

La souveraine habileté consiste a bien connaitre le prix des choses.
La Rochefoucauld (1613-1680)

A Quasilinear Parabolic Type Variational Solution for Fourier's Irreversible Heat Conduction Process with Minimum Principles in Case of Finite Signal Rate

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Abstract. The linear parabolic type pde for heat conduction process is here analysed with Fourier's constitutive equation. It is an old and well-known problem that the signal spreading velocity of temperature disturbances is infinite with constant phenomenological coefficient for the Fourier heat conduction process law. Here is shown a quasilinear solution for this problem with a finite signal rate. Connecting to parabolic pde it is shown the minimum principle solution of Onsager, Prigogine and Gyarmati type for the Fourier irreversible heat conduction process in energy, entropy and Fourier representation pictures too. For the stationary state of irreversible heat conduction process there is an interesting form for the variational minimum solution with the aid of the so-called "naive" variational procedure [1]. This procedure is equivalent for the irreversible heat conduction process with the Euler-Lagrange pde method's results. As to the phenomenological solutions of quasilinear irreversible heat conduction process the least dissipation of energy in stationary state leads for some materials in solid state physics to phonon theory (umklapp processes for dielectrics and semiconductors) and to the conduction electrons (in metals) for which heat conduction coefficients variate in hyperbolic way depending on the temperature. Solutions are subharmonic type [2].

INTRODUCTION

The Fourier heat conduction equation is mathematically a second order, parabolic and linear pde. The basic law \vec{q} known as Fourier constitutive equation \vec{q} gives the relationship between the heat flow and the temperature gradient [6]

$$q(r, t) = \lambda \nabla T(r, t), \quad w/m^2 \quad (1)$$

Although the thermal conductivity λ varies with temperature it is often treated as constant causing theoretical problems for the explanation of the differential equation of heat conduction process.

ENERGY-BALANCE EQUATION OF THE HEAT CONDUCTION PROCESS FOR A SMALL CONTROL VOLUME V

The energy - balance equation states that

$$\begin{aligned} & \text{[The rate of heat entering through the bounding surface of } V\text{]} + \\ & + \text{[The rate of energy generation in } V\text{]} = \text{[The rate of storage of energy in } V\text{]} \end{aligned} \quad (2)$$

where

$$\begin{aligned} & \text{[The rate of heat entering through the bounding surface of } V\text{]} = \\ & = - \int_A \mathbf{q} \cdot \mathbf{n} dA = - \int_V \nabla \cdot \mathbf{q} dV \end{aligned} \quad (3)$$

Here the divergence theorem is used to convert the surface integral to volume integral according to Gauss. And

$$\text{[The rate of energy generation in } V\text{]} = \int_V g(r, t) dV \quad (4)$$

as well as

$$\text{[The rate of energy storage in } V\text{]} = \int_V \rho c \frac{\partial T(r, t)}{\partial t} dV \quad (5)$$

Substituting (3), (4) and (5) into equation (2) then for an arbitrary small volume element V to remove the integral sign obtains

$$-\nabla \cdot \mathbf{q}(r, t) + g(r, t) = \rho c_V \frac{\partial T(r, t)}{\partial t} \quad (6)$$

We substitute $\mathbf{q}(r, t)$ from eqn (1) to (6) and with constant thermal conductivity results in a simplified form to

$$\nabla^2 T(r, t) + \frac{1}{\lambda} g(r, t) = \frac{1}{\alpha} \frac{\partial T(r, t)}{\partial t} \quad (7)$$

where

$$\alpha = \frac{\lambda}{\rho c_V} \quad (8)$$

is the thermal diffusivity. Equation (7) became the diffusion or the Fourier equation for a medium with uniform thermal conductivity and no heat generation in the medium

$$\nabla^2 T(r, t) = \frac{1}{\alpha} \frac{\partial T(r, t)}{\partial t} \quad (9)$$

and for stationary state (9) simplifies to the classical Laplace equation

$$\nabla^2 T(r, t) = 0 \quad (10)$$

The thermal diffusivity α is the property of the material. The higher the thermal diffusivity, the faster is the propagation of heat in the solid body. The Fourier equation with $\lambda = \text{const}$ apparently supposes such a solid which had so a great thermal diffusivity α by which the time of penetration of heat would be infinitely small. But such a solid does not exist in reality.

ON THE CRITICISM OF THE FOURIER EQUATION

1. Meixner [4] in 1970 made remarks on the theory of heat conduction in solids.
 - a. One of his objections was against Fourier's law is that it leads to the infinite velocity of propagation of temperature disturbances and therefore contradicts the basic laws of special relativity.
 - b. On the other hand it is a requirement for the classical theory that the propagation of the temperature disturbances cannot be faster than the average velocity of molecules.
2. It is a very important fact that eqn (9) is not a special representative of Onsager's strictly linear theory because Onsager always wrote the heat flux in the form of

$$J_q = L \nabla \frac{1}{T} \quad (11)$$

This means that according to the equality of flux [7] in different representational pictures the irreversible heat conduction process in Fourier picture must be inevitable way quasilinear with respect to Onsager's linear theory.

