

# Maximum likelihood separation of spatially autocorrelated images using a Markov model

Shahram Hosseini<sup>\*</sup>, Rima Guidara<sup>\*</sup>, Yannick Deville<sup>\*</sup> and Christian Jutten<sup>†</sup>

<sup>\*</sup>*Laboratoire d'Astrophysique de Toulouse-Tarbes (LATT), Observatoire Midi-Pyrénées - Université Paul Sabatier, 14 Avenue Edouard Belin, 31400 Toulouse, France. Emails: shosseini or rguidara or ydeville@ast.obs-mip.fr*

<sup>†</sup>*Laboratoire des Images et des Signaux (LIS), UMR CNRS-INPG-UJF, Grenoble, France. Email: Christian.Jutten@inpg.fr*

**Abstract.** We recently proposed a quasi-efficient maximum likelihood approach for blindly separating Markovian time series. In the present paper, we extend this idea to bi-dimensional sources (in particular images), where the spatial autocorrelation of each source is described using a second-order Markov random field. The experimental results using artificial and real images prove the advantage of the method with respect to the maximum likelihood approaches which do not take into account the source autocorrelation, and the autocorrelation-based methods which ignore the source non-Gaussianity.

**Keywords:** Blind source separation, Markov random fields, Maximum likelihood

**PACS:** 42.30.Wb

## INTRODUCTION

Linear instantaneous blind source separation consists in recovering unobserved source signals from several observed signals which are supposed to be linear instantaneous mixtures of these sources. It has been shown that this goal can be achieved by exploiting non-Gaussianity, autocorrelation or non-stationarity of sources [1], leading to numerous algorithms [2].

We recently proposed [3] a quasi-efficient Maximum Likelihood (ML) approach for blindly separating mixtures of temporally correlated, independent sources where a Markov model was used to simplify the joint Probability Density Functions (PDF) of successive samples of each source. This approach exploits both source non-Gaussianity and autocorrelation in a quasi-optimal manner. The theoretical analysis and the experimental results proved its advantage with respect to the ML methods which ignore the source autocorrelation [4] and the autocorrelation-based methods which ignore the source non-Gaussianity [5], [6].

In this paper, our objective is to extend this idea to bi-dimensional sources (in particular images), where the spatial autocorrelation of each source is described using a second-order Markov Random Field (MRF). The idea of using MRF for image separation has recently been exploited by other authors [7], where the source PDF are supposed to be known, and are used to choose the Gibbs priors. In the present work, however, we make no assumption about the source PDF so that the method remains quasi-efficient whatever the source distributions.

## PROBLEM STATEMENT

In its simplest form, blind separation of bi-dimensional sources (in particular images) can be formulated as follows. Assume we have  $N = N_1 \times N_2$  samples of a  $K$ -dimensional vector  $\mathbf{x}(n_1, n_2)$  resulting from a linear transformation  $\mathbf{x}(n_1, n_2) = \mathbf{A}\mathbf{s}(n_1, n_2)$ , where  $\mathbf{s}(n_1, n_2)$  is the vector of independent image sources  $s_i(n_1, n_2)$ , each one of dimension  $N_1 \times N_2$  and possibly spatially autocorrelated, and  $\mathbf{A}$  is a  $K \times K$  invertible matrix. Our objective is to estimate the separating matrix  $\mathbf{B} = \mathbf{A}^{-1}$  up to a diagonal matrix and a permutation matrix.

One of the separation approaches consists in maximizing the likelihood function of the observations. This approach has the advantage of providing an asymptotically efficient estimator (smallest error covariance matrix among unbiased estimators). For i.i.d. sources, this method has been used by Pham and Garat [4].

It is known that the autocorrelation of each source may be used for improving the estimation [2]. This additional information can actually make the estimation of the model possible in cases where the basic ICA methods cannot estimate it, for example, if the sources are Gaussian but autocorrelated. However, most of the methods exploiting the autocorrelation are second-order methods [5], [6] which generally provide unbiased but non-efficient estimators.

