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Information Intrinsic Geometric Flows : Kähler-Ricci & Calabi Flows on Siegel & Hyper-Abelian Metrics of Complex Autoregressive Models

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1. Preambule

Geometric Flow Theory is cross fertilized by diverse elements coming from Pure Mathematic (geometry, algebra, analyse, PDE) and Mathematical Physic (calculus of variations, General Relativity, Einstein Manifold, String Theory), but its foundation is mainly based on Riemannian Geometry, as explained by M. Berger in a recent panoramic view of this discipline [Berger], its extension to complex manifolds, the Erich Kähler's Geometry [Kähler1], vaunted for its unabated vitality by J.P. Bourguignon [Bourguignon] in [Kähler2], and Minimal Surface Theory recently synthetized by F. Hélein [Helein]. This paper would like to initiate seminal studies for applying intrinsic geometric flows in the framework of information geometry theory. More specifically, after having introduced Information metric deduced for Complex Auto-Regressive (CAR) models from Fisher Matrix (Siegel Metric and Hyper-Abelian Metric from Entropic Kähler Potential), we study asymptotic behaviour of PARCOR parameters (reflexion coefficients of CAR models) driven by intrinsic Information geometric Kähler-Ricci and Calabi flows. These Information geometric flows can be used in different contexts to define distance between CAR models interpreted as geodesics of Entropy Manifold (e.g : distance between plane curves parametrized by CAR models).

2. Siegel Metric for Complex Autoregressive Model

Chentsov has defined main axioms of Information Geometry. In this Theory, we consider families of parametric density functions $G_\Theta = \{p(./\theta):\theta\in\Theta\}$ with $\Theta = [\theta_1 \ \cdots \ \theta_n]^T$, from which we can define a Riemannian Manifold Structure by meam of Fisher Information matrix $(g_{ij}(\theta))_{ij}$:

$$g_{ij}(\theta) = E_\theta\left[\frac{\partial \ln p(./\theta)}{\partial\theta_i}.\frac{\partial \ln p(./\theta)}{\partial\theta_j^*}\right], \text{ with the Riemannian metric : } ds^2 = \sum_{i,j=1}^{n} g_{ij}(\theta).d\theta_i.d\theta_j^*$$

This metric can also be naturally introduced by a Taylor expansion of Kullback Divergence :

$$K(\theta,\tilde\theta)\Big|_{\tilde\theta=\theta+d\theta} \cong K(\theta,\theta) + \left(\frac{\partial K(\theta,\tilde\theta)}{\partial\tilde\theta}\right)^+_{\tilde\theta=\theta}(\tilde\theta-\theta) + \frac{1}{2}(\tilde\theta-\theta)^+\left(\frac{\partial^2 K(\theta,\tilde\theta)}{\partial\tilde\theta\partial\tilde\theta^*}\right)(\tilde\theta-\theta) \cong \frac{1}{2}\sum_{i,j} g_{ij}(\theta)d\theta_i.d\theta_j^*$$

We demonstrate easily that this Fisher metric is equivalent to the Siegel metric, introduced by Siegel in the 60's in the framework of Symplectic Geometry. Indeed, if we consider a Complex Multivariate Gaussian Law :

$$p(X/R_n,m_n) = (2\pi)^{-n}.|R_n|^{-1}.e^{-Tr[\hat R_n.R_n^{-1}]} \quad \text{with} \quad \hat R_n = (X-m_n).(X-m_n)^+ \quad \text{such that} \quad E[\hat R_n] = R_n$$

it is well-known that the Fisher Information matrix is given by : $\mathrm{g}_{\mathrm{ij}}(\theta) = -Tr[\partial_i R_n.\partial_j R_n^{-1}] + \partial_i m_n^+.R_n^{-1}.\partial_j m_n$

In the following, we will only consider random process with zero mean $m_n = E[X] = 0$, and so if we apply the following relation $R_n.R_n^{-1} = I_n \Rightarrow \partial R_n = -R_n.\partial R_n^{-1}.R_n$, the Fisher matrix is reduced to : $g_{ij}(\theta) = Tr[(R_n.\partial_i R_n^{-1}).(R_n.\partial_j R_n^{-1})]$ with the associated Riemannian metric :

$$ds^2 = \sum_{i,j} g_{ij}(\theta).d\theta_i.d\theta_j = \mathrm{Tr}\left[\mathrm{R_n}.\left(\sum_i \partial_i R_n^{-1}.d\theta_i\right).R_n.\left(\sum_j \partial_j R_n^{-1}.d\theta_j\right)\right] \quad \text{with} \quad \mathrm{dR_n^{-1}} = \sum_k \partial_k R_n^{-1}.d\theta_k$$

