ARRAY PROCESSING TECHNIQUES AND SHAPE RECONSTRUCTION IN TOMOGRAPHY

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ABSTRACT

We consider the problem of the shape reconstruction of a compact object in X ray tomography when the contour of the object is modeled by a polygon. The problem is then to estimate the vertices of that polygon from a limited number of projections. The main objectives of this paper are:
– to show how this reconstruction problem becomes equivalent to a generic mathematical inversion problem which arises also in linear antenna Array Processing (AP);
– to evaluate the performances of the classical AP techniques to handle with this reconstruction problem, and,
– to propose a new method based on Bayesian estimation approach for the resolution of this inverse problem.

1. INTRODUCTION

Image reconstruction in X ray tomography consists of determining an object $f(x,y)$ from its projections:

$$p(r,\phi)=\iint f(x,y)\delta(r-x\cos\phi-y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y \qquad (1)$$

In many image reconstruction applications, especially in non destructive testing (NDT) and evaluation (NDE), we know that $f(x,y)$ has a constant value c_1 inside a region (default region P) and another constant value c_2 outside that region (safe region), e.g. metal & air. The image reconstruction problem becomes then the determination of the shape of the default region. In this communication, without loss of generality, we assume that $c_1=1$ and $c_2=0$ and model the object by a polygonal disc:

$$f(x,y)=\begin{cases}1 & \text{if } (x,y)\in P,\\ 0 & \text{elsewhere}\end{cases} \qquad (2)$$

where P is a polygonal region characterized by the coordinates $\{z_j=(x_j+iy_j), j=1,\cdots,N\}$ of its vertices.

The problem of estimating z_j from projections has first been addressed by Milanfar *et al.* [1, 2] who proposed a reconstruction method using the geometrical moments of the projections and showed a link with array processing methods. This work can be considered as an extension to this work which completes some of their conclusions. So, the goals of this paper are twofold:

1. To perform an up to date evaluation of the performances of the classical AP techniques to handle with this reconstruction problem. For this we used three methods: Least square Prony's methods (both weighted and regularized) and the matrix pencil method and;

2. To propose a new method based on Bayesian estimation approach for the resolution of this inverse problem.

2. POLYGON RECONSTRUCTION FROM ITS LINE PROJECTIONS

Following the Radon transform (1) relating $p(r,\phi)$ and $f(x,y)$ and the special model for $f(x,y)$ in (2), it is easily shown that there is a relation between the geometrical moments of the projections:

$$h_k(\phi)=\int p(r,\phi)r^k\,\mathrm{d}r \qquad (3)$$

and the geometrical moments of the object:

$$\mu_{p,q}=\iint_P f(x,y)x^p y^q\,\mathrm{d}x\,\mathrm{d}y \qquad (4)$$

which is

$$h_k(\phi)=\sum_{j=0}^{k}\binom{k}{j}\cos^{k-j}(\phi)\sin^j(\phi)\mu_{k-j,j} \qquad (5)$$

These moments are themselves related to the harmonic moments c_k:

$$c_k=\iint_P f(x,y)z^k\,\mathrm{d}x\,\mathrm{d}y,\ \text{with } z=x+iy \qquad (6)$$

by

$$c_k=\sum_{j=0}^{k}\binom{k}{j}\, i^j\mu_{k-j,j} \qquad (7)$$

which are themselves related to the coordinates z_j by:

$$\tau_k=\sum_{j=1}^{N}a_j z_j^k \quad \text{with} \quad \tau_k=k(k-1)c_{k-2}. \qquad (8)$$

There is also another relation between a_j and z_j coming from the Green's theorem in the complex plane and the Cauchy-Riemann equations for analytic functions [3, 4], which is:

$$a_j=\frac{i}{2}\left(\frac{\bar{z}_{j-1}-\bar{z}_j}{z_{j-1}-z_j}-\frac{\bar{z}_j-\bar{z}_{j+1}}{z_j-z_{j+1}}\right). \qquad (9)$$

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One can then consider a reconstruction procedure as follows:

1. From projections $p(r,\phi)$ calculate $h_k(\phi)$ using (3);
2. Calculate $\mu_{k-j,j}, j=0,\cdots,k$ from $h_k(\phi)$ using (5);
3. Calculate c_k from $\mu_{k-j,j}$ using (7);
4. Calculate z_j from c_k using (8);
5. If the polygon is convex then stop, because there is only one convex polygon with given vertices. If not we have to choose an ordering for z_j to construct the non convex polygon. For this, first calculate a_j using (8). Call this estimate $\boldsymbol{a}_0$. Then, calculate a_j using (9) for all possible orderings $\{\boldsymbol{a}_m\}$, and finally, choose the ordering set of vertices z_j which gives the best fit between $\boldsymbol{a}_0$ and $\boldsymbol{a}_m$. We can use either an L_2 or an L_1 distances to measure the fitness.

Note that, for the steps 1 and 3 we have direct relations and they do not create any difficulties, but steps 2 and 4 are inversion and we have to study the conditions of their inversion.

Note also that the mathematical problem at the fourth step (eq. 8) is exactly the same as one in antenna Array Processing (AP) where a_j are the amplitudes of the sources, z_j are related to the Direction of Arrival (DAO) of these sources and τ_k are the measurements obtained by antennas.

This idea has been first addressed by Milanfar *et al.* [1, 2] who derived these relations and the theoretical necessary and sufficient conditions of existence and uniqueness of these two inversion steps 2 and 4, which can be resumed as follows:

1. The geometrical moments up to order N of an object $(\{\mu_{k-j,j}, j=0,\cdots,k,\ k=0,\cdots N\})$ are uniquely determined from the $N+1$ moments of its $N+1$ projections $\{h_k(\phi_j), k,j=0,\cdots,N\}$.
This property is derived by looking for the inversibility conditions of the following system of equations derived from the eq. (5) for $k=0,\cdots,N$ and for different projection angles $\phi_j, j=0,\cdots,N$:

$$h_k(\phi_j)=\sum_{j=0}^{k}\binom{k}{j}\cos^{k-j}(\phi_j)\sin^j(\phi_j)\mu_{k-j,j},\qquad k,j=0,\cdots,N.$$

2. The N vertices $\{z_j, j=1,\cdots,N\}$ of any non degenerate polygon are uniquely determined from $2N-1$ moments τ_k. This property is derived by looking for the inversibility conditions of the following system of equations derived from the eq. (8) for $k=0,\cdots,2N-1$:

$$\tau_k=\sum_{j=1}^{n}a_j z_j^k,\quad k=0,\cdots,2N-1$$

Now, combining these two conditions, and noting that $\tau_k, k=0,\cdots,2N-1$ are uniquely determined from $2N-3$ moments c_k which are uniquely determined from $(\mu_{k-j,j}, j=0,\cdots,k,\ k=0,\cdots 2N-2)$ we deduce that $\{z_j, j=1,\cdots,N\}$ are uniquely determined from the $2N-2$ moments of the projections $\{h_k(\phi_j), k,j=0,\cdots,2N-3\}$.

3. ARRAY PROCESSING METHODS USED FOR EVALUATION

We have seen that for a non degenerate polygon with vertices $\{z_j, j=1,\cdots,N\}$ we have

$$\tau_k=\sum_{j=1}^{N}a_j z_j^k. \tag{10}$$

Writing down this equation for $k=0,\cdots,M-1$ we obtain

$$\begin{pmatrix}1 & 1 & \cdots & 1\\ z_1 & z_2 & \cdots & z_N\\ z_1^2 & z_2^2 & \cdots & z_N^2\\ \vdots & \vdots & & \vdots\\ z_1^{M-1} & z_2^{M-1} & \cdots & z_N^{M-1}\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\\ \vdots\\ a_N\end{pmatrix}=\begin{pmatrix}\tau_0\\ \tau_1\\ \tau_2\\ \vdots\\ \tau_{M-1}\end{pmatrix}$$

or

$$\boldsymbol{Z}\boldsymbol{a}=\boldsymbol{\tau}. \tag{11}$$

Prony's method [5] consists of showing that the z_j are the zeros of the polynomial