In [3], we proposed an extension of the Pham-Garat ML algorithm to the case of temporally correlated sources represented by Markov models. We now want to extend our method to 2-dimensional signals.

The ML method consists in maximizing the joint PDF of all the samples of all the components of the vector  $\mathbf{x}$  (all the observations), with respect to the separating matrix  $\mathbf{B}$ . We denote this PDF

$$f_{\mathbf{x}}(x_1(1, 1), \dots, x_K(1, 1), \dots, x_1(N_1, N_2), \dots, x_K(N_1, N_2)) \quad (1)$$

Under the assumption of independence of the sources, this function is equal to

$$\left( \frac{1}{|\det(\mathbf{B}^{-1})|} \right)^N \prod_{i=1}^K f_{s_i}(s_i(1, 1), \dots, s_i(N_1, N_2)) \quad (2)$$

where  $f_{s_i}(\cdot)$  represents the joint PDF of  $N$  samples of the source  $s_i$ . Each joint PDF can be decomposed using Bayes rule in many different manners following different sweeping trajectories within the image corresponding to source  $s_i$ . Contrary to the temporal case, there are several logical sweeping schemes preserving the continuity and so, allowing the exploitation of the spatial autocorrelation. Some of them are shown in Fig. 1. These schemes being essentially equivalent, we chose the first one (horizontal sweeping). Then, the source joint PDF  $f_{s_i}(s_i(1, 1), s_i(1, 2), \dots, s_i(1, N_2), s_i(2, 1), \dots, s_i(N_1, N_2))$  can be decomposed using Bayes rule to obtain

$$\begin{aligned} & f_{s_i}(s_i(1, 1)) f_{s_i}(s_i(1, 2) | s_i(1, 1)) f_{s_i}(s_i(1, 3) | s_i(1, 2), s_i(1, 1)) \dots \dots \\ & f_{s_i}(s_i(1, N_2) | s_i(1, N_2 - 1), \dots, s_i(1, 1)) f_{s_i}(s_i(2, 1) | s_i(1, N_2), \dots, s_i(1, 1)) \dots \dots \\ & f_{s_i}(s_i(N_1, N_2) | s_i(N_1, N_2 - 1), \dots, s_i(1, 1)) \end{aligned} \quad (3)$$

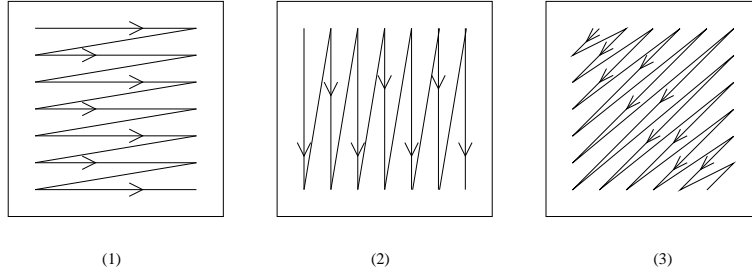


FIGURE 1. Different sweeping possibilities.

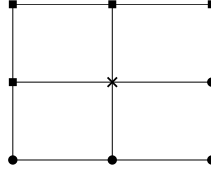


FIGURE 2. second-order Markov random field.

## SIMPLIFYING THE LIKELIHOOD FUNCTION USING A MARKOV MODEL

Equation (3) may be simplified by assuming a Markov model for the sources. We suppose hereafter that the sources are second-order Markov random fields, *i.e.* the conditional PDF of a pixel  $s(n_1, n_2)$  given all the other pixels is equal to its conditional PDF given its 8 nearest neighbors (Fig. 2). From this assumption, it is clear that the conditional PDF of a pixel not situated on the boundaries, given all its predecessors (in the sense of sweeping trajectory) is equal to its conditional PDF given its three top neighbors and its left neighbor (squares in Fig. 2). In other words, if  $D_{n_1, n_2}$  is the set of pixel values  $s_i(k, l)$  such that  $\{k < n_1\}$  or  $\{k = n_1, l < n_2\}$ , then

$$f_{s_i}(s_i(n_1, n_2) | D_{n_1, n_2}) = f_{s_i}(s_i(n_1, n_2) | s_i(n_1, n_2 - 1), s_i(n_1 - 1, n_2 + 1), s_i(n_1 - 1, n_2), s_i(n_1 - 1, n_2 - 1)) \quad (4)$$