We can then observe that it is completely equivalent with Siegel Metric : $\boxed{\mathrm{ds}^2 = Tr[(R_n.dR_n^{-1})^2]}$ introduced by Karl Ludwig Siegel in his book « Symplectic Geometry ». This metric is invariant under the action of the following group $(GL_n(C),.)$: $R_n \to W_n.R_n.W_n^+$, $\mathrm{W_n} \in GL_n(C)$, and geodesics are given by :

$$\begin{cases} \dot{S}(s) = S(s).H \\ S(s) = R_1^{-1/2}.e^{H.s}.R_1^{-1/2} \end{cases} \quad \text{with} \quad S(0) = R_1^{-1}$$

From this metric, if we take the Frobenius Norm $\|X\| = \sqrt{\langle X, X\rangle}$ avec $\langle \mathrm{X}, \mathrm{Y}\rangle = Tr\left[X.Y^+\right]$, distance between 2 covariances matrices $R_n^{(1)-1}$ and $R_n^{(2)-1} \in P_{n+1}(C)$ is given by Jensen distance :

$$d^2(R_n^{(1)-1}, R_n^{(2)-1}) = \left\|\ln\left(R_n^{(1)1/2}.R_n^{(2)-1}.R_n^{(1)1/2}\right)\right\|^2 = \sum_{i=1}^{n} \ln^2 \lambda_i^{(n)} \quad \text{with} \quad \det\left(R_n^{(1)1/2+}.R_n^{(2)-1}.R_n^{(1)1/2} - \lambda_i^{(n)}.I_n\right) = 0$$

In case of Complex Autoregressive models, we can exploit this specific blocks structure of covariances matrices and prove that :

$$\Omega_n = R_n^{(1)1/2+}.R_n^{(2)-1}.R_n^{(1)1/2} = \begin{bmatrix} \beta_{n-1} & \beta_{n-1}.W_{n-1}^+ \\ \beta_{n-1}.W_{n-1} & \Omega_{n-1} + \beta_{n-1}.W_{n-1}.W_{n-1}^+ \end{bmatrix} \quad \text{with} \quad \mathrm{W}_{\text{n-1}} = \sqrt{\alpha_{n-1}^{(1)}}.R_{n-1}^{(1)1/2+}.\left[A_{n-1}^{(2)} - A_{n-1}^{(1)}\right]$$

$$\text{and} \quad \beta_{\text{n-1}} = \frac{\alpha_{n-1}^{(2)}}{\alpha_{n-1}^{(1)}} \quad \text{where} \quad \alpha_n^{-1} = \left[1 - |\mu_n|^2\right]\alpha_{n-1}^{-1} \quad \text{and} \quad A_n = \begin{bmatrix} A_{n-1} \\ 0 \end{bmatrix} + \mu_n.\begin{bmatrix} A_{n-1}^{(-)} \\ 1 \end{bmatrix}$$

We easily observe that Jensen Metric can then be easily computed recursively to the CAR model order by using this following equations giving interleaving eigenvalues $\Lambda_{\text{n}} = diag\left\{\cdots \lambda_i^{(n)} \cdots\right\}$ of $\Omega_n = R_n^{(1)1/2+}.R_n^{(2)-1}.R_n^{(1)1/2}$ at successive order :

$$\boxed{F^{(n)}\left(\lambda_k^{(n)}\right) = \lambda_k^{(n)} - \beta_{n-1} + \beta_{n-1}.\lambda_k^{(n)}.\sum_{i=1}^{n-1} \frac{\left|W_{n-1}^+.X_i^{(n-1)}\right|^2}{\left(\lambda_i^{(n-1)} - \lambda_k^{(n)}\right)} = 0 \quad \text{and} \quad \frac{X_k^{(n)}}{X_{k,1}^{(n)}} = \begin{bmatrix} 1 \\ -\lambda_k^{(n)}.U_{n-1}.\left(\Lambda_{n-1} - \lambda_k^{(n)}.I_{n-1}\right)^{-1}.U_{n-1}^+.W_{n-1} \end{bmatrix}}$$

