Inverse Problems in Signal and Image processing,
Imaging systems and Computer Vision
From Deterministic Regularization to
Probabilistic Bayesian Approaches

Ali Mohammad-Djafari

Groupe Problèmes Inverses
Laboratoire des signaux et systèmes (L2S)
UMR 8506 CNRS - SUPELEC - UNIV PARIS SUD 11
Supélec, Plateau de Moulon, 91192 Gif-sur-Yvette, FRANCE.

djafari@lss.supelec.fr
http://djafari.free.fr
http://www.lss.supelec.fr

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Content

- Inverse problems: Examples and general formulation
- Inversion methods:
  - analytical, parametric and non parametric
- Deterministic methods:
  - Data matching, Least Squares, Regularization
- Probabilistic methods:
  - Probability matching, Maximum likelihood, Bayesian inference
- Bayesian inference approach
- Prior models for images
- Bayesian computation
- Applications:
  - Computed Tomography, Image separation, Superresolution, SAR Imaging
- Conclusions
- Questions and Discussion
Inverse problems : 3 main examples

- **Example 1:**
  Measuring variation of temperature with a thermometer
  - \( f(t) \) variation of temperature over time
  - \( g(t) \) variation of length of the liquid in thermometer

- **Example 2:**
  Making an image with a camera, a microscope or a telescope
  - \( f(x, y) \) real scene
  - \( g(x, y) \) observed image

- **Example 3:** Making an image of the interior of a body
  - \( f(x, y) \) a section of a real 3D body \( f(x, y, z) \)
  - \( g_\varphi(r) \) a line of observed radiograph \( g_\varphi(r, z) \)

- **Example 1:** Deconvolution
- **Example 2:** Image restoration
- **Example 3:** Image reconstruction
Measuring variation of temperature with a thermometer

- $f(t)$ variation of temperature over time
- $g(t)$ variation of length of the liquid in thermometer

**Forward model: Convolution**

$$g(t) = \int f(t') h(t - t') \, dt' + \epsilon(t)$$

$h(t)$: impulse response of the measurement system

**Inverse problem: Deconvolution**

Given the forward model $\mathcal{H}$ (impulse response $h(t)$) and a set of data $g(t_i), i = 1, \cdots, M$

find $f(t)$
Measuring variation of temperature with a thermometer

Forward model: Convolution

\[ g(t) = \int f(t') h(t - t') \, dt' + \epsilon(t) \]

Inversion: Deconvolution
Making an image with a camera, a microscope or a telescope

- \( f(x, y) \) real scene
- \( g(x, y) \) observed image
- **Forward model:** Convolution
\[
g(x, y) = \int \int f(x', y') h(x - x', y - y') \, dx' \, dy' + \epsilon(x, y)
\]

\( h(x, y) \): Point Spread Function (PSF) of the imaging system
- **Inverse problem:** Image restoration

Given the forward model \( \mathcal{H} (\text{PSF} \ h(x, y)) \)
and a set of data \( g(x_i, y_i), i = 1, \ldots, M \)
find \( f(x, y) \)
Making an image with an unfocused camera

Forward model: 2D Convolution

\[ g(x, y) = \int \int f(x', y') h(x - x', y - y') \, dx' \, dy' + \epsilon(x, y) \]

Inversion: Deconvolution
Making an image of the interior of a body

Different imaging systems:

**Active Imaging**
- Incident wave
- Measurement

**Passive Imaging**
- Incident wave
- Measurement

**Transmission**
- Incident wave
- Measurement

**Reflection**
- Incident wave
- Measurement

**Forward problem:** Knowing the object predict the data

**Inverse problem:** From measured data find the object
Making an image of the interior of a body

- \( f(x, y) \) a section of a real 3D body \( f(x, y, z) \)
- \( g_\phi(r) \) a line of observed radiograph \( g_\phi(r, z) \)

- **Forward model:**
  Line integrals or Radon Transform

\[
g_\phi(r) = \int_{L_{r,\phi}} f(x, y) \, dl + \epsilon_\phi(r)
\]

\[
= \int \int f(x, y) \delta(r - x \cos \phi - y \sin \phi) \, dx \, dy + \epsilon_\phi(r)
\]

- **Inverse problem:** Image reconstruction

Given the forward model \( \mathcal{H} \) (Radon Transform) and
a set of data \( g_\phi_i(r), i = 1, \ldots, M \)
find \( f(x, y) \)
2D and 3D Computed Tomography

\[ g_\phi(r_1, r_2) = \int_{L_{r_1, r_2, \phi}} f(x, y, z) \, dl \]
\[ g_\phi(r) = \int_{L_{r, \phi}} f(x, y) \, dl \]

Forward problem: \( f(x, y) \) or \( f(x, y, z) \rightarrow g_\phi(r) \) or \( g_\phi(r_1, r_2) \)

Inverse problem: \( g_\phi(r) \) or \( g_\phi(r_1, r_2) \rightarrow f(x, y) \) or \( f(x, y, z) \)
Microwave or ultrasound imaging

Measurs: diffracted wave by the object $g(r_i)$

Unknown quantity: $f(r) = k_0^2(n^2(r) - 1)$

Intermediate quantity : $\phi(r)$

$$g(r_i) = \int\int_D G_m(r_i, r') \phi(r') f(r') \, dr', \ r_i \in S$$

$$\phi(r) = \phi_0(r) + \int\int_D G_o(r, r') \phi(r') f(r') \, dr', \ r \in D$$

Born approximation ({$\phi(r') \simeq \phi_0(r')$}): 

$$g(r_i) = \int\int_D G_m(r_i, r') \phi_0(r') f(r') \, dr', \ r_i \in S$$

Discretization : 

$$\begin{cases} 
g = G_m F \phi \\
\phi = \phi_0 + G_o F \phi 
\end{cases} \quad \Rightarrow \quad \begin{cases} 
g = H(f) \\
\text{with } F = \text{diag}(f) \\
H(f) = G_m F (I - G_o F)^{-1} \phi_0 
\end{cases}$$
Fourier Synthesis in X ray Tomography

\[
g(r, \phi) = \int \int f(x, y) \delta(r - x \cos \phi - y \sin \phi) \, dx \, dy
\]

\[
G(\Omega, \phi) = \int g(r, \phi) \exp \{-j\Omega r\} \, dr
\]

\[
F(\omega_x, \omega_y) = \int \int f(x, y) \exp \{-j\omega_x x, \omega_y y\} \, dx \, dy
\]

\[
F(\omega_x, \omega_y) = G(\Omega, \phi) \quad \text{for} \quad \omega_x = \Omega \cos \phi \quad \text{and} \quad \omega_y = \Omega \sin \phi
\]