If  $N$  is sufficiently large, the conditional PDF of the pixels located on the left, top and right image boundaries (for which, the 4 mentioned neighbors are not available) may be neglected in (3). Supposing that the sources are stationary so that the conditional PDF (4) does not depend on  $n_1$  and  $n_2$ , it follows from (4) that the decomposed joint PDF (3) can be rewritten as

$$f_{s_i}(s_i(1, 1), s_i(1, 2), \dots, s_i(1, N_2), s_i(2, 1), \dots, s_i(N_1, N_2)) \simeq \prod_{n_1=2}^{N_1} \prod_{n_2=2}^{N_2-1} f_{s_i}(s_i(n_1, n_2) | s_i(n_1, n_2 - 1), s_i(n_1 - 1, n_2 + 1), s_i(n_1 - 1, n_2), s_i(n_1 - 1, n_2 - 1)) \quad (5)$$

The likelihood function may be then obtained by replacing (5) in (2). Finally, taking the

logarithm, the log-likelihood function can be obtained as

$$N \log(|\det(\mathbf{B})|) + \sum_{i=1}^K \sum_{n_1=2}^{N_1} \sum_{n_2=2}^{N_2-1} \log f_{s_i}(s_i(n_1, n_2) | s_i(n_1, n_2 - 1), s_i(n_1 - 1, n_2 + 1), s_i(n_1 - 1, n_2), s_i(n_1 - 1, n_2 - 1)) \quad (6)$$

Dividing the above cost function by  $N$  and defining the spatial average operator  $E_N[\cdot] = \frac{1}{N} \sum_{n_1=2}^{N_1} \sum_{n_2=2}^{N_2-1} [\cdot]$ , Equation (6) may be rewritten in the following simpler form

$$L_1 = \log(|\det(\mathbf{B})|) + E_N \left[ \sum_{i=1}^K \log f_{s_i}(s_i(n_1, n_2) | s_i(n_1, n_2 - 1), s_i(n_1 - 1, n_2 + 1), s_i(n_1 - 1, n_2), s_i(n_1 - 1, n_2 - 1)) \right] \quad (7)$$

Maximizing the above function with respect to the separating matrix  $\mathbf{B}$  requires the computation of its gradient. Computing the derivative of (7) with respect to the separating matrix  $\mathbf{B}$ , we obtain

$$\frac{\partial L_1}{\partial \mathbf{B}} = \mathbf{B}^{-T} + E_N \left[ \sum_{i=1}^K \frac{\partial}{\partial \mathbf{B}} \log f_{s_i}(s_i(n_1, n_2) | s_i(n_1, n_2 - 1), s_i(n_1 - 1, n_2 + 1), s_i(n_1 - 1, n_2), s_i(n_1 - 1, n_2 - 1)) \right] \quad (8)$$

Defining the set  $\Upsilon = \{(0, 0), (0, 1), (1, -1), (1, 0), (1, 1)\}$ , and using the chain rule, we can write

$$\begin{aligned} \frac{\partial}{\partial \mathbf{B}} \log f_{s_i}(s_i(n_1, n_2) | s_i(n_1, n_2 - 1), s_i(n_1 - 1, n_2 + 1), s_i(n_1 - 1, n_2), s_i(n_1 - 1, n_2 - 1)) = \\ \sum_{(k,l) \in \Upsilon} \frac{\partial \log f_{s_i}(s_i(n_1, n_2) | s_i(n_1, n_2 - 1), s_i(n_1 - 1, n_2 + 1), s_i(n_1 - 1, n_2), s_i(n_1 - 1, n_2 - 1))}{\partial s_i(n_1 - k, n_2 - l)} \cdot \frac{\partial s_i(n_1 - k, n_2 - l)}{\partial \mathbf{B}} \end{aligned}$$