Always based on Blocks structure of covariance matrices, we can deduce a recursive expression of the metric : $ds_n^2 = ds_{n-1}^2 + \left(\frac{d\alpha_{n-1}}{\alpha_{n-1}}\right)^2 + \alpha_{n-1}.dA_{n-1}^+.R_{n-1}.dA_{n-1}$. From which, we define a new hyperbolic distance betwwen CAR models as Inferior Bound of this metric : $\boxed{ds_n^2 > \sum_{k=0}^{n-1}\left(\frac{d\alpha_k}{\alpha_k}\right)^2 + \sum_{k=1}^{n-1}\frac{|d\mu_k|^2}{1-|\mu_k|^2}}$

3. Erich Kähler Geometry with Information metric based on Entropic Hyper-Abelian Kähler Potential

Natural extension of Riemannian Geometry to Complex Manifold has been introduced by a seminal paper of Erich Kähler during 30th 's of last century. We can easily apply this geometric framework for information metric definition.

Let a complex Manifold M^n of dimension n, we can associate a Kählerian metric, which can be locally defined by its definite positive Riemannian form : $ds^2 = 2\sum_{i,j=1}^{n} g_{i\bar{j}}.dz^i dz^{\bar{j}}$ with $\left[g_{i\bar{j}}\right]_{i,j}$ an Hermitian definite positive matrix. Kähler assumption assumes that we can define a Kähler potential Φ, such that $g_{i\bar{j}} = \frac{\partial^2 \Phi}{\partial z^i \partial z^{\bar{j}}}$.

Classical tensors of Riemannian Geometry can be also extended by the following expressions : $\Gamma_{ij}^k = \sum_{l=1}^{n} g^{k\bar{l}} \frac{\partial g_{i\bar{l}}}{\partial z^j}$ and $\Gamma_{\bar{i}\bar{j}}^{\bar{k}} = \sum_{l=1}^{n} g^{\bar{k}l} \frac{\partial g_{l\bar{i}}}{\partial z^{\bar{j}}}$, $R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial z^{\bar{l}}} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial z^{\bar{l}}}$

Main relation, given by Erich Kähler, is that Ricci tensor can be expressed by : $\boxed{R_{i\bar{j}} = -\frac{\partial^2 \log\left(\det g_{k\bar{l}}\right)}{\partial z_i \partial \bar{z}_j}}$ with the associated scalar curvature $R = \sum_{k,l=1}^{n} g^{k\bar{l}}.R_{k\bar{l}}$. One important geometric flow, in physic & mathematic, is the

Kähler-Ricci flow which drive the evolution of the metric by : $\boxed{\dfrac{\partial g_{ij}}{\partial t} = -R_{ij} + \dfrac{1}{n} R g_{ij}}$. This flow converges to a Kähler-Einstein metric $R_{i\bar{j}} = k_0 . g_{k\bar{l}}$, which is also equivalent to : $-\dfrac{\partial^2 \log(\det g_{k\bar{l}})}{\partial z_i \partial \bar{z}_j} = k_0 . \dfrac{\partial^2 \Phi}{\partial z^i \partial z^{\bar{j}}}$, known as Monge-Ampère equation : $\boxed{\det(g_{k\bar{l}}) = |\psi|^2 e^{-k_0 \Phi}}$ where Φ is a Kähler potential and ψ a non specified holomorphic function, but that could be reduced to unity : if $k_0 \neq 0$) by choice of a new Φ potential, or if k_0=0 by local holomorphic coordinates selection so that volume $\det(g_{k\bar{l}})$ is reduced to 1 (cancellation of Ricci tensor is existence condition of this coordinates system).

In case of Complex Auto-Regressive (CAR) models, if we choose as Kähler potential Φ ***with*** $g_{i\bar{j}} = \dfrac{\partial^2 \Phi}{\partial z^i \partial z^{\bar{j}}}$***, the Entropy of the process expressed according to PARCOR coefficients (reflexion coefficients) in the unit Poincaré Polydisk*** $\{ z / |z_k| < 1 \ \forall k = 1, \ldots n \}$***, the Kähler potential is given by :***

$\boxed{\Phi = \sum_{k=1}^{n-1} \rho_k . \ln\left[1 - |z_k|^2\right] = \ln K_D(z,z)}$***, with Bergman kernel*** $K_D(z,z) = \prod_{k=1}^{n-1} \left(1 - |z_k|^2\right)^{\rho_k}$