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Fourier Synthesis in X ray tomography

\[ G(\omega_x, \omega_y) = \int \int f(x, y) \exp \{-j(\omega_x x + \omega_y y)\} \, dx \, dy \]

Forward problem: Given \( f(x, y) \) compute \( G(\omega_x, \omega_y) \)

Inverse problem: Given \( G(\omega_x, \omega_y) \) on those lines estimate \( f(x, y) \)
Fourier Synthesis in Diffraction tomography

Incident plane wave

$f(x, y)$

Diffraacted wave

$\psi(r, \phi)$

FT

$f^\wedge(\omega_x, \omega_y)$

$\omega_x$

$\omega_y$
Fourier Synthesis in Diffraction tomography

\[ G(\omega_x, \omega_y) = \iint f(x, y) \exp \{-j(\omega_x x + \omega_y y)\} \, dx \, dy \]

**Forward problem:** Given \( f(x, y) \), compute \( G(\omega_x, \omega_y) \)

**Inverse problem:** Given \( G(\omega_x, \omega_y) \) on those semi cercles, estimate \( f(x, y) \)

A. Mohammad-Djafari, Inverse Problems in Imaging & Computer vision, Master ITEMS 2012, UPB, Bucarest, Nov 2012, 15/79
Fourier Synthesis in different imaging systems

\[ G(\omega_x, \omega_y) = \int \int f(x, y) \exp \left\{ -j (\omega_x x + \omega_y y) \right\} \, dx \, dy \]

X ray Tomography  
Diffraction  
Eddy current  
SAR & Radar

**Forward problem:** Given \( f(x, y) \) compute \( G(\omega_x, \omega_y) \)

**Inverse problem:** Given \( G(\omega_x, \omega_y) \) on those algebraic lines, cercles or curves, estimate \( f(x, y) \)
Invers Problems: other examples and applications

- X ray, Gamma ray Computed Tomography (CT)
- Microwave and ultrasound tomography
- Positron emission tomography (PET)
- Magnetic resonance imaging (MRI)
- Photoacoustic imaging
- Radio astronomy
- Geophysical imaging
- Non Destructive Evaluation (NDE) and Testing (NDT) techniques in industry
- Hyperspectral imaging
- Earth observation methods (Radar, SAR, IR, ...)
- Survey and tracking in security systems
Computed tomography (CT)

A Multislice CT Scanner

\[ g(s_i) = \int_{L_i} f(r) \, dl_i + \epsilon(s_i) \]

Discretization

\[ g = Hf + \epsilon \]
Positron emission tomography (PET)
Magnetic resonance imaging (MRI)

Nuclear magnetic resonance imaging (NMRI), Para-sagittal MRI of the head
Radio astronomy (interferometry imaging systems)

The Very Large Array in New Mexico, an example of a radio telescope.
General formulation of inverse problems

- General non linear inverse problems:
  \[ g(s) = [\mathcal{H} f(r)](s) + \epsilon(s), \quad r \in \mathcal{R}, \quad s \in S \]

- Linear models:
  \[ g(s) = \int f(r) h(r, s) \, dr + \epsilon(s) \]
  If \( h(r, s) = h(r - s) \longrightarrow \) Convolution.

- Discrete data:
  \[ g(s_i) = \int h(s_i, r) f(r) \, dr + \epsilon(s_i), \quad i = 1, \ldots, m \]

- Inversion: Given the forward model \( \mathcal{H} \) and the data \( g = \{g(s_i), i = 1, \ldots, m\} \) estimate \( f(r) \)

- Well-posed and Ill-posed problems (Hadamard): exisitance, uniqueness and stability

- Need for prior information
Analytical methods (mathematical physics)

\[ g(s_i) = \int h(s_i, r) f(r) \, dr + \epsilon(s_i), \quad i = 1, \ldots, m \]

\[ g(s) = \int h(s, r) f(r) \, dr \]

\[ \hat{f}(r) = \int w(s, r) g(s) \, ds \]

\( w(s, r) \) minimizing a criterion:

\[ Q(w(s, r)) = \left\| g(s) - [\mathcal{H} \hat{f}(r)](s) \right\|_2^2 = \int \left| g(s) - [\mathcal{H} \hat{f}(r)](s) \right|_2^2 \, ds \]

\[ = \int \left| g(s) - \int h(s, r) \hat{f}(r) \, dr \right|_2^2 \, ds \]

\[ = \int \left| g(s) - \int \int h(s, r) w(s, r) g(s) \, ds \, dr \right|_2^2 \, ds \]

Trivial solution: \( h(s, r)w(s, r) = \delta(r)\delta(s) \)
Analytical methods

- Trivial solution:

\[ w(s, r) = h^{-1}(s, r) \]

Example: Fourier Transform:

\[ g(s) = \int f(r) \exp\{-js.r\} \, dr \]

\[ h(s, r) = \exp\{-js.r\} \rightarrow w(s, r) = \exp\{+js.r\} \]

\[ \hat{f}(r) = \int g(s) \exp\{+js.r\} \, ds \]

- Known classical solutions for specific expressions of \( h(s, r) \):
  - 1D cases: 1D Fourier, Hilbert, Weil, Melin, ...
  - 2D cases: 2D Fourier, Radon, ...
**X-ray Tomography**

\[ g(r, \phi) = -\ln \left( \frac{I}{I_0} \right) = \int_{L_{r,\phi}} f(x, y) \, dl \]

\[ g(r, \phi) = \iint_{D} f(x, y) \delta(r - x \cos \phi - y \sin \phi) \, dx \, dy \]

\[ f(x, y) \rightarrow \text{RT} \rightarrow g(r, \phi) \]