The solution of (9) gives only good approximation for systems whose non-equilibrium state are close to the adequate equilibrium position, i.e., when ∇T is not large. But near to equilibrium state $L = const$ or $L' = const$ are the right phenomenological coefficients.

3. It can be seen that Fourier's prescribes only in this law the form of the flux as a constitutive equation. Onsager's linear theory speaks from different content as does it Fourier about his own.

The heat conduction equation in different representational picture according to minimum variational principles of Onsager, Prigogine and Gyarmati [7]:

The Fourier representational picture shows that the heat conduction process is quasilinear with respect to Onsager's linear theory.

Fourier picture

$$\rho C_V \frac{\partial T}{\partial t} = \nabla \cdot (\lambda(T) \nabla T) \quad (12)$$

In energy and entropy representational pictures the heat conduction process is linear according to Onsager's theory.

Energy picture

$$\rho C_V \frac{\partial T}{\partial t} = -L' \nabla \cdot (\nabla \ln T) \quad (13)$$

Entropy picture

$$\rho C_V \frac{\partial T}{\partial t} = -L \nabla \cdot \left(\nabla \frac{1}{T} \right) \quad (14)$$

Eqns (12) , (13) and (14) shows the important mathematical fact that the divergence operations must be in right place. This is consistent of the quasilinearity and linearity in the different representations, respectively. The three equations are equal in content but in form are different.

THE EVALUATION OF THE HEAT CONDUCTION PROCESS WITH THE AID OF MATHEMATICAL MEANS

Stationary state

The physical meanings of the Laplace expression [8,9] is that

$$\tilde{T} - T^0 = \frac{R^2}{\delta} (\Delta T)_0 \quad (15)$$

Therefore one of the point values of the Laplace expression Δu is proportional to the difference, of the mean value of the function in question concerning of the range surrounding this point and the value of the function in this point. In case of the Laplace equation

$$\Delta T = \frac{8(\tilde{T} - T^0)}{RT} = 0 \quad (16)$$

Then $\frac{\partial T}{\partial t} = 0$ is the stationary state where inside the territory of R .

$$\tilde{T} - T^0 = 0 \quad (17)$$

This the case of equilibrium i.e., zero order stationarity [2,8,10] with harmonic solution.

In case of Laplace expression with subharmonic solution for irreversible processes of stationary state

$$\tilde{T} - T^0 = \frac{R^2}{8} \frac{(\nabla T)^2}{T} \quad (18)$$

For the non-stationary state

I. Parabolic linear heat conduction processes:

$$\rho C_V \frac{\partial T}{\partial t} = \nabla \cdot (\lambda \nabla T) \quad (19)$$

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \Delta T = \frac{8(\tilde{T} - T^0)}{R^2} \quad (20)$$

Here we suppose that for $R = \infty$ it is valid

$$\tilde{T} - T \approx 0 \quad (21)$$

This suggestion is compatible with the infinite velocity of propagation of the temperature disturbances. This means that in the linear case there is not any entropy production and energy dissipation, respectively. The rate of energy storage is too large which is not realistic.

II. Parabolic quasilinear irreversible heat conduction process where

$$\rho C_V(T) \frac{\partial T}{\partial t} = \nabla \cdot (\lambda(T) \nabla T) = 0 \quad (22)$$

$$\rho_0 C_V(T) \frac{\partial T}{\partial t} = \lambda'(T) (\nabla T)^2 + \lambda(T) \Delta T \quad (23)$$

Expressing from (23)

$$\Delta T = \frac{\rho C_V(T) \frac{\partial T}{\partial t} - \lambda'(T)(\nabla T)^2}{\lambda(T)} \quad (24)$$

$$\tilde{T} - T_0 = \frac{R^2}{8} \left(\frac{\rho C_V(T) \frac{\partial T}{\partial t}}{\lambda(T)} - \frac{\lambda'(T)(\nabla T)^2}{\lambda(T)} \right) \quad (25)$$

$$\tilde{T} - T_0 = \frac{R^2}{8} \left(\frac{1}{\alpha^*} \frac{\partial T}{\partial t} - \frac{\lambda'(T)(\nabla T)^2}{\lambda(T)} \right) \quad (26)$$

where α^* is the reduced diffusivity and the second term in parenthesis is a reduced dissipative flux, so the velocity of propagation of temperature disturbances has a finite value in consequence of $\lambda(T)$. Therefore the energy storage in volume V will be a real one. Thus $\tilde{T} - T \neq 0$ and it is changing depending on the different solids. The reduced dissipative flux also shows that according to the quasilinear heat conduction equation in Fourier picture the entropy production and energy dissipation cause quasilinear information loss.

THE LEAST DISSIPATION OF ENERGY IN STATIONARY STATE [2,7] FOR DIELECTRICS AND METALS.