Very surprisingly, this case was the first example of potential studied by Erich Kähler in his seminal paper, named by Erich Kähler* Hyper-Abelian case*, relatively to the other case studied by him as* Hyper-Fuchsian *Case $\Phi = \rho . \ln\left(1 - \sum_{k=1}^{n} |z_k|^2\right)$ ***in unit hyper-ball*** $\left\{ z / \sum_{k=1}^{n} |z_k|^2 < 1 \right\}$**.**

This choice of Kähler potential as Entropy of CAR model, can be justified by remarking that Entropy Hessian along one direction in the tangent plane of parametric manifold is a definite positive form that can be considered as a Kählerian differential metric : $g_{ij}^{(H)}(\theta) = -\dfrac{\partial^2 H(P_\theta)}{\partial \theta_i . \partial \theta_j} \quad \Rightarrow ds_H^2 = \sum_{i,j=1}^{n} g_{ij}^{H}(\theta) . d\theta_i . d\theta_j$

It is proved by considering the following $\gamma - entropy$: $H_\gamma(p) = \int \gamma[p(x)] . dx$

If we derive *H* in the direction *f*, then : $dH_\gamma(p:f) = \dfrac{d}{dt} H\gamma(p + tf)\big|_{t=0} = -\int \gamma'[p(x)] . f(x) . dx$

and the second derivative in the direction g : $d^2 H_\gamma(p:f,g) = -\int \gamma''[p(x)] . f(x) . g(x) . dx$

Hessian is then given by : $\Delta_f H_\gamma(p) = d^2 H_\gamma(p:f,f) = -\int \gamma''[p(x)] . f(x)^2 . dx$

Let $F_\Omega = \{ p(./\theta) \in P_\gamma : \theta \in \Omega \}$ Manifold in parametric space and $dp = dp(\theta) = \sum_{i=1}^{n} \dfrac{\partial p(./\theta)}{\partial \theta_i} . d\theta_i$

We can develop Hessian relation by : $\Delta_\gamma H_\gamma(p) = d^2 \{ H_\gamma(p) \} (\theta) = -\int \gamma''(p) [dp]^2 . dx$

As long as γ is convex, we find the final result $\boxed{\mathrm{ds}_\gamma^2 = -\Delta_\gamma H_\gamma(p)}$ with :

$ds_\gamma^2 = \sum_{i,j=1}^{n} g_{ij}^{(\gamma)}(\theta) . d\theta_i . d\theta_j \quad \text{avec} \quad g_{ij}^{(\gamma)}(\theta) = \int \gamma''(p) . \dfrac{\partial p}{\partial \theta_i} . \dfrac{\partial p}{\partial \theta_j} . dx$

By choosing : $\begin{cases} \gamma_\alpha(x) = (\alpha - 1)^{-1} . (x^\alpha - x) & \alpha \neq 0 \\ \gamma_\alpha(x) = x . \ln(x) & \alpha = 0 \end{cases}$, we have : $g_{ij}^{(\alpha)}(\theta) = \int p^\alpha . \dfrac{\partial \ln p}{\partial \theta_i} . \dfrac{\partial \ln p}{\partial \theta_j} . d\theta_i . d\theta_j$

So, in case of Complex Autoregressive models, with as previously Whishart density, Entropy is given by : $H_n = -\int P(X_n / m_n, R_n) . \ln[P(X_n / m_n, R_n)] dX_n = \ln|R_n| + n . \ln(\pi . e)$. If we use the blocks structure of covariance matrix in case of CAR models and relation : if $G = \begin{bmatrix} a & V^+ \\ W & B \end{bmatrix}$ then $|G| = |a| . |B - a^{-1} . W . V^+|$, we obtain the Entropy expressed according to PARCOR coefficients :

$$\boxed{H_n = \sum_{k=1}^{n-1}(k-n).\ln\left[1-|\mu_k|^2\right]+n.\ln\left[\pi.e.\alpha_0^{-1}\right] \text{ with } \begin{cases}\alpha_0^{-1}=P_0=\frac{1}{n}\sum_{k=1}^{n}|x_k|^2 \\ X_n=\left[x_1 \ \cdots \ x_n\right]^T\end{cases}}$$

The Kähler metric is then given by Hessien of Entropy, where Entropy is considered as Kähler potential.