The first derivative in the sum is defined as the opposite of the conditional score function of the source  $s_i$  with respect to the term  $s_i(n_1 - k, n_2 - l)$ , which will be denoted by  $\psi_{s_i}^{(k,l)}(n_1, n_2)$ :

$$\psi_{s_i}^{(k,l)}(n_1, n_2) = \frac{-\partial \log f_{s_i}(s_i(n_1, n_2) | s_i(n_1, n_2 - 1), s_i(n_1 - 1, n_2 + 1), s_i(n_1 - 1, n_2), s_i(n_1 - 1, n_2 - 1))}{\partial s_i(n_1 - k, n_2 - l)}$$

The second derivative in the sum is equal to

$$\frac{\partial s_i(n_1 - k, n_2 - l)}{\partial \mathbf{B}} = \frac{\partial \mathbf{e}_i^T \mathbf{B} \mathbf{x}(n_1 - k, n_2 - l)}{\partial \mathbf{B}} = \mathbf{e}_i \mathbf{x}^T(n_1 - k, n_2 - l) \quad (9)$$

where  $\mathbf{e}_i$  is the  $i$ -th column of the identity matrix. Thus, we can write

$$\frac{\partial L_1}{\partial \mathbf{B}} = \mathbf{B}^{-T} - E_N \left[ \sum_{(k,l) \in \Upsilon} \left( \sum_{i=1}^K \psi_{s_i}^{(k,l)}(n_1, n_2) \mathbf{e}_i \right) \cdot \mathbf{x}^T(n_1 - k, n_2 - l) \right] \quad (10)$$

However,  $\sum_{i=1}^K \psi_{s_i}^{(k,l)}(n_1, n_2) \mathbf{e}_i$  is nothing but a column vector containing the score functions  $\psi_{s_i}^{(k,l)}(n_1, n_2)$  of the  $K$  sources. Denoting this vector by  $\Psi_s^{(k,l)}(n_1, n_2)$ , the gradient (8) can be rewritten as

$$\frac{\partial L_1}{\partial \mathbf{B}} = \mathbf{B}^{-T} - E_N \left[ \sum_{(k,l) \in \Upsilon} \Psi_s^{(k,l)}(n_1, n_2) \cdot \mathbf{x}^T(n_1 - k, n_2 - l) \right] \quad (11)$$

## PRACTICAL IMPLEMENTATION

In practice, the actual sources being unknown, their densities could be estimated only via the reconstructed sources  $\hat{\mathbf{s}}(n_1, n_2) = \hat{\mathbf{B}}\mathbf{x}(n_1, n_2)$ . It is clear that this estimation is not correct at the first steps of the optimization procedure. However, as we showed in [3] for the temporal case, under mild conditions the method converges rapidly toward the actual sources so that the estimation becomes more and more accurate. The score functions of the reconstructed sources are estimated using a non-parametric method proposed in [8] involving the estimation of joint entropies using a discrete Riemann sum and the third-order cardinal spline kernels. The estimation of the separating matrix  $\mathbf{B}$  is done using a batch iterative approach. At each iteration, using the current value of the matrix  $\mathbf{B}$ , the conditional score functions of the estimated sources are estimated and the gradient (11) is computed. Afterwards, the matrix  $\mathbf{B}$  is updated for maximizing the cost function (7) using a relative gradient ascent scheme for achieving an equivariant estimation [9]:

$$\mathbf{B}_{new} = \left( \mathbf{I} + \mu \frac{\partial L_1(\mathbf{B})}{\partial \mathbf{B}} \mathbf{B}_{old}^T \right) \mathbf{B}_{old} \quad (12)$$

