An amended MaxEnt formulation for deriving Tsallis factors, and associated issues

J.-F. Bercher

Équipe Signal et Information, MOSIM, ESIEE, France.

February 28, 2006

The formalism of nonextensive statistical mechanics [1, 2] leads to a generalized Boltzmann factor in the form of a Tsallis distribution (or factor) that depends on an entropic index and recovers the classical Boltzmann factor as a special limit case [1]. This distribution may behave as a power law. In a wide variety of fields, experiments, numerical results and analytical derivations fairly agree with the description by a Tsallis distribution.

Tsallis' distributions (sometimes called Levy distributions) are derived by maximization of Tsallis entropy [3], under suitable constraints. However, these distributions do not coincide with those derived for the classical MaxEnt and consequently will not be justified from a probabilistic point of view, because of the uniqueness of the rate function in the large deviations theory [4, 5]. In view of the success of nonextensive statistics, there should exist a probabilistic setting that provides a justification for the maximization of Tsallis entropy. There are now several indications that results of nonextensive statistics are physically relevant for partially equilibrated or nonequilibrated systems, with a stationary state characterized by fluctuations of an intensive parameter [6, 7].

In this work I propose an amended MaxEnt formulation for systems with a displaced equilibrium, find that the relevant entropy in this setting is the Rényi entropy, interpret the mean constraints, derive the correct form of solutions, propose numerical procedures for estimating the parameters of the Tsallis factor and characterize the associated entropies.

I show that a Tsallis-like distribution can be derived as the minimizer of the Kullback-Leibler information divergence with respect to a reference distribution Q (or equivalently as the maximizer of Shannon Q-entropy), where the minimization is carried under a mean log-likelihood constraint and a (mean) observation constraint. The mean log-likelihood constraint characterizes the 'displacement' from the conventional equilibrium. This corresponds to an amended MaxEnt formulation, where one looks for an intermediate distribution between two references P_1 and Q, with an additional constraint that tunes the new equilibrium. The solution P^* is analog to the escort distribution, that appears quite arbitrary [4, 5] in non-extensive statistics.

Then I present two scenarii for the mean observation constraint: the observable is taken as the mean under P_1, the distribution of a "subsystem", or under P^*, the apparent distribution of the system. In the two cases, I show that the amended MaxEnt formulation leads to the maximization of the Rényi α Q-entropy, subject to the corresponding mean constraint. So doing, we recover two of the classical choices of constraint in the nonextensive literature. These two scenarii lead to two Tsallis-like distributions with opposite exponents, and the entropic index α appears to be simply the Lagrange parameter associated to the log-likelihood constraint.

A difficulty comes from the determination of the parameter of the Tsallis-like distributions, that are self-referential. In order to identify the value of their natural parameter, I propose two 'alternate' (but effectively computable) dual functions, whose maximizations enable to exhibit the optimum parameters.

In the conference, I will illustrate these results for several references $Q(x)$ present the results of numerical evaluations, and recover some well-known entropy functionals in the classical Gibbs-Boltzmann-Shannon limit. We will also give further results concerning symmetry and duality between the solutions associated with classical and generalized mean constraint, and between entropy functionals. Finally I will discuss the underlying Legendre structure of generalized thermodynamics associated to this setting.

References

[1] C. Tsallis, "Nonextensive statistics: Theoretical, experimental and computational evidences and connections," *Brazilian Journal of Physics*, vol. 29, pp. 1–35, March 1999.

[2] C. Tsallis, "Entropic nonextensivity: a possible measure of complexity," *Chaos, Solitons,& Fractals*, vol. 13, pp. 371–391, 2002.

[3] C. Tsallis, "Possible generalization of Boltzmann-Gibbs statistics," *Journal of Statistical Physics*, vol. 52, no. 1-2, pp. 479–487, 1988.

[4] B. R. La Cour and W. C. Schieve, "Tsallis maximum entropy principle and the law of large numbers," *Phys. Rev. E*, vol. 62, pp. 7494 – 7496, November 2000.

[5] J. Grendar, M. and M. Grendar, "Maximum entropy method with non-linear moment constraints: challenges," in *Bayesian inference and maximum entropy methods in science and engineering* (G. Erickson, ed.), AIP, 2004.

[6] C. Beck, "Generalized statistical mechanics of cosmic rays," *Physica A*, vol. 331, pp. 173–181, january 2004.

[7] G. Wilk and Z. Wodarczyk, "Interpretation of the nonextensitivity parameter q in some applications of Tsallis statistics and Lévy distributions," *Physical Review Letters*, vol. 84, pp. 2770–2773, March 2000.