Let $\theta^{(n)}=\left[P_0 \quad \mu_1 \qquad \mu_{n-1}\right]^T=\left[\theta_1^{(n)} \quad \cdots \quad \theta_n^{(n)}\right]^T$, we have then $\boxed{g_{11}=nP_0^{-2}}$, $\boxed{g_{ij}=\frac{(n-i).\delta_{ij}}{\left(1-|\mu_i|^2\right)^2}}$

From which, we deduce the final expression of Kähler metric of this Hyper-Abelian Case :

$$\boxed{ds_n^2=n.\left(\frac{dP_0}{P_0}\right)^2+\sum_{i=1}^{n-1}(n-i)\frac{|d\mu_i|^2}{\left(1-|\mu_i|^2\right)^2}}$$

4. Information Kähler-Ricci & Calabi Flows for Complex Autoregressive models

First of Intrinsic Geometric Flow is Ricci Flow. Historical Root of Ricci flow can be found in Hilbert work on General Relativity, where the Minimal Action Principal is defined with tool of Calculus of variations. Einstein Equation was derived by Hilbert from Functional Minimisation, where the "Hilbert Action" S is defined as the integral of scalar Curvature R on the Manifold M^n : $S(g)=\int_{M^n} R.\sqrt{\det(g)}.d^n x=\int_{M^n} R.d\eta$ with volume $V(g)=\int_{M^n} d\eta$ and $R=\sum_\mu\sum_\nu g^{\mu\nu}.R_{\mu\nu}$ the scalar curvature defined by mean of Ricci Tensor $R_{\mu\nu}$ and the metric tensor $g^{-1}=\left[g^{ij}\right]$. Fundamental theorem of Hilbert said that for $n\geq 3$, $S(g)$ is minimal with $V(g)=cste$ if $R(g)$ is constant and g is an Einstein metric : $R_{ij}=\frac{1}{n}R.g_{ij}$. So the more natural geometric flow that converges to Einstein metric is given by :

$\frac{\partial g_{ij}}{\partial t}=2\left[-R_{ij}+\frac{1}{n}Rg_{ij}\right]$, but unfortunatelly this flow exhibits some convengence problems in finite time. That the main raison why R. Hamilton has introduced the following Ricci flow :

$\frac{\partial g_{ij}}{\partial t}=2\left[-R_{ij}+\frac{1}{n}r.g_{ij}\right]$ avec $r=\left(\int_{M^n} Rd\eta\right)/\left(\int_{M^n} d\eta\right)$ where R has been replaced by r the mean scalar curvature on the Manifold or equivalently this one after coordinate change $\frac{\partial g_{ij}}{\partial t}=-2.R_{ij} \quad \forall i,j$. This flow can be interpreted as Fourrier Heat Operator acting on metric g, by using isothermal coordinates introduced by G. Darmois. In such local isothermal coordinates system $\left\{x^i\right\}_{i=1,2}$, following relations are cancelled : $F^k=\Delta_g x^i=-\sum_{\lambda,\mu} g^{\lambda\mu}\Gamma_{\lambda\mu}^k$ for $k=1,2$, with : $\Delta_g f=\sum_{\lambda,\mu} g^{\lambda\mu}\left(\partial_{\lambda\mu}^2 f-\sum_k \Gamma_{\lambda\mu}^k\partial_k f\right)$ Laplace-Beltrami operator. More specifically A. Lichnerowicz has proved that we can expressed Ricci tensor as $R_{ij}=-G_{ij}-L_{ij}$ with $G_{ij}=\frac{1}{2}\sum_{\lambda\mu} g_{\lambda\mu}\partial_{\lambda\mu}^2 g_{ij}+H_{ij}$ and $L_{ij}=\frac{1}{2}\left(\sum_\mu g_{i\mu}\partial_j F^\mu+\sum_\mu g_{j\mu}\partial_i F^\mu\right)$

Then in isothermal coordinates, we have : $-2.R_{ij}=\sum_k\sum_l g^{kl}.\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}+Q_{ij}\left(g^{-1},\partial g\right)$. So, Ricci Flow can be interpreted as a Diffusion Equation action on metric g : $\frac{\partial g_{ij}}{\partial t}=\Delta_g g_{ij}$.

