# RECONSTRUCTION OF THE SHAPE OF A COMPACT OBJECT FROM FEW PROJECTIONS

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## ABSTRACT

*In many applications such as non destructive testing (NDT), we search for an anomaly (air hole, inclusion) inside a homogeneous region (metal). This problem classically is done in three steps: detection, localization and characterization. The present industrial techniques do well in the two first steps, but the exact determination of the shape of the default region is still in the research domain. The computational cost of the classical tomographic reconstruction using the pixel (2D) or the voxel (3D) representation of the examined body still hinders its use in industrial applications. In this work, we propose modelling the contour (or the surface) of the default region by a polygon (or by a polyhedron) and estimating directly the coordinates of its vertices from very limited tomographic projection data.*

## 1. INTRODUCTION

Tomographic image reconstruction, which has recently been applied to non destructive testing (NDT), consists of determining an object $f(x,y,z)$ from its projections. In many NDT applications, we know that $f(x,y,z)$ has a constant value $c_1$ inside a region (default region D) and another constant value $c_2$ outside that region (safe region), e.g. metal & air.

$$f(x,y,z) = \begin{cases} c_1 & \text{if} \quad (x,y,z) \in D, \\ c_2 & \text{elsewhere} \end{cases} \qquad (1)$$

The image reconstruction problem then becomes the determination of the shape of the default region. In this work, without loss of generality, we assume that $c_1 = 1$ and $c_2 = 0$ and model the shape of the object either by its contour in 2D case and by its surface in 3D case.

There has been much work in image reconstruction dealing with this problem. To emphasize the originality and the context of this work, we give here a summary of the different approaches for this problem:

• The first approach consists of modelling the whole body by a 2D array of pixels (or 3D array of voxels) and then relating the projection data $\boldsymbol{p}$ to them by the linear equation $\boldsymbol{p} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{n}$, where $\boldsymbol{f}$ is a vector containing all the pixel or voxel values, $\boldsymbol{p}$ is a vector containing the projection data, $\boldsymbol{n}$ is a vector representing the measurement errors and $\boldsymbol{H}$ is the discretized transform operator. Then the solution is defined as the minimizer of a compound criterion

$$J(\boldsymbol{f}) = Q(\boldsymbol{p} - \boldsymbol{H}\boldsymbol{f}) + \lambda\Omega(\boldsymbol{f}), \qquad (2)$$

where $\lambda$ is the regularization parameter and $Q$ and $\Omega$ have to be chosen appropriately to reflect our prior knowledge of the noise and the image. This is the classical regularization approach to the general image reconstruction problem. One can also interpret $J(\boldsymbol{f})$ as the maximum a posteriori (MAP) criterion in the Bayesian estimation framework where $-Q(\boldsymbol{f})$ represents the log-likelihood term and $-\Omega(\boldsymbol{f})$ is the log of the *a priori* prior probability law. This approach has been used with success in many applications [1] but the cost of its calculation is great due to the dimension of $\boldsymbol{f}$.

• The second approach starts by giving a parametric model for the object and then tries to estimate the parameters using least squares (LS) or maximum likelihood (ML) methods. In general, in this approach one chooses a parametric model such as a superposition of circles and/or ellipses (spheres and ellipsoids) to be able to relate analytically the projections to the parameters. It is then easy to calculate analytically the projection data $p(\boldsymbol{\theta})$ and to define the solution either as the ML or as the LS estimate :

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \left\{ \|p - p(\boldsymbol{\theta})\|^2 \right\} \qquad (3)$$

This approach has also been used with success in image reconstruction [2, 3], but the range of applicability of these methods is limited to cases in which the parametric models are actually appropriate.

• The third approach, which is more appropriate to our problem of shape reconstruction, consists of using a function to directly model the contour of the object and estimating it directly from the data. This method has been used in image restoration [4], but appears to be new in image reconstruction applications. Our proposed method falls into this category.