Denoting  $\mathbf{H} = \frac{\partial L_1(\mathbf{B})}{\partial \mathbf{B}} \mathbf{B}_{old}^T$  and Using (11), we can write

$$\mathbf{H} = \mathbf{I} - E_N \left[ \sum_{(k,l) \in \Upsilon} \psi_s^{(k,l)}(n_1, n_2) \cdot \hat{\mathbf{s}}^T(n_1 - k, n_2 - l) \right] \quad (13)$$

Because of the scaling indeterminacy, the diagonal entries of the matrix  $\mathbf{H}$  have no importance. Thus, we can replace  $\mathbf{H}$  by only the second term of the right hand of (13), denoted  $\mathbf{G}$ :

$$\mathbf{G} = -E_N \left[ \sum_{(k,l) \in \Upsilon} \psi_s^{(k,l)}(n_1, n_2) \cdot \hat{\mathbf{s}}^T(n_1 - k, n_2 - l) \right] \quad (14)$$

Hence, the update formula (12) becomes

$$\mathbf{B}_{new} = (\mathbf{I} + \mu \mathbf{G}) \mathbf{B}_{old} \quad (15)$$

To remove the ambiguity due to the scaling indeterminacy, the rows of the separating matrix  $\mathbf{B}$  are normalized at each iteration so that the estimated sources have unit variance.

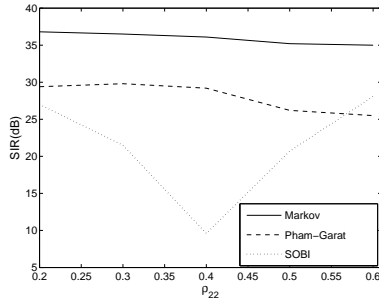


FIGURE 3. Mean of SIR as a function of the coefficient  $\rho_{1,-1}$  of the second AR filter.

## EXPERIMENTAL RESULTS

In the first experiment, we use artificial image sources of size  $50 \times 50$  which satisfy exactly the considered Markov model. Two independent white and uniformly distributed image noises,  $e_1(n_1, n_2)$  and  $e_2(n_1, n_2)$ , are filtered by two autoregressive (AR) filters using the following formula:

$$s_i(n_1, n_2) = e_i(n_1, n_2) + \rho_{0,1}s_i(n_1, n_2 - 1) + \rho_{1,-1}s_i(n_1 - 1, n_2 + 1) + \rho_{1,0}s_i(n_1 - 1, n_2) + \rho_{1,1}s_i(n_1 - 1, n_2 - 1) \quad (16)$$

The coefficients  $\rho_{i,j}$  are chosen to guarantee a sufficient stability condition proposed in [10]. Thus, the coefficients of the first and the second filters are respectively fixed to  $\{-0.5, 0.4, 0.5, 0.3\}$  and  $\{-0.5, \rho_{1,-1}, 0.5, 0.3\}$ . The coefficient  $\rho_{1,-1}$  of the second filter may change in its stability interval, *i.e.*  $[0.2, 0.6]$ . Then, the source images  $s_i(n_1, n_2)$  are mixed by the mixing matrix  $\mathbf{A} = \begin{pmatrix} 1 & 0.99 \\ 0.99 & 1 \end{pmatrix}$ .

We compare our method with two well-known algorithms: SOBI [6] and Pham-Garat [4]. SOBI is a second-order method which consists in jointly diagonalizing several covariance matrices evaluated at different lags. The Pham-Garat algorithm is based on a maximum likelihood approach which supposes that the sources are i.i.d. and therefore does not take into account their possible autocorrelation.

For each method, the experiment was repeated 100 times corresponding to 100 different seed values of the random variable generator. For each experiment, the output Signal to Interference Ratio (in dB) was computed by  $SIR = 0.5 \sum_{i=1}^2 10 \log_{10} \frac{E[s_i^2]}{E[(\hat{s}_i - s_i)^2]}$ , after normalizing the estimated sources,  $\hat{s}_i(n_1, n_2)$ , so that they have the same variances and signs as the source signals,  $s_i(n_1, n_2)$ . The mean of SIR as a function of the coefficient  $\rho_{1,-1}$  of the second AR filter is shown in Fig. 3. Our algorithm outperforms the other two, whatever  $\rho_{1,-1}$ . It can be remarked that the SOBI algorithm fails to separate the sources when  $\rho_{1,-1} = 0.4$ . It is not surprising because for this value of  $\rho_{1,-1}$ , the two filtered sources are generated by the same AR filter and have the same spectral densities. It is well-known that this second-order method is not able to separate such sources.