This geometric flow has been extended in the framework of Kählerian Geometry with the Kähler-Ricci Flow : $\boxed{\dfrac{\partial g_{ij}}{\partial t} = -R_{ij} \quad \forall i, j}$. If we use definition of metric g as previously, as derived from Entropic Kähler potential in case of a Complex Autoregressive models, then we can express Ricci tensor : $\left[R_{k\bar{l}}\right]_{k,l} = \left[-\dfrac{\partial^2 \log\det\left(g_{i\bar{j}}\right)}{\partial z_k \partial \bar{z}_l}\right]_{k,l}$:

$$\boxed{R_{k\bar{l}} = -2\delta_{kl}\left(1-|\mu_k|^2\right)^{-2} \quad \text{for } k = 2,\ldots,n-1 \text{ and } R_{11} = -2P_0^{-2}}$$

Identically, we can deduce scalar curvature $R = \sum_{k,l} g^{k\bar{l}}.R_{k\bar{l}}$ of CAR models :

$$\boxed{R = -2.\left[n^{-1} + \sum_{j=1}^{n-1}(n-j)^{-1}\right] = -2.\left[\sum_{j=0}^{n-1}(n-j)^{-1}\right]}$$ (this curvature diverges when n tends to infinity).

We can observe that we have an Einstein metric but more generally defined as :

$\boxed{\left[R_{ij}\right] = B^{(n)}\left[g_{ij}\right] \text{ with } R = Tr\left[B^{(n)}\right]}$ where $B^{(n)} = -2diag\left\{..,(n-i)^{-1},..\right\}$

If we exame Kähler-Ricci flow acting on PARCOR parameters, we have : $\dfrac{\partial \ln\left(1-|\mu_i|^2\right)}{\partial t} = -\dfrac{1}{(n-i)}$ and $\dfrac{\partial \ln P_0}{\partial t} = \dfrac{1}{n}$. From which, we obtain the behaviour of PARCOR coefficient in asymptotic case :

$$\boxed{|\mu_i(t)|^2 = 1 - \left(1-|\mu_i(0)|^2\right)e^{-\frac{t}{(n-i)}} \xrightarrow[t\to\infty]{} |\mu_i(t)|^2 = 1}$$ that converges to unit circle.

We can introduce another Intrinsic Geometric Flow, which has been introduced by Calabi in the framework of Kähler geometry. This flow is related to the notion of "extreme" metrics. This metrics are obtained by a Calabi flow and are deduced from a new functional that depends on the square of the scalar curvature and no longer the scalar curvature itself : $\boxed{\Theta(g) = \int_M R^2 d\eta}$. According to Schwarz inequality, Kähler-Einstein metric is also an extreme metric that minimises $\Theta(g)$. Solution is defined as steady state of the PDE equation : $\boxed{\dfrac{\partial \psi}{\partial t} = R_\psi - \bar{R}}$ with ψ Kähler potential associated to the Kähler metric $g_{i\bar{j}} = \dfrac{\partial^2 \psi}{\partial z^i \partial z^{\bar{j}}}$, R_ψ scalar curvature and $\bar{R}$ its mean value on the Manifold. This can be used to defined shortest path between two parametric models. Indeed, if we consider the path $\psi(t)$ $\left(0 \le t < 1\right)$ and if we assume the existence of Kähler potential $\left\{\phi(s,t) \,:\, 0 \le t < 1\right\}$ driven by Calabi Flow :

$$\frac{\partial \phi}{\partial t} = R(\phi) - \bar{R} \quad \text{and} \quad \phi(0,t) = \psi(t).$$

Length of path $L(s)$ is then given by : $\boxed{L(s) = \int_0^1 \left(\int_V \left(\dfrac{\partial \phi(s,t)}{\partial t}\right)^2 .d\eta_{\phi(s,t)}\right)^{\frac{1}{2}} .dt}$.

If we apply same approach as previously for CAR models, Calabi flow will act on Entropy $-H(p)$ defined as Kähler potential : $-\sum_{k=1}^{n-1}(k-n).\dfrac{\partial \ln\left[1-|\mu_k|^2\right]}{\partial t} - n.\dfrac{\partial \ln\left[\pi.e.P_0\right]}{\partial t} = -2.\left[\sum_{k=1}^{n}\dfrac{1}{n-k} + \dfrac{1}{n}\right]$

We then deduce the asymptotic behaviour of PARCOR coefficients submitted to Calabi Flow :

$$\boxed{\frac{\partial \ln\left(1-|\mu_k|^2\right)}{\partial t} = -\frac{2}{(n-k)^2} \quad \text{with} \quad \frac{\partial \ln P_0}{\partial t} = \frac{2}{n^2}}$$

We can observe, as previsouly, that Calabi flow will drive PARCOR coefficients evolutions to unit circle.

5. Références

In Final paper, we will illustrate the use of these metrics and intrinsic geometric flows to define new robuste distance between planar shapes that are parameterised by complex Auto-Regressive models. This can be used for Planar Shape Recognition or classification

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