IRT \quad ? \quad \Rightarrow
Analytical Inversion methods

Radon:

\[ g(r, \phi) = \int_L f(x, y) \, dl \]

\[ g(r, \phi) = \int \int_D f(x, y) \delta(r - x \cos \phi - y \sin \phi) \, dx \, dy \]

\[ f(x, y) = \left( -\frac{1}{2\pi^2} \right) \int_0^\pi \int_{-\infty}^{+\infty} \frac{\partial}{\partial r} g(r, \phi) \frac{dr \, d\phi}{(r - x \cos \phi - y \sin \phi)} \]
Filtered Backprojection method

\[ f(x, y) = \left( -\frac{1}{2\pi^2} \right) \int_0^\pi \int_{-\infty}^{+\infty} \frac{\partial}{\partial r} g(r, \phi) \frac{dr}{(r - x \cos \phi - y \sin \phi)} \ d\phi \ d\phi \]

Derivation \( \mathcal{D} \):
\[ \bar{g}(r, \phi) = \frac{\partial g(r, \phi)}{\partial r} \]

Hilbert Transform \( \mathcal{H} \):
\[ g_1(r', \phi) = \frac{1}{\pi} \int_0^\infty \frac{\bar{g}(r, \phi)}{r - r'} dr \]

Backprojection \( \mathcal{B} \):
\[ f(x, y) = \frac{1}{2\pi} \int_0^\pi g_1(r' = x \cos \phi + y \sin \phi, \phi) d\phi \]

\[ f(x, y) = \mathcal{B} \mathcal{H} \mathcal{D} g(r, \phi) = \mathcal{B} \mathcal{F}_1^{-1} |\Omega| \mathcal{F}_1 g(r, \phi) \]

- Backprojection of filtered projections:

\[
\begin{array}{c|c|c|c|c|c}
\hline
\text{g(r,}\phi) & \text{FT} & \text{Filter} & \text{IFT} & \text{g1(r,}\phi) & \text{Backprojection} \\
\hline
\mathcal{F}_1 & & |\Omega| & \mathcal{F}_1^{-1} & & \mathcal{B} \\
\hline
\end{array}
\]

A. Mohammad-Djafari, Inverse Problems in Imaging & Computer vision, Master ITEMS 2012, UPB, Bucarest, Nov 2012, 27
Limitations: Limited angle or noisy data

- Limited angle or noisy data
- Accounting for detector size
- Other measurement geometries: fan beam, ...

Original 64 proj. 16 proj. 8 proj. \([0, \pi/2]\)
Limitations: Limited angle or noisy data

Original Data Backprojection Filtered Backprojection
Parametric methods

- $f(r)$ is described in a parametric form with a very few number of parameters $\theta$ and one searches $\hat{\theta}$ which minimizes a criterion such as:
  - Least Squares (LS): $Q(\theta) = \sum_i |g_i - [\mathcal{H} f(\theta)]_i|^2$
  - Robust criteria: $Q(\theta) = \sum_i \phi(|g_i - [\mathcal{H} f(\theta)]_i|)$ with different functions $\phi$ ($L_1$, Hubert, ...).
  - Likelihood: $\mathcal{L}(\theta) = -\ln p(g|\theta)$
  - Penalized likelihood: $\mathcal{L}(\theta) = -\ln p(g|\theta) + \lambda \Omega(\theta)$

Examples:

- Spectrometry: $f(t)$ modelled as a sum og gaussians $f(t) = \sum_{k=1}^{K} a_k \mathcal{N}(t|\mu_k, \nu_k)$ $\theta = \{a_k, \mu_k, \nu_k\}$
- Tomography in CND: $f(x, y)$ is modelled as a superposition of circular or elliptical discs $\theta = \{a_k, \mu_k, r_k\}$

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Non parametric methods

\[ g(s_i) = \int h(s_i, r) f(r) \, dr + \epsilon(s_i), \quad i = 1, \cdots, M \]

- \( f(r) \) is assumed to be well approximated by

\[
\int h(s_i, r) f(r) \, dr \approx \sum_{j=1}^{N} f_j \, b_j(r)
\]

with \( \{b_j(r)\} \) a basis or any other set of known functions

\[
g(s_i) = g_i \approx \sum_{j=1}^{N} f_j \int h(s_i, r) b_j(r) \, dr, \quad i = 1, \cdots, M
\]

\[
g = Hf + \epsilon \quad \text{with} \quad H_{ij} = \int h(s_i, r) b_j(r) \, dr
\]

- \( H \) is huge dimensional

- LS solution : \( \hat{f} = \arg \min_{f} \{ Q(f) \} \) with

\[
Q(f) = \sum_i |g_i - [Hf]_i|^2 = \|g - Hf\|^2
\]

does not give satisfactory result.
Algebraic methods: Discretization

\[ f(x, y) = \sum_j f_j b_j(x, y) \]
\[ b_j(x, y) = \begin{cases} 
1 & \text{if } (x, y) \in \text{pixel } j \\
0 & \text{else} 
\end{cases} \]

\[ g_i = \sum_{j=1}^{N} H_{ij} f_j + \epsilon_i \]

\[ g = H f + \epsilon \]
Inversion: Deterministic methods

Data matching

- Observation model
  \[ g_i = h_i(f) + \epsilon_i, \quad i = 1, \ldots, M \rightarrow g = H(f) + \epsilon \]

- Misatch between data and output of the model \( \Delta(g, H(f)) \)
  \[ \hat{f} = \arg \min_f \{ \Delta(g, H(f)) \} \]

- Examples:
  - LS  \( \Delta(g, H(f)) = \| g - H(f) \|^2 = \sum_i |g_i - h_i(f)|^2 \)
  - \( L_p \)  \( \Delta(g, H(f)) = \| g - H(f) \|^p = \sum_i |g_i - h_i(f)|^p, \quad 1 < p < 2 \)
  - KL  \( \Delta(g, H(f)) = \sum_i g_i \ln \frac{g_i}{h_i(f)} \)

- In general, does not give satisfactory results for inverse problems.
Regularization theory

Inverse problems = Ill posed problems
→ Need for prior information

Functional space (Tikhonov):

\[ g = \mathcal{H}(f) + \epsilon \rightarrow J(f) = \|g - \mathcal{H}(f)\|^2_2 + \lambda \|Df\|^2_2 \]

Finite dimensional space (Philips & Towmey):

\[ g = H(f) + \epsilon \]

• Minimum norme LS (MNLS):
  \[ J(f) = \|g - H(f)\|^2 + \lambda \|f\|^2 \]

