A NEW BOUND FOR DISCRETE DISTRIBUTIONS BASED ON MAXIMUM ENTROPY

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Abstract

In this paper we compare some classical and well known bounds (as Chernoff's bound or moment bounds) for nonnegative integer-valued random variables for estimating the survival probability and a new tighter bound stemming from Maximum Entropy technique constrained by fractional moments given by $\mathbb{E}(X^\alpha)$, $\alpha \in \mathbb{R}^+$.

Because the classical bounds are usually given in terms of integer moments or in terms of moment generating function, they may be able to exploit only partially the information contained in the data: for this reason these bounds are not very tight although they can be easily calculated.

We exploit a result of Lin (1992) which supports the characterization of a distribution through its fractional moments and we show (Novi Inverardi and Tagliani (2003)) that the Maximum Entropy probability mass function $P_M^{(\mathrm{fm})}$ recovered involving M fractional moments converges in entropy to the true probability mass function P. This last result means that if we are interested in approximating a discrete distribution and/or some its characteristic constants (think to expected values, tails, probabilities or other) the equivalent counterparts evaluated on $P_M^{(\mathrm{fm})}$ are as close as we like to the true values and the closeness depends on the (increasing) value of M.

But usually the available knowledge on a distribution is expressed by integer moments and not by fractional moments. This means that we need a link between moment generating function and/or integer moments and fractional moments.

Traditionally the moment generating function of a random variable X is used to generate positive integer moments of X. But it is clear that the moment generating function also contains a wealth of knowledge about arbitrary real moments and hence, on fractional moments. Taking this into account, to obtain fractional

moments, Cressie and Borkent (1986) exploit some properties of the moment generating function and its fractional derivatives; Klar (2003), in addition to moment generating function, considers the knowledge of a set of integer moments which can be obtained by proper integration of the moment generating function on a contour $\mathcal{C}$ of the complex plane. In particular, the results due to Klar (2003) and Kammler (1977) give the key to obtain fractional moments from integer moments and moment generating function. Hence the new tighter bound which we propose is able to capture optimally all the information content summarized by few fractional moments.

As numerical examples clearly show, fractional moments are definitely better than integer moments for recovering a probability distribution via Maximum Entropy setup; as a consequence, the bounds for P obtained via fractional moments are tighter than the classical bounds built involving integer moments. But, as is easy to see, they are not always easy to evaluate, at least analytically. However, in the computer-era, this should not to be perceived as a serious drawback.

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