\ln\left[\frac{n\alpha}{\theta}\right] + [\alpha(n-i+1)+1] \\ & \left[\frac{\Psi(n+1)-\Psi(n-i+1)}{\alpha}\right] + (i-1)\left[\Psi(n+1)-\Psi(i)\right]. \end{aligned}$$

In particular case if:
1)

$$i = n = 1 \Longrightarrow H_{1,1}(Y) = -\ln\left(\frac{\alpha}{\theta}\right) + \frac{\alpha+1}{\alpha} = H_X.$$

2) $i = n \Longrightarrow$ we obtain the useful result as:

$$\begin{aligned} H_{n,n} &= -\ln\left(\frac{n\alpha}{\theta}\right) + \left[\frac{\Psi(1)-\Psi(\alpha(1+l))}{1+l}\right] + \\ & \left(\frac{n-1}{n}\right) + \left(\frac{\alpha+1}{\alpha}\right)\left[\Psi(n+1)-\Psi(1)\right]. \end{aligned}$$

3) $i = 1 \Longrightarrow H_{1,n}(Y) = \ln\left(\frac{\theta}{n\alpha}\right) + \frac{n\alpha+1}{n\alpha} = H_X(n\alpha).$

If one sets $\gamma = 1$ and $\mu = \theta$ in relation (1) then one gets a Pareto distribution of type (I).
Since the entropy expression of Pareto family is not depends on location parameter μ, thus, one finds that the entropy
H_X(Pareto(II))= H_X(Pareto(I)),
and
$H_{i,n}$(Pareto(II))= $H_{i,n}$(Pareto (I)).

BURR (XII) DISTRIBUTION

This distribution is a special case of Pareto (IV) with $\mu = 0$, $\gamma \longrightarrow \frac{1}{\gamma}$,
thus; by setting $\mu = 0$ and replacing γ by $\frac{1}{\gamma}$ in relations (1) and (2), the cumulative distribution and probability density function of Burr type (XII) distribution is dervived:

$$F_X(x) = 1 - \left(1 + \left(\frac{x}{\theta}\right)^{\gamma}\right)^{-\alpha}, \quad x > 0, \quad \alpha, \gamma > 0,$$

$$f_X(x) = \left(\frac{\alpha\gamma}{\theta}\right)\left(\frac{x}{\theta}\right)^{\gamma-1}\left(1 + \left(\frac{x}{\theta}\right)^{\gamma}\right)^{-(\alpha+1)}, \quad x > 0, \quad \alpha, \gamma > 0.$$

By replacing γ by $\frac{1}{\gamma}$ in relation (9) the entropy expression of Burr type (XII) distribution is derived:

$$H_X = \ln\left(\frac{\theta}{\alpha\gamma}\right) + (\gamma-1)\left[\frac{\Psi(\alpha)-\Psi(1)}{\gamma}\right] + (\frac{\alpha+1}{\alpha}).$$

The entropy expression of the i^{th} order statistics corresponding to the sample size of n of this distribution, by replacing γ by $\frac{1}{\gamma}$ in relation (20) is derived as:

$$\begin{aligned} H_{i,n}(Y) &= -\ln\left[\frac{\alpha\gamma}{\theta}n\binom{n-1}{i-1}\right] + (\gamma-1)n\binom{n-1}{i-1}\sum_{l=0}^{i-1}(-1)^l\binom{i-1}{l} \\ &\quad \left[\frac{\Psi(1)-\Psi(\alpha(n-i+1+l))}{\gamma(n-i+1+l)}\right] + [\alpha(n-i+1)+1] \\ &\quad \left[\frac{\Psi(n+1)-\Psi(n-i+1)}{\alpha}\right] + (i-1)\left[\Psi(n+1)-\Psi(i)\right]. \end{aligned}$$

In particular case if:
1) $i = 1 \Longrightarrow$

$$H_{1,n} = -\ln\left(\frac{n\alpha\gamma}{\theta}\right) + \left(\frac{\gamma-1}{\gamma}\right)\left[\frac{\Psi(1)-\Psi(n\alpha)}{n}\right] + \frac{n\alpha+1}{n\alpha} = H_X(n\alpha).$$

2) $i = n \Longrightarrow$ we obtain the useful result as:

$$\begin{aligned} H_{n,n}(Y) &= -\ln\left(\frac{n\alpha\gamma}{\theta}\right) + (\gamma-1)n\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l} \\ &\quad \left[\frac{\Psi(1)-\Psi(\alpha(1+l))}{\gamma(1+l)}\right] + \left(\frac{\alpha+1}{\alpha}\right)\left[\Psi(n+1)-\Psi(1)\right] + \left(\frac{n-1}{n}\right). \end{aligned}$$

3) $i = n = 1 \Longrightarrow H_{1,1}(Y) = -\ln\left(\frac{\alpha\gamma}{\theta}\right) + \left(\frac{\gamma-1}{\gamma}\right)\left[\Psi(1)-\Psi(\alpha)\right].$

CONCLUSION

In this paper we obtained the exact form of entropy expression for Pareto (IV) and related distributions. Entropy for i^{th} order statistics of these distributions corresponding to the random samples size n has been derived.
Deriving entropy expression for remainder Burr family distributions and i^{th} order statistics of these distributions corresponding to the random samples size n are my further works.

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