The originality of our work is to model the shape of the object by a polygon in 2D and by a polyhedral surface in 3D and to estimate the coordinates of its vertices directly from the projection data. Some similar work in 2D has been published by Milanfar, Karl & Wilsky [5, 6, 7], but their method is based on specific relations between the moments of the projection data and coordinates of the vertices of the polygon. In recent work [8], we showed limitations of their approach, which result from the fact that using only the low-order moments of the projection data omits important information.

## 2. PROPOSED METHOD

In this work we propose to model the contour or the surface of the object (default region) as a polygon or a polyhedron with a large number $N$ of vertices to be able to approximate any practical shape. Then we propose directly estimating the coordinates $\{(x_j, y_j, z_j), j = 1, \cdots, N\}$ of its vertices from the projection data.

The idea of modeling the shape of the object as a polygonal disc is not new and some work has been done in image reconstruction applications, but, in general in these works, a hypothesis of convexity of the polygonal disc has been used, which is very restrictive in real applications. In our work we do not make this assumption.

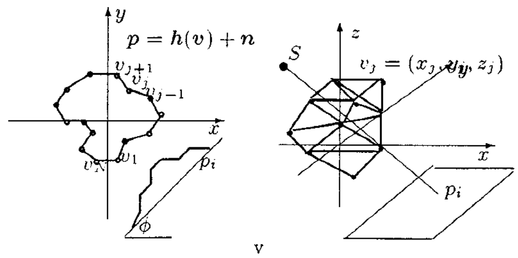


Figure 1: Proposed shape reconstruction modeling.

We define the solution of the problem as the minimizer the criterion

$$J(v) = ||p - h(v)||^2 + \lambda\Omega(v), \tag{4}$$

where $v = \{(x_j, y_j, z_j), j = 1, \cdots, N\}$ is a vector whose components $v_j = (x_j, y_j, z_j)$ represent the spatial coordinates of the polygon vertices, $h(v)$ represents the direct operator which calculates the projections for any given $v$ and $\Omega(v)$ is chosen to be a function which reflects the regularity of the object contour.

In 2D case, we can represent $v = \{(x_j, y_j), j = 1, \cdots, N\}$ by a complex vector. In this case we use the following regularizing function:

$$\begin{aligned} \Omega(v) &= \sum_{j=1}^{N} |v_{j-1} - 2v_j + v_{j+1}|^2 \\ &= \frac{1}{4}\sum_{j=1}^{N} \left|v_j - \frac{1}{2}(v_{j-1} + v_{j+1})\right|^2. \end{aligned} \tag{5}$$

Note that $\left|v_j - \frac{1}{2}(v_{j-1} + v_{j+1})\right|$ is just the Euclidian distance between the point $v_j$ and the mid-point of the line segment joining $v_{j-1}$ and $v_{j+1}$, so this choice favors a shape whose local curvature is limited. We can also give a probabilistic interpretation to this choice. In fact we can consider $v_j$ as complex random variables with the following Markovian law:

$$\begin{aligned} p(v_j|v) &= p(v_j|v_{j-1}, v_{j+1}) \\ &\propto \exp\left[-\frac{1}{2\sigma^2}|v_{j-1} - 2v_j + v_{j+1}|^2\right]. \end{aligned} \tag{6}$$

Other functions are possible and are studied in this work.

In 3D case, by the same reasoning, we use

$$\Omega(v) = \frac{1}{4}\sum_{j=1}^{N} \left|v_j - \frac{1}{K}\sum_{j\in\mathcal{V}i} v_j\right|^2, \tag{7}$$

where $j \in \mathcal{V}i$ stands for neighborhood, $K$ is the number of vertices $v_j$ in the neighborhood of the vertice $v_i$ and $\frac{1}{K}\sum_{j\in\mathcal{V}i} v_j$ gives the coordinates of a point which is the geometric center of all the vertices $v_j$ in the neighborhood of the vertice $v_i$. Thus, the defined $\Omega(v)$ in this case also favors a shape whose local curvature is limited.