• Classical regularization:
  \[ J(f) = \|g - H(f)\|^2 + \lambda \|Df\|^2 \]

• More general regularization:
  \[ J(f) = Q(g - H(f)) + \lambda \Omega(Df) \]
  or
  \[ J(f) = \Delta_1(g, H(f)) + \lambda \Delta_2(f, f_{\infty}) \]

Limitations:

• Errors are considered implicitly white and Gaussian
• Limited prior information on the solution
• Lack of tools for the determination of the hyperparameters
Inversion: Probabilistic methods

Taking account of errors and uncertainties $\rightarrow$ Probability theory

- Maximum Likelihood (ML)
- Minimum Inaccuracy (MI)
- Probability Distribution Matching (PDM)
- Maximum Entropy (ME) and Information Theory (IT)
- Bayesian Inference (BAYES)

Advantages:

- Explicit account of the errors and noise
- A large class of priors via explicit or implicit modeling
- A coherent approach to combine information content of the data and priors

Limitations:

- Practical implementation and cost of calculation
Bayesian estimation approach

\[ \mathcal{M} : \quad g = H f + \epsilon \]

- Observation model \( \mathcal{M} \) + Hypothesis on the noise \( \epsilon \):
  \[ p(g|f; \mathcal{M}) = p_\epsilon(g - Hf) \]

- A priori information:
  \[ p(f|\mathcal{M}) \]

- Bayes:
  \[ p(f|g; \mathcal{M}) = \frac{p(g|f; \mathcal{M}) p(f|\mathcal{M})}{p(g|\mathcal{M})} \]

Link with regularization:

Maximum A Posteriori (MAP):

\[ \hat{f} = \arg \max_f \{ p(f|g) \} = \arg \max_f \{ p(g|f) p(f) \} \]
\[ = \arg \min_f \{ -\ln p(g|f) - \ln p(f) \} \]

with

\[ Q(g, Hf) = -\ln p(g|f) \quad \text{and} \quad \lambda \Omega(f) = -\ln p(f) \]
Case of linear models and Gaussian priors

\[ g = Hf + \epsilon \]

- Hypothesis on the noise: \[ \epsilon \sim \mathcal{N}(0, \sigma^2_\epsilon I) \]
  \[ p(g|f) \propto \exp \left\{ -\frac{1}{2\sigma^2_\epsilon} \| g - Hf \|^2 \right\} \]

- Hypothesis on \( f \): \[ f \sim \mathcal{N}(0, \sigma^2_f (D^t D)^{-1}) \]
  \[ p(f) \propto \exp \left\{ -\frac{1}{2\sigma^2_f} \| Df \|^2 \right\} \]

- A posteriori:
  \[ p(f|g) \propto \exp \left\{ -\frac{1}{2\sigma^2_\epsilon} \| g - Hf \|^2 - \frac{1}{2\sigma^2_f} \| Df \|^2 \right\} \]

- MAP:
  \[ \hat{f} = \arg \max_f \{ p(f|g) \} = \arg \min_f \{ J(f) \} \]
  with \[ J(f) = \| g - Hf \|^2 + \lambda \| Df \|^2, \quad \lambda = \frac{\sigma^2_\epsilon}{\sigma^2_f} \]

- Advantage: characterization of the solution
  \[ f|g \sim \mathcal{N}(\hat{f}, \hat{P}) \quad \text{with} \quad \hat{f} = \hat{P}H^t g, \quad \hat{P} = (H^t H + \lambda D^t D)^{-1} \]
MAP estimation with other priors:

\[
\hat{f} = \arg \min_{f} \{ J(f) \} \quad \text{with} \quad J(f) = \| g - H f \|^2 + \lambda \Omega(f)
\]

Separable priors:

- **Gaussian:** \( p(f_j) \propto \exp\{ -\alpha |f_j|^2 \} \longrightarrow \Omega(f) = \alpha \sum_j |f_j|^2 \)
- **Gamma:** \( p(f_j) \propto f_j^\alpha \exp\{ -\beta f_j \} \longrightarrow \Omega(f) = \alpha \sum_j \ln f_j + \beta f_j \)
- **Beta:**
  \[
p(f_j) \propto f_j^\alpha (1 - f_j)^\beta \longrightarrow \Omega(f) = \alpha \sum_j \ln f_j + \beta \sum_j \ln(1 - f_j)
\]
- **Generalized Gaussian:**
  \[
p(f_j) \propto \exp\{ -\alpha |f_j|^p \}, \quad 1 < p < 2 \longrightarrow \Omega(f) = \alpha \sum_j |f_j|^p,
\]

Markovian models:

\[
p(f_j | f) \propto \exp \left\{ -\alpha \sum_{i \in N_j} \phi(f_j, f_i) \right\} \longrightarrow \Omega(f) = \alpha \sum_j \sum_{i \in N_j} \phi(f_j, f_i),
\]
MAP estimation with markovien priors:

\[
\hat{f} = \arg \min_f \{ J(f) \} \quad \text{with} \quad J(f) = \| g - H f \|^2 + \lambda \Omega(f)
\]

\[
\Omega(f) = \sum_j \phi(f_j - f_{j-1})
\]

with \( \phi(t) \):

Convex functions:

\[ |t|^\alpha, \ \sqrt{1 + t^2} - 1, \ \log(\cosh(t)), \ \begin{cases} 
    t^2 \\
    2T|t| - T^2
\end{cases} \quad |t| \leq T \]

\[ |t| > T \]

or Non convex functions:

\[ \log(1 + t^2), \ \frac{t^2}{1 + t^2}, \ \arctan(t^2), \ \begin{cases} 
    t^2 \\
    T^2
\end{cases} \quad |t| \leq T \]

\[ |t| > T \]
Main advantages of the Bayesian approach

- MAP = Regularization
- Posterior mean ? Marginal MAP ?
- More information in the posterior law than only its mode or its mean
- Meaning and tools for estimating hyper parameters
- Meaning and tools for model selection
- More specific and specialized priors, particularly through the hidden variables
- More computational tools:
  - Expectation-Maximization for computing the maximum likelihood parameters
  - MCMC for posterior exploration
  - Variational Bayes for analytical computation of the posterior marginals
  - ...