In the second experiment, two  $250 \times 250$  astrophysical image sources are mixed and the three above-mentioned algorithms are used for separating them. It is clear that the working hypotheses are no longer true because on the one hand the images are not stationary and on the other hand they cannot be described by a second-order MRF. Two mixing matrix  $\mathbf{A}_1 = \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}$  and  $\mathbf{A}_2 = \begin{pmatrix} 1 & 0.99 \\ 0.99 & 1 \end{pmatrix}$ , corresponding respectively to weakly mixed and highly mixed sources, are successively used for this experiment. Our method led to 70-dB SIR with the first mixing matrix but it failed to separate the sources with the second matrix. A bad initial estimation of the conditional score functions in the second case may explain this result. In fact, the actual sources being unknown, the score functions are estimated from the reconstructed sources which are initialized to the mixtures. In the case of highly mixed sources, the mixtures are completely different from the sources so that the score functions are badly estimated at the first steps of the iterative algorithm. As we have shown in [3], when the sources satisfy the working hypotheses (stationarity, Markov model, ...), the algorithm is robust to this bad initialization and the outputs converge rapidly toward actual sources where a good estimation of the score functions is possible. However, when the working hypotheses are not satisfied, it seems that the algorithm is sensitive to the initial estimation of the score functions. That is why it works well when the sources are weakly mixed: the mixtures are somewhat similar to the sources so that the initial estimation of the score functions is acceptable.

This analysis suggests a solution to our problem when the sources are highly mixed: initializing our method with a sub-optimal method like SOBI to obtain a low-ratio mixture, then applying our Markov method to finish with a quasi-optimal estimation. Using this procedure, we obtained a 70-dB SIR while SOBI led to 36dB and the Pham-Garat algorithm to 13dB. The initial sources, the mixtures and the reconstructed sources using SOBI and using Markov (initialized by SOBI) are shown in Fig. 4. It can be easily verified that the sources separated by our algorithm are extremely similar to the actual sources while each of the sources separated by SOBI contains some residual of the other source.

## CONCLUSION

In this paper, we proposed a maximum likelihood approach for blind image separation which takes into account both non-Gaussianity and spatial autocorrelation of the sources using a Markov model. The first simulations using artificial and real images confirm the good performance of our method in comparison to the classical methods which only take advantage of one of the above source properties. Our current method uses a non-parametric estimation of the conditional score functions and a gradient algorithm for maximizing the likelihood function. As a result, it is very time consuming. We are currently working on a parametric polynomial estimator of the conditional score functions, and on a modified equivariant Newton optimization algorithm to reduce the computational cost.