In energy picture the quasilinear pde in stationary state for heat conduction is

$$\frac{\lambda(T)}{T} (\nabla T)^2 - \lambda(T) \Delta T = 0 \quad (27)$$

In case of phonon collision for umklapp processes Fig. 1 shows for the heat conduction coefficient of germanium the temperature dependency. The upper curve represents the measured points while the lower one is the curve of fitting with very good agreement for least dissipation of energy. The agreement is similarly good for silicon too [2]. For metals by conductive electrons Fig. 2 represents the measurements of berillium's heat conductivity with also good agreement for the fitting curve. The max-min property is shown by the harmonic function

$$z(x, y) = \ln(x^2 + y^2) \quad (28)$$

The max-min property is shown by the subharmonic function

$$z(x, y) = x^2 + y^2 \quad (29)$$

THE "NAIVE" VARIATION PROCEDURE

As was stated before there is a discrepancy between Onsager's linear theory and the Fourier's constitutive equation. The picture constants can be expressed formally [1]

$$\begin{aligned} \lambda &= T^{-1} L^* = T^{-2} L && \text{in Fourier picture} \\ L^* &= T \lambda = T^{-1} L && \text{in Energy picture} \end{aligned} \quad (30)$$

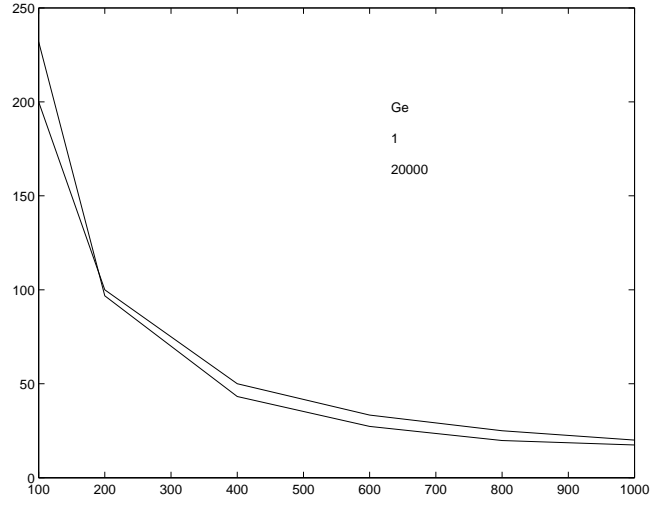


FIGURE 1. Temperature dependence of germanium's heat conduction coefficient with a hyperbola fitted.

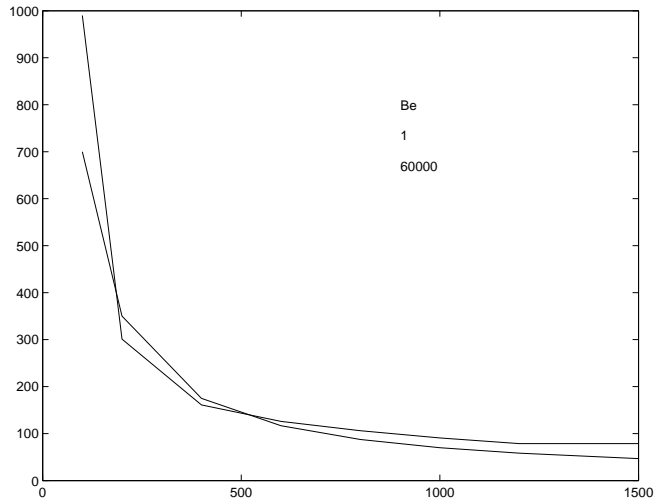


FIGURE 2. Temperature dependence of berillium's heat conduction coefficient with a hyperbola fitted.

$$L = T^2 \lambda = T L^* \quad \text{in Entropy picture}$$

According to Onsager's theory the phenomenological coefficients are constant near equilibrium. But in the stationary case of heat conduction we must take care with them. They are constant only their own picture and become temperature dependent if they enter a foreign picture i.e., [1,7]

$$\begin{aligned} \lambda &= T^{-1} L_{\lambda}^*(T) = T^{-2} L_{\lambda}(T) && \text{in Fourier picture} \\ L^* &= T \lambda_{L^*}(T) = T^{-1} L_{L^*}(T) && \text{in Energy picture} \\ L &= T^2 \lambda_L(T) = T L_L^*(T) && \text{in Entropy picture} \end{aligned} \quad (31)$$

On the base of (35) were built the nonlinear Euler-Lagrange differential equations from the nonlinear Lagrange densities among others

$$(\nabla T)^2 - T\Delta T = 0 \quad \text{and} \quad 2(\nabla T)^2 - T\Delta T = 0 \quad (32)$$

and

$$(\nabla T)^2 + T\Delta T = 0 \quad \text{and} \quad 2(\nabla T)^2 + T\Delta T = 0 \quad (33)$$

The detailed procedure is written in [1,7].

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