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Full Bayesian approach

\[ M : \quad g = Hf + \epsilon \]

- **Forward & errors model:** \[ \rightarrow p(g|f, \theta_1; M) \]
- **Prior models** \[ \rightarrow p(f|\theta_2; M) \]
- **Hyperparameters** \[ \theta = (\theta_1, \theta_2) \rightarrow p(\theta|M) \]
- **Bayes:** \[ \rightarrow p(f, \theta|g; M) = \frac{p(g|f, \theta; M)p(f|\theta; M)p(\theta|M)}{p(g|M)} \]
- **Joint MAP:** \[ (\hat{f}, \hat{\theta}) = \arg \max_{(f, \theta)} \{ p(f, \theta|g; M) \} \]
- **Marginalization:** \[ \begin{align*}
    p(f|g; M) &= \int p(f, \theta|g; M) \, d\theta \\
p(\theta|g; M) &= \int p(f, \theta|g; M) \, df
\end{align*} \]
- **Posterior means:** \[ \begin{align*}
    \hat{f} &= \int f \ p(f, \theta|g; M) \, d\theta \ d f \\
    \hat{\theta} &= \int \theta \ p(f, \theta|g; M) \, df \ d\theta
\end{align*} \]
- **Evidence of the model:**

\[ p(g|M) = \int \int p(g|f, \theta; M)p(f|\theta; M)p(\theta|M) \, df \ d\theta \]
Two main steps in the Bayesian approach

- Prior modeling
  - Separable:
    - Gaussian, Generalized Gaussian, Gamma,
    - mixture of Gaussians, mixture of Gammas, ...
  - Markovian: Gauss-Markov, GGM, ...
  - Separable or Markovian with hidden variables
    (contours, region labels)

- Choice of the estimator and computational aspects
  - MAP, Posterior mean, Marginal MAP
  - MAP needs optimization algorithms
  - Posterior mean needs integration methods
  - Marginal MAP needs integration and optimization
  - Approximations:
    - Gaussian approximation (Laplace)
    - Numerical exploration MCMC
    - Variational Bayes (Separable approximation)
Which images I am looking for?
Which signals I am looking for?

- **Gaussian**
  \[ p(f_j) \propto \exp\{-\alpha |f_j|^2\} \]

- **Generalized Gaussian**
  \[ p(f_j) \propto \exp\{-\alpha |f_j|^p\}, \quad 1 \leq p \leq 2 \]

- **Gamma**
  \[ p(f_j) \propto f_j^{\alpha} \exp\{-\beta f_j\} \]

- **Beta**
  \[ p(f_j) \propto f_j^{\alpha} (1 - f_j)^{\beta} \]
Different prior models for signals and images

- **Separable**
  \[ p(f) = \prod_j p_j(f_j) \propto \exp\left\{ -\beta \sum_j \phi(f_j) \right\} \]

  \[ p(f) \propto \exp\left\{ -\beta \sum_{r \in \mathcal{R}} \phi(f(r)) \right\} \]

- **Markovien (simple)**
  \[ p(f_j|f_{j-1}) \propto \exp\left\{ -\beta \phi(f_j - f_{j-1}) \right\} \]

  \[ p(f) \propto \exp\left\{ -\beta \sum_{r \in \mathcal{R}} \sum_{r' \in \mathcal{V}(r)} \phi(f(r), f(r')) \right\} \]

- **Markovien with hidden variables**
  \[ z(r) \text{ (lines, contours, regions)} \]

  \[ p(f|z) \propto \exp\left\{ -\beta \sum_{r \in \mathcal{R}} \sum_{r' \in \mathcal{V}(r)} \phi(f(r), f(r'), z(r), z(r')) \right\} \]
Different prior models for images: Separable

- **Gaussian:**
  \[
p(f_j) \propto \exp\left\{ -\alpha |f_j|^2 \right\} \quad \Rightarrow \quad \Omega(f) = \alpha \sum_j |f_j|^2
  \]

- **Generalized Gaussian (GG):**
  \[
p(f_j) \propto \exp\left\{ -\alpha |f_j|^p \right\}, \quad 1 \leq p \leq 2 \quad \Rightarrow \quad \Phi(f) = \alpha \sum_j |f_j|^p
  \]

- **Gamma:** \( f_j > 0 \)
  \[
p(f_j) \propto f_j^\alpha \exp\left\{ -\beta f_j \right\} \quad \Rightarrow \quad \Omega(f) = \alpha \sum_j \ln f_j + \beta \sum_j f_j
  \]

- **Beta:** \( 1 > f_j > 0 \)
  \[
p(f_j) \propto f_j^\alpha (1 - f_j)^\beta \quad \Rightarrow \quad \Omega(f) = \alpha \sum_j \ln f_j + \beta \sum_j \ln(1 - f_j)
  \]
Different prior models for images: Separable

Gaussian
\[ p(f_j) \propto \exp\{-\alpha |f_j|^2\} \]

Generalized Gaussian
\[ p(f_j) \propto \exp\{-\alpha |f_j|^p\}, \quad 1 \leq p \leq 2 \]

Gamma
\[ p(f_j) \propto f_j^\alpha \exp\{-\beta f_j\} \]

Beta
\[ p(f_j) \propto f_j^\alpha (1 - f_j)^\beta \]
Different prior models: Simple Markovian

\[ p(f_j | f) \propto \exp \left\{ -\alpha \sum_{i \in V_j} \phi(f_j, f_i) \right\} \rightarrow \Phi(f) = \alpha \sum_j \sum_{i \in V_j} \phi(f_j, f_i) \]

- **1D case and one neighbor** \( V_j = j - 1 \):
  \[ \Phi(f) = \alpha \sum_j \phi(f_j - f_{j-1}) \]

- **1D Case and two neighbors** \( V_j = \{j - 1, j + 1\} \):
  \[ \Phi(f) = \alpha \sum_j \phi(f_j - \beta(f_{j-1} + f_{j-1})) \]

- **2D case with 4 neighbors:**
  \[ \Phi(f) = \alpha \sum_{r \in \mathcal{R}} \phi \left( f(r) - \beta \sum_{r' \in \mathcal{N}(r)} f(r') \right) \]

- \( \phi(t) = |t|^\gamma \): Generalized Gaussian
Different prior models: Simple Markovian

IID Gaussian
\[ p(f_j) \propto \exp\left\{-\alpha |f_j|^2\right\} \]

\[ p(f_j|f_{j-1}) \propto \exp\left\{-\alpha |f_j - f_{j-1}|^2\right\} \]

