Inverse Problems in Signal and Image processing, Imaging systems and Computer Vision

From Deterministic Regularization to Probabilistic Bayesian Approaches

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- Invers problems : Examples and general formulation
- Inversion methods :
  analytical, parametric and non parametric
- Determinitic methods:
  Data matching, Least Squares, Regularization
- Probabilistic methods:
  Probability matching, Maximum likelihood, Bayesian inference
- Bayesian inference approach
- Prior models for images
- Bayesian computation
- Applications:
  Computed Tomography, Image separation, Superresolution, SAR Imaging
- Conclusions
- Questions and Discussion

Inverse problems : 3 main examples

- Example 1:
  Measuring variation of temperature with a therometer
  - $f(t)$ variation of temperature over time
  - $g(t)$ variation of length of the liquid in thermometer
- Example 2:
  Making an image with a camera, a microscope or a telescope
  - $f(x, y)$ real scene
  - $g(x, y)$ observed image
- Example 3: Making an image of the interior of a body
  - $f(x, y)$ a section of a real 3D body $f(x, y, z)$
  - $g_\phi(r)$ a line of observed radiographe $g_\phi(r, z)$

- Example 1: Deconvolution
- Example 2: Image restoration
- Example 3: Image reconstruction

Measuring variation of temperature with a therometer

- $f(t)$ variation of temperature over time
- $g(t)$ variation of length of the liquid in thermometer
- Forward model: Convolution

$$g(t) = \int f(t')\, h(t - t')\, \mathrm{d}t' + \epsilon(t)$$

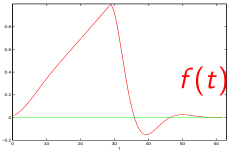
  $h(t)$: impulse response of the measurement system
- Inverse problem: Deconvolution

  Given the forward model $\mathcal{H}$ (impulse response $h(t)$))
  and a set of data $g(t_i), i = 1, \cdots, M$
  find $f(t)$

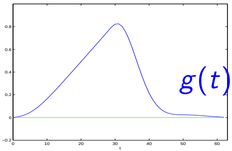
Measuring variation of temperature with a therometer

Forward model: Convolution