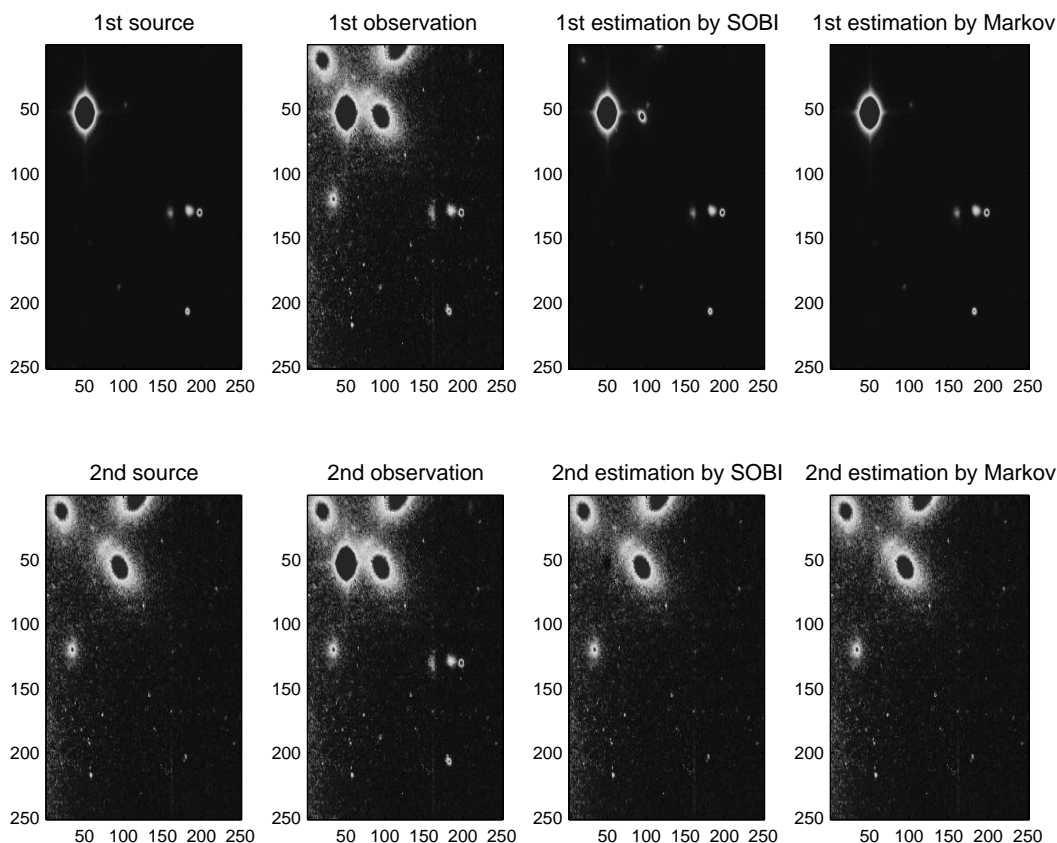


FIGURE 4. Experiment using astrophysical images.

## REFERENCES

1. J.-F. Cardoso, The three easy routes to independent component analysis: contrast and geometry, in *Proc. ICA2001*, San Diego, 2001, pp. 1-6.
2. A. Hyvarinen, J. Karhunen, E. Oja, *Independent component analysis*, John Wiley and Sons, 2001.
3. S. Hosseini, C. Jutten, and D.-T. Pham, Markovian source separation, *IEEE Trans. on Signal Processing*, vol. 51, pp. 3009-3019, 2003.
4. D.-T. Pham and P. Garat, Blind separation of mixture of independent sources through a quasi-maximum likelihood approach, *IEEE Trans. on Signal Processing*, vol. 45, pp. 1712-1725, July 1997.
5. L. Tong, R. Liu, V. Soon, and Y. Huang, Indeterminacy and identifiability of blind identification, *IEEE Trans. on Circuits Syst.*, vol. 38, pp. 499-509, May 1991.
6. A. Belouchrani, K. Abed Meraim, J.-F. Cardoso, and E. Moulines, A blind source separation technique based on second order statistics, *IEEE Trans. on Signal Processing*, vol. 45, pp. 434-444, Feb. 1997.
7. E. E. Kuruoglu, A. Tonazzini and L. Bianchi, Source separation in noisy astrophysical images modelled by markov random fields, in *Proc. ICIP'04*, pp. 2701-2704.
8. D.-T. Pham, Fast algorithms for mutual information based independent component analysis, *IEEE Trans. on Signal Processing*, vol. 52, no. 10, Oct. 2004.
9. J.-F. Cardoso and B. Laheld, Equivariant adaptive source separation, *IEEE Trans. on Signal Processing*, vol. 44, pp. 3017-3030, Dec. 1996.
10. M. Benidir and M. Barret, *Stabilité des filtres et des systèmes linéaires*, Edit. DUNOD, Paris, 1999.