The criterion $J(v)$ is multimodal essentially due to the fact that $h(v)$ is a nonlinear function of $v$. Calculating the optimal solution corresponding to its global minimum requires carefully designed algorithms. For this we propose the two following strategies:

• The first is a global optimization technique such as simulated annealing. This technique gives satisfactory results, as can be seen from the simulations in the next section, but requires a great number of iterations and some skill in choosing the first temperature and cooling schedule. Still, the overall computational cost is not very large, due to the fact that we do not need to calculate the gradient of the criterion.

• The second is to find an initial solution in the attractive region of the global optimum and to use a local descent type algorithm such as ICM (Iterated Conditional Modes) to find the solution. The main problem here is how to find this initial solution. We use a moment based method proposed by Milanfar, Karl & Wilsky [5, 6] which is accurate enough to obtain an initial solution which is not very far from the optimum. The basic idea of this method is to relate the moments of the projections to the moments of a class of polygons obtained by an affine transformation of a regular polygon, and so to estimate a polygon whose vertices are on an ellipse and whose moments up to the second order match those of the projections.

However, there is no theoretical proof that this initial solution will be in the attractive region of the global optimum. The next section will show some results comparing the performances of these two methods as well as a comparison with some other classical methods.

## 3. SIMULATION RESULTS

To measure the performances of the proposed method and keeping the objective of using this method for NDT applications where the number of projections are very limited, we simulated a case where the object is a polygon with $N = 40$ corners (hand-made) and calculated its projections for only 5 directions

($\phi = -45, -22.5, 0, 22.5, 45$ degrees) (see Figure 1). We added noise (zero-mean, white, Gaussian) to simulate measurement errors, with S/N ratio of 20dB. Figure 2 shows the original object and the simulated projection data. Finally, from these data we estimated the solution by both of the two proposed methods.

In Figure 3, we give the reconstruction results obtained by simulated annealing (SA) algorithm. In this figure we show the original object, the initial solution, the intermediate solutions during the SA iterations and the final solution obtained after 200 iterations.

In Figure 4, we give the reconstruction results obtained by ICM algorithm initialized by a solution obtained from the two first moments of the projections. In this figure we show the original object, the initial solution, the intermediate solutions during the SA iterations and the final solution obtained after 200 iterations.

In Figure 5 we show a comparison between the results obtained by the proposed method and those obtained by a classical backprojection, and pixel-based estimation approaches with different regularization functionals $\Omega(f)$, more specifically:

- the *Entropic laws* with $\phi(x) = -x\log x$ where we called it *Maximum entropy regularized method*,
- the *Gaussian Markovian laws:* with the potential function $\phi(x, y) = |x - y|^2$ which can also be considered as a quadratic regularization method; and
- the *Markovian laws with non convex potential functions:* $\phi(x, y) = \min\left\{|x - y|^2, 1\right\}$ and $\phi(x, y) = \frac{-1}{1+|x-y|^2}$. In these two last cases we used a Graduated Non Convexity (GNC) based optimization algorithm to find the solution.

For the sake of curiosity, we also show binary segmented images obtained by thresholding these last images.

In Figures 6 and 7 we show a result obtained in 3D case. Figure 6 shows the object and the 9 simulated projections. Figure 7 shows the reconstructed object.

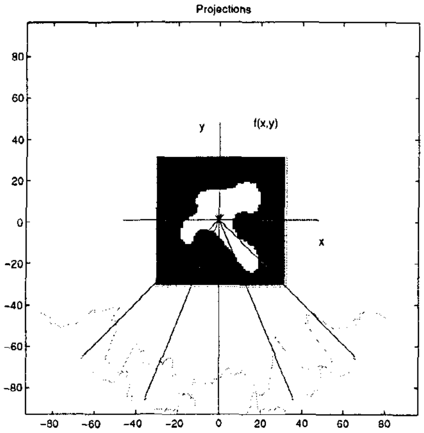


Figure 2: Original image and simulated projections data.