IID GG
\[ p(f_j) \propto \exp\left\{-\alpha |f_j|^p\right\} \]

\[ p(f_j|f_{j-1}) \propto \exp\left\{-\alpha |f_j - f_{j-1}|^p\right\} \]

Gauss-Markov

Different prior models: Non-stationary signals

- Modulated Variances IID
  \[ p(f_j | z_j) = \mathcal{N}(0, \nu(z_j)) \]

- Modulated Variances Gauss-Markov
  \[ p(f_j | f_{j-1}, z_j) = \mathcal{N}(f_{j-1}, \nu(z_j)) \]

- Modulated amplitudes IID
  \[ p(f_j | z_j) = \mathcal{N}(a(z_j), 1) \]

- Modulated amplitudes Gauss-Markov
  \[ p(f_j | f_{j-1}, z_j) = \mathcal{N}(a(f_{j-1}, z_j), 1) \]
Different prior models: Markovian with hidden variables

- **Piecewise Gaussians**
  
  \[ p(f_j | q_j, f_{j-1}) = \mathcal{N} \left( (1 - q_j)f_{j-1}, \sigma_f^2 \right) \]

- **Mixture of Gaussians (MoG)**
  
  \[ p(f_j | z_j = k) = \mathcal{N} \left( m_k, \sigma_k^2 \right) \& z_j \text{ markovian} \]

\[ p(f | q) \propto \exp \left\{ -\alpha \sum_j |f_j - (1 - q_j)f_{j-1}|^2 \right\} \]

\[ p(f | z) \propto \exp \left\{ -\alpha \sum_k \sum_{j \in R_k} \left( \frac{f_j - m_k}{\sigma_k} \right)^2 \right\} \]
Particular case of Gauss-Markov models

\[
\begin{cases}
  g = H f + \epsilon & \text{with} \\
  f \sim \mathcal{N} (0, \sigma_f^2 (DD^t)^{-1})
\end{cases}
\]

\[
\begin{cases}
  g = H f + \epsilon \\
  f = C f + z & \text{with} \\
  z \sim \mathcal{N}(0, \sigma_f^2 I)
\end{cases}
\]

and \( D = (I - C) \)

\[
f \mid g \sim \mathcal{N}(\hat{f}, \hat{P}) \quad \text{with} \quad \hat{f} = \hat{P} H^t g, \quad \hat{P} = (H^t H + \lambda D^t D)^{-1}
\]

\[
\hat{f} = \arg \min_f \{ J(f) = \| g - H f \|^2 + \lambda \| D f \|^2 \}
\]

\[
\begin{cases}
  g = H f + \epsilon \\
  \text{with} \quad f \sim \mathcal{N} (0, \sigma_f^2 (DD^t))
\end{cases}
\]

\[
\begin{cases}
  g = H f + \epsilon \\
  f = D z & \text{with} \\
  z \sim \mathcal{N}(0, \sigma_f^2 I)
\end{cases}
\]

\[
z \mid g \sim \mathcal{N}(\hat{z}, \hat{P}) \quad \text{with} \quad \hat{z} = \hat{P} D^t H^t g, \quad \hat{P} = (D^t H^t HD + \lambda I)^{-1}
\]

\[
\hat{z} = \arg \min_z \{ J(z) = \| g - HD z \|^2 + \lambda \| z \|^2 \} \quad \rightarrow \quad \hat{f} = D \hat{z}
\]

\( z \) Decomposition coeff on a basis (column of \( D \))

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Which images I am looking for?

Gauss-Markov

Generalized GM

Piecewise Gaussian

Mixture of GM
Markovien prior models for images

\[ \Omega(f) = \sum_j \phi(f_j - f_{j-1}) \]

- Gauss-Markov: \( \phi(t) = |t|^2 \)
- Generalized Gauss-Markov: \( \phi(t) = |t|^\alpha \)
- Piecewise Gauss-Markov or GGM: \( \phi(t) = \begin{cases} t^2 & |t| \leq T \\ T^2 & |t| > T \end{cases} \)

or equivalently:

\[ \Omega(f|q) = \sum_j (1 - q_j)\phi(f_j - f_{j-1}) \]

- \( q \) line process (contours)
- Mixture of Gaussians:

\[ \Omega(f|z) = \sum_k \sum_{\{j:z_j=k\}} \left( \frac{f_j - m_k}{v_k} \right)^2 \]

- \( z \) region labels process.
Gauss-Markov-Potts prior models for images

\[ f(r) \]
\[ z(r) \]
\[ c(r) = 1 - \delta(z(r) - z(r')) \]

\[
p(f(r) | z(r) = k, m_k, v_k) = \mathcal{N}(m_k, v_k)
\]
\[
p(f(r)) = \sum_k P(z(r) = k) \mathcal{N}(m_k, v_k) \quad \text{Mixture of Gaussians}
\]

- Separable iid hidden variables: \( p(z) = \prod_r p(z(r)) \)
- Markovian hidden variables: \( p(z) \) Potts-Markov:

\[
p(z(r) | z(r'), r' \in \mathcal{V}(r)) \propto \exp \left\{ \gamma \sum_{r' \in \mathcal{V}(r)} \delta(z(r) - z(r')) \right\}
\]

\[
p(z) \propto \exp \left\{ \gamma \sum_r \sum_{r' \in \mathcal{V}(r)} \delta(z(r) - z(r')) \right\}
\]
Four different cases

To each pixel of the image is associated 2 variables \( f(r) \) and \( z(r) \)

- \( f \mid z \) Gaussian iid, \( z \) iid :
  Mixture of Gaussians

- \( f \mid z \) Gauss-Markov, \( z \) iid :
  Mixture of Gauss-Markov

- \( f \mid z \) Gaussian iid, \( z \) Potts-Markov :
  Mixture of Independent Gaussians (MIG with Hidden Potts)

- \( f \mid z \) Markov, \( z \) Potts-Markov :
  Mixture of Gauss-Markov (MGM with hidden Potts)
Case 1: \( f \mid z \) Gaussian iid, \( z \) iid