$$P(z)=\prod_{i=0}^{N}(z-z_i)=\sum_{i=0}^{N}p_i z^{N-i},\quad p_0=1, \tag{12}$$

where the coefficients $\{p_i, i=1,\cdots,N\}$ are the solution of

$$\begin{pmatrix}\tau_0 & \cdots & \tau_{N-1}\\ \tau_1 & \cdots & \tau_N\\ \vdots & \ddots & \vdots\\ \tau_{M-N-1} & \cdots & \tau_{M-2}\end{pmatrix}\begin{pmatrix}p_N\\ p_{N-1}\\ \vdots\\ p_1\end{pmatrix}=-\begin{pmatrix}\tau_N\\ \tau_{N+1}\\ \vdots\\ \tau_{M-1}\end{pmatrix},$$

or

$$\boldsymbol{T}\boldsymbol{p}=-\boldsymbol{h}. \tag{13}$$

It is then easy to show that if $M\geq 2N-1$ the matrix $\boldsymbol{T}$ has rank N. The equation (13) forms the basis of the Least Squares Prony's (LSP) methods which consists of forming the matrix $\boldsymbol{T}$ and the vector $\boldsymbol{h}$ from the data τ_k, estimating $\boldsymbol{p}$ from this equation and calculating z_j by solving $P(z)=0$. To solve (13) we used either the ordinary LSP (OLSP):

$$\widehat{\boldsymbol{p}}=-(\boldsymbol{T}^t\boldsymbol{T})^{-1}\boldsymbol{T}^t\boldsymbol{h} \tag{14}$$

or the total LSP (TLSP):

$$\widehat{\boldsymbol{p}}=-(\boldsymbol{T}^t\boldsymbol{T}+\sigma_{min}\boldsymbol{I})^{-1}\boldsymbol{T}^t\boldsymbol{h}, \tag{15}$$

where σ_{min} is the minimum singular value of the concatenated matrix $[\boldsymbol{T},\boldsymbol{h}]$.

The matrix pencil method [6, 7, 8] is based on the observation that if we define

$$\boldsymbol{T}_0=\begin{pmatrix}\tau_{N-1} & \tau_{N-2} & \cdots & \tau_0\\ \tau_N & \tau_{N-1} & \cdots & \tau_1\\ \vdots & \vdots & \ddots & \vdots\\ \tau_{2M-2} & \tau_{2M-3} & \cdots & \tau_{2M-N-1}\end{pmatrix},$$

$$\boldsymbol{T}_1=\begin{pmatrix}\tau_N & \tau_{N-1} & \cdots & \tau_1\\ \tau_{N-1} & \tau_N & \cdots & \tau_2\\ \vdots & \vdots & \ddots & \vdots\\ \tau_{2M-1} & \tau_{2M-2} & \cdots & \tau_{2M-N}\end{pmatrix}$$

then z_j are the generalized eigenvalues of the pencil matrix $T_0 - zT_1$ or directly the eigenvalues of the matrix $(T_1^t T_1)^{-1} T_1^t T_0$.

4. EVALUATION OF THE PERFORMANCES OF AP METHODS

To evaluate the performances of the AP methods in our reconstruction problem we adopted the following procedure:

1. Generate polygonal objects;
2. Calculate projections and add noise;
3. Calculate the geometric moments of the projections;
4. Calculate the geometric moments of the object using those of the projections;
5. Apply the array processing techniques to estimate the vertices coordinates of the polygon and construct the polygon;
6. Compare the results with original.

Three main parameters influencing the results are: the number of projections N_p, the number of moments calculated for each projection N_m and the signal to noise ratio SNR. We considered three objects; a triangle ($N = 3$) and two non convex polygonal shapes with $N = 4$ and $N = 6$ (See Fig. 1). In any of these cases we considered three situations;

a) $N_p = N_m = 2N - 2$ and $SNR = 50dB$,
b) $N_p > N_m = 2N - 2$ and $SNR = 30dB$,
c) $N_p > N_m > 2N - 2$ and $SNR = 20dB$.