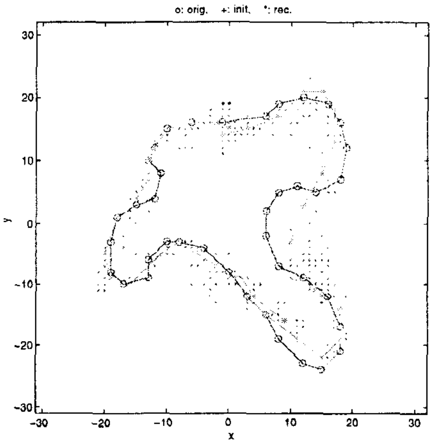


Figure 3: Reconstruction using simulated annealing.
o) Original object, +) Initialization,
.) Evolution of the solution during the iterations and
⋆) Final reconstructed object.

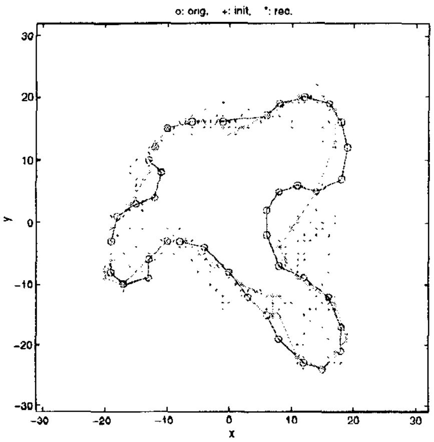


Figure 4: Reconstruction obtained by ICM algorithm.
o) Original object, +) Initialization,
.) Evolution of the solution during the iterations and
⋆) Final reconstructed object.

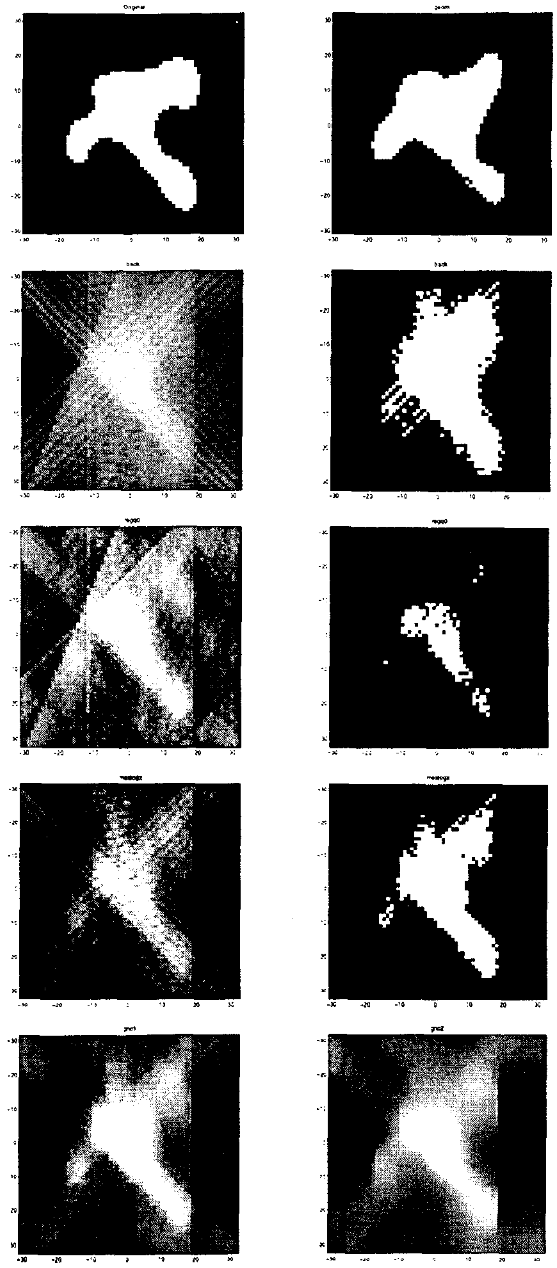

Figure 5: A comparison with backprojection and some other classical methods:
a) Original, b) Proposed method,
c) Backprojection, d) Binary threshold of c,
e) Gaussian Markov modeling MAP reconstruction,
f) Binary threshold of e,
g) Maximum entropy regularized reconstruction,
h) Binary threshold of g,
i) Compound Markov modeling and GNC optimization algorithm using truncated quadratic potential function,
j) GNC with Lorentzian potential function.