Independent Mixture of Independent Gaussians (IMIG):

\[
p(f(r) \mid z(r) = k) = \mathcal{N}(m_k, v_k), \quad \forall r \in \mathcal{R}
\]

\[
p(f(r)) = \sum_{k=1}^{K} \alpha_k \mathcal{N}(m_k, v_k), \text{ with } \sum_k \alpha_k = 1.
\]

\[
p(z) = \prod_r p(z(r) = k) = \prod_r \alpha_k = \prod_k \alpha_k^{n_k}
\]

Noting

\[
m_z(r) = m_k, \quad v_z(r) = v_k, \quad \alpha_z(r) = \alpha_k, \quad \forall r \in \mathcal{R}_k
\]

we have:

\[
p(f \mid z) = \prod_{r \in \mathcal{R}} \mathcal{N}(m_z(r), v_z(r))
\]

\[
p(z) = \prod_r \alpha_z(r) = \prod_k \alpha_k^{\sum_{r \in \mathcal{R}} \delta(z(r) - k)} = \prod_k \alpha_k^{n_k}
\]
Case 2: $f \sim \text{Gauss-Markov}, \quad z \sim \text{iid}$

Independent Mixture of Gauss-Markov (IMGM):

$$p(f(r)|z(r), z(r'), f(r'), r' \in \mathcal{V}(r)) = \mathcal{N}(\mu_z(r), \nu_z(r)), \forall r \in \mathcal{R}$$

$$\mu_z(r) = \frac{1}{|\mathcal{V}(r)|} \sum_{r' \in \mathcal{V}(r)} \mu^*_z(r')$$
$$\mu^*_z(r') = \delta(z(r') - z(r)) f(r') + (1 - \delta(z(r') - z(r)) m_z(r')$$
$$= (1 - c(r')) f(r') + c(r') m_z(r')$$

$$p(f|z) \propto \prod_r \mathcal{N}(\mu_z(r), \nu_z(r)) \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \Sigma_k)$$
$$p(z) = \prod_r \nu_z(r) = \prod_k \alpha^k$$

with $1_k = 1, \forall r \in \mathcal{R}_k$ and $\Sigma_k$ a covariance matrix ($n_k \times n_k$).
Case 3: \( f \mid z \) Gauss iid, \( z \) Potts

Gauss iid as in Case 1:

\[
p(f \mid z) = \prod_{r \in \mathcal{R}} \mathcal{N}(m_z(r), \nu_z(r)) = \prod_{k} \prod_{r \in \mathcal{R}_k} \mathcal{N}(m_k, \nu_k)
\]

Potts-Markov

\[
p(z(r) \mid z(r'), r' \in \mathcal{V}(r)) \propto \exp \left\{ \gamma \sum_{r' \in \mathcal{V}(r)} \delta(z(r) - z(r')) \right\}
\]

\[
p(z) \propto \exp \left\{ \gamma \sum_{r \in \mathcal{R}} \sum_{r' \in \mathcal{V}(r)} \delta(z(r) - z(r')) \right\}
\]
Case 4: $f|z$ Gauss-Markov, $z$ Potts

Gauss-Markov as in Case 2:

$$p(f(r)|z(r), z(r'), f(r'), r', r' \in \mathcal{V}(r)) = \mathcal{N}(\mu_z(r), \nu_z(r)), \forall r \in \mathcal{R}$$

$$\mu_z(r) = \frac{1}{|\mathcal{V}(r)|} \sum_{r' \in \mathcal{V}(r)} \mu_z^*(r')$$

$$\mu_z^*(r') = \delta(z(r') - z(r)) \cdot f(r') + (1 - \delta(z(r') - z(r))) \cdot m_z(r')$$

$$p(f|z) \propto \prod_r \mathcal{N}(\mu_z(r), \nu_z(r)) \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \Sigma_k)$$

Potts-Markov as in Case 3:

$$p(z) \propto \exp \left\{ \gamma \sum_{r \in \mathcal{R}} \sum_{r' \in \mathcal{V}(r)} \delta(z(r) - z(r')) \right\}$$
Summary of the two proposed models

\( f \mid z \) Gaussian iid
\( z \approx \text{Potts-Markov} \)

\( f \mid z \) Markov
\( z \approx \text{Potts-Markov} \)

(MIG with Hidden Potts)  (MGM with hidden Potts)
Bayesian Computation

\[ p(f, z, \theta | g) \propto p(g | f, z, \nu_\epsilon) \, p(f | z, m, v) \, p(z | \gamma, \alpha) \, p(\theta) \]

\[ \theta = \{ \nu_\epsilon, (\alpha_k, m_k, v_k), k = 1, \ldots, K \} \quad p(\theta) \quad \text{Conjugate priors} \]

- Direct computation and use of \( p(f, z, \theta | g; M) \) is too complex
- Possible approximations:
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)

- Main idea in Variational Bayesian methods:
  Approximate
  \[ p(f, z, \theta | g; M) \quad \text{by} \quad q(f, z, \theta) = q_1(f) \, q_2(z) \, q_3(\theta) \]
  - Choice of approximation criterion : \( KL(q : p) \)
  - Choice of appropriate families of probability laws
    for \( q_1(f) \), \( q_2(z) \) and \( q_3(\theta) \)
MCMC based algorithm

\[ p(f, z, \theta | g) \propto p(g | f, z, \theta) \, p(f | z, \theta) \, p(z) \, p(\theta) \]

General scheme:

\[ \hat{f} \sim p(f | \hat{z}, \hat{\theta}, g) \rightarrow \hat{z} \sim p(z | \hat{f}, \hat{\theta}, g) \rightarrow \hat{\theta} \sim (\theta | \hat{f}, \hat{z}, g) \]

- Estimate \( f \) using 
  \[ p(f | \hat{z}, \hat{\theta}, g) \propto p(g | f, \theta) \, p(f | \hat{z}, \hat{\theta}) \]
  Needs optimisation of a quadratic criterion.

- Estimate \( z \) using 
  \[ p(z | \hat{f}, \hat{\theta}, g) \propto p(g | \hat{f}, \hat{z}, \hat{\theta}) \, p(z) \]
  Needs sampling of a Potts Markov field.