Fig. 1 shows a typical reconstruction results. From these results we can see that when SNR is very high and the necessary conditions on number of projections and number of moments are satisfied then the reconstruction results are satisfactory, but when SNR is low, the results are no more very satisfactory. In this case, even increasing the number of projections or the number of moments can not be of any help.

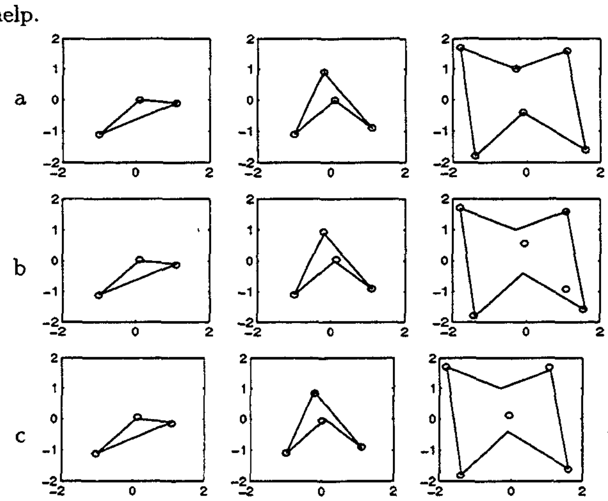

Fig. 1: Three different polygons and their reconstructions using the following parameters :
a) $N_p = N_m = 2N - 2$ and $SNR = 50dB$,
b) $N_p > N_m = 2N - 2$ and $SNR = 30dB$,
c) $N_p > N_m > 2N - 2$ and $SNR = 20dB$.
- original polygons, o reconstructed vertices.

5. CONCLUSIONS AND PROPOSITION OF A NEW METHOD

All the simulations showed that the classical AP methods are very sensitive to the noise. Our main conclusion is that this approach can not be used in practical situations of image reconstruction. The main reason is that this procedure does not use all the information contained in the projection data. It only uses the moments of these projections.

Recently we proposed a reconstruction method ([9, 10]) using directly the projection data (not only the moments) and estimate z_j by minimizing the following combined criterion:

$$J(z) = ||p - h(z)||^2 + \lambda\Omega(z), \qquad (16)$$

where $z = x + iy$ is a complex vector whose real and imaginary parts represent the x and the y coordinates of the polygon vertices, p is a vector containing all the projection data, $h(z)$ represents the direct operator which calculates the projections for any given z and $\Omega(z)$ is chosen to be a function which reflects the regularity of the object contour.

This criterion can be interpreted as a regularization criterion based on the Bayesian approach and the solution as the MAP estimate when the noise is assumed to be zero-mean, white and Gaussian and $\exp[-\lambda\Omega(z)]$ as a prior probability distribution for z.

In this work we used the following function:

$$\Omega(z) = \sum_{j=1}^{N} |z_{j-1} - 2z_j + z_{j+1}|^2 \qquad (17)$$

which favors a shape whose local curvature is limited. This is due to the fact that $|z_{j-1} - 2z_j + z_{j+1}|^2$ is proportional to the distance between the vertices z_j and the mid-point of the segment joining the two adjacent vertices z_{j-1} and z_{j+1}. So, trying to minimize $\Omega(z)$ favors the shapes with low local curvature. Other functions are possible and have been studied in this work.

What is important here is to note that, in this approach, in opposition to the AP methods described before, we do not have any limitations for the number of projections. For example, Fig. 2.a shows an example of simulation of a typical situation in non destructive testing where the object is modeled by a polygon with $N = 40$ vertices and where we have only 5 noisy projections with SNR=20dB and limited in angles between -45 and 45 degrees. Fig. 2.b shows the result of reconstruction which is very satisfactory.

The main difficulty for implementing this method is to use an appropriate global optimization algorithm to minimize the non convex criterion (16). For this we proposed the following strategies:

1. The first is to use a global optimization technique such as simulated annealing (SA). This technique has given satisfactory result as it can be seen from the simulations in the next section. However, this algorithm needs a great number of iterations and some skills for choosing the first temperature and cooling schedule, but the overall calculations is not very important due to the fact that the criterion $J(z)$ depends locally on z.