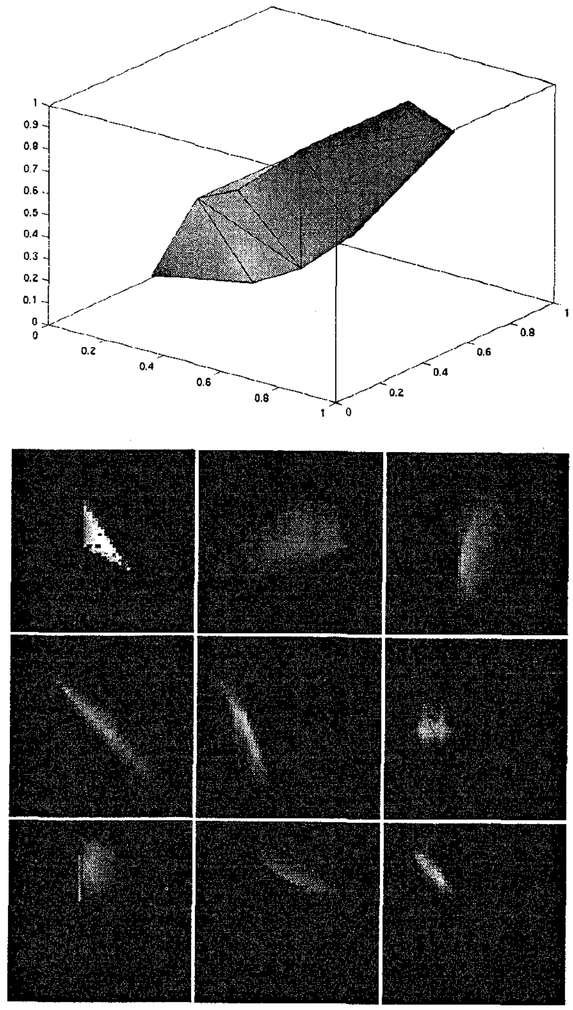

Figure 6: A 3D object and its 9 projections.

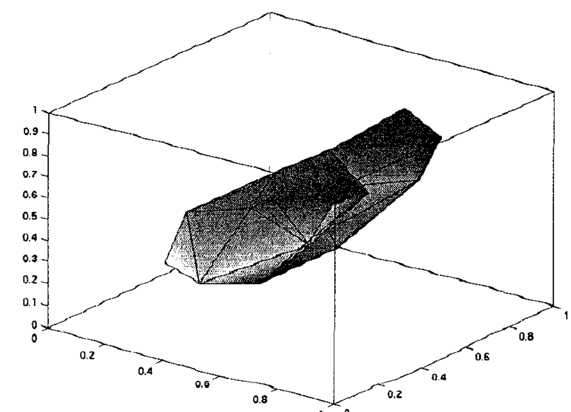

Figure 7: The result of reconstruction obtained by the proposed method.

## 4. CONCLUSION

A new method for tomographic image reconstruction of a compact object from a few number of its projections is proposed. The basic idea of the proposed method is modelling the compact object as a polygonal or a polyhedral body and to estimate the coordinates of its vertices directly from the projections using the Bayesian MAP estimation framework or equivalently by optimizing a regularized criterion.

Unfortunately, this criterion is not unimodal. To find the optimized solution two methods are examined: – a global optimization method based on simulated annealing (SA) and – a local descent-based method with a good initialization obtained using a moment based method. This method can be compared to the Iterated Conditional Modes (ICM) algorithm proposed by Besag.

The first method seems to yield the best results at present. The second can also give satisfactory estimates, but may be trapped in local minima. In both methods the main computational cost is due to the calculation of the variation of the criterion when one of the vertices' coordinates is changed. We have written an efficient program to do this.

An extension of this work to 3D image reconstruction from a small number of conic projections is underway. The final objective of the proposed method is non destructive testing (NDT) image reconstruction applications where we can use not only X-rays but also ultrasound or Eddy currents or a combination of them.

## 5. REFERENCES


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