- Estimate \( \theta \) using 
  \[ p(\theta | \hat{f}, \hat{z}, g) \propto p(g | \hat{f}, \sigma^2 I) \, p(\hat{f} | \hat{z}, (m_k, v_k)) \, p(\theta) \]
  Conjugate priors \( \rightarrow \) analytical expressions.
Application of CT in NDT

Reconstruction from only 2 projections

\[ g_1(x) = \int f(x, y) \, dy, \quad g_2(y) = \int f(x, y) \, dx \]

- Given the marginals \( g_1(x) \) and \( g_2(y) \) find the joint distribution \( f(x, y) \).
- Infinite number of solutions: \( f(x, y) = g_1(x) g_2(y) \Omega(x, y) \)
\( \Omega(x, y) \) is a Copula:

\[ \int \Omega(x, y) \, dx = 1 \quad \text{and} \quad \int \Omega(x, y) \, dy = 1 \]
Application in CT

\[ g|f = Hf + \epsilon \]
\[ g|f \sim \mathcal{N}(Hf, \sigma^2 I) \]
Gaussian

\[ f|z \]
\[ f|z \sim \text{iid Gaussian} \]
Gauss-Markov

\[ z \]
\[ z \sim \text{iid} \]
Gauss-Markov

\[ c \]
\[ c(r) \in \{0, 1\} \]
binary

\[ 1 - \delta(z(r) - z(r')) \]

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Proposed algorithm

\[ p(f, z, \theta | g) \propto p(g | f, z, \theta) p(f | z, \theta) p(\theta) \]

General scheme:

\[ \hat{f} \sim p(f | \hat{z}, \hat{\theta}, g) \rightarrow \hat{z} \sim p(z | \hat{f}, \hat{\theta}, g) \rightarrow \hat{\theta} \sim (\theta | \hat{f}, \hat{z}, g) \]

Iterative algorithm:

- **Estimate** \( f \) using \( p(f | \hat{z}, \hat{\theta}, g) \propto p(g | f, \theta) p(f | \hat{z}, \hat{\theta}) \)
  Needs optimisation of a quadratic criterion.

- **Estimate** \( z \) using \( p(z | \hat{f}, \hat{\theta}, g) \propto p(g | \hat{f}, \hat{z}, \hat{\theta}) p(z) \)
  Needs sampling of a Potts Markov field.

- **Estimate** \( \theta \) using
  
  \[ p(\theta | \hat{f}, \hat{z}, g) \propto p(g | \hat{f}, \sigma^2 \epsilon I) p(\hat{f} | \hat{z}, (m_k, v_k)) p(\theta) \]
  Conjugate priors \(\rightarrow\) analytical expressions.
Results

Original Backprojection Filtered BP LS
Gauss-Markov+pos GM+Line process GM+Label process

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Application in Microwave imaging

\[ g(\omega) = \int f(r) \exp \{-j(\omega \cdot r)\} \, dr + \epsilon(\omega) \]

\[ g(u, v) = \iint f(x, y) \exp \{-j(ux + vy)\} \, dx \, dy + \epsilon(u, v) \]

\[ g = Hf + \epsilon \]
Application in Microwave imaging
Conclusions

- Bayesian Inference for inverse problems
- Approximations (Laplace, MCMC, Variational)
- Gauss-Markov-Potts are useful prior models for images incorporating regions and contours
- Separable approximations for Joint posterior with Gauss-Markov-Potts priors
- Application in different CT (X ray, US, Microwaves, PET, SPECT)

Perspectives:

- Efficient implementation in 2D and 3D cases
- Evaluation of performances and comparison with MCMC methods
- Application to other linear and non linear inverse problems: (PET, SPECT or ultrasound and microwave imaging)
Color (Multi-spectral) image deconvolution

\[ f_i(x, y) \xrightarrow{h(x, y)} + \varepsilon_i(x, y) \xrightarrow{g_i(x, y)} \]

Observation model: \[ g_i = H f_i + \varepsilon_i, \quad i = 1, 2, 3 \]
Images fusion and joint segmentation

(with O. Féron)

\[
\begin{align*}
  g_i(r) &= f_i(r) + \epsilon_i(r) \\
  p(f_i(r) | z(r) = k) &= \mathcal{N}(m_{i,k}, \sigma_{i,k}^2) \\
  p(f | z) &= \prod_i p(f_i | z)
\end{align*}
\]
Data fusion in medical imaging
(with O. Féron)

\[
\begin{align*}
    g_i(r) &= f_i(r) + \epsilon_i(r) \\
    p(f_i(r)|z(r) = k) &= \mathcal{N}(m_{ik}, \sigma_{ik}^2) \\
    p(f|z) &= \prod_i p(f_i|z)
\end{align*}
\]

\(g_1\) \rightarrow \hat{f}_1

\(g_2\) \rightarrow \hat{f}_2

\hat{z}
Super-Resolution

(with F. Humblot)

Low Resolution images

High Resolution image
Joint segmentation of hyper-spectral images

(with N. Bali & A. Mohammadpour)

\[
\begin{aligned}
g_i(r) &= f_i(r) + \epsilon_i(r) \\
p(f_i(r)|z(r) = k) &= \mathcal{N}(m_{i,k}, \sigma_{i,k}^2), \quad k = 1, \ldots, K \\
p(f|z) &= \prod_i p(f_i|z) \\
m_{i,k} & \text{ follow a Markovian model along the index } i
\end{aligned}
\]
Segmentation of a video sequence of images

(with P. Brault)

\[
\begin{align*}
  g_i(r) &= f_i(r) + \epsilon_i(r) \\
  p(f_i(r) | z_i(r) = k) &= \mathcal{N}(m_{ik}, \sigma_{ik}^2), \quad k = 1, \cdots, K \\
  p(f | z) &= \prod_i p(f_i | z_i) \\
  z_i(r) &\text{ follow a Markovian model along the index } i
\end{align*}
\]
Source separation

(with H. Snoussi & M. Ichir)

\[
\begin{align*}
g_i(r) &= \sum_{j=1}^{N} A_{ij} f_j(r) + \epsilon_i(r) \\
p(f_j(r)|z_j(r) = k) &= \mathcal{N}(m_{jk}, \sigma_{jk}^2) \\
p(A_{ij}) &= \mathcal{N}(A_{0ij}, \sigma_{0ij}^2)
\end{align*}
\]
Some references

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Questions and Discussions

A. Mohammad-Djafari, Inverse Problems in Imaging & Computer vision, Master ITEMS 2012, UPB, Bucarest, Nov 2012, 79