2. The second is to find an initial solution in the attractive region of the global optimum and to use a local descent type algorithm to find the solution.

The main problem here is how to find this initial solution. For this, we used a moment based method proposed by Milanfar, Karl & Wilsky [1, 2] which is accurate enough to obtain an initial solution which is not very far from the optimum. The basic idea of this method is to relate the moments of the projections to the moments of a class of polygonal discs obtained by an affine transformation of a centered regular polygonal disc, and so to estimate a polygonal disc whose vertices are on an ellipse and whose moments up to the second order matches those of the projections. However, there is no theoretical proof that this initial solution will be in the attractive region of the global optimum. In [9, 10] we showed some results comparing the performances of these two methods as well as a comparison with some other classical methods.

For more details on the implementation of this new method and a comparison of it's results with classical image reconstruction methods see [9, 10].

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a

b

c

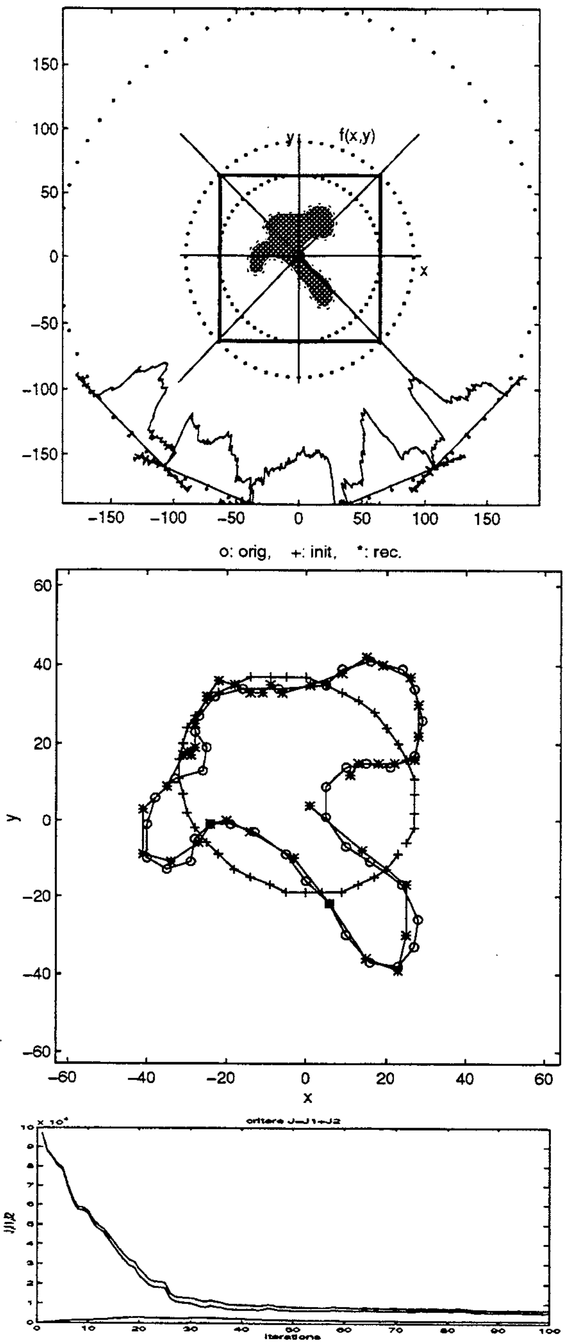

Fig. 2: Tomographic imaging geometry in non destructive testing applications.
a) Original image and 5 noisy projections with limited angles between -45 and 45 degrees.
b) Reconstruction using a polygonal model with $N = 40$ vertices and a simulated annealing algorithm for optimization.
o: Original object vertices, +: Initialization and
$\star$: Reconstructed object vertices.
c) Evolution of the criterion $J(z) = J_1(z) + \lambda J_2(z)$ with $J_1(z) = ||p - h(z)||^2$ and $J_2(z) \sum_{j=1}^{N} |z_{j-1} - 2z_j + z_{j+1}|^2$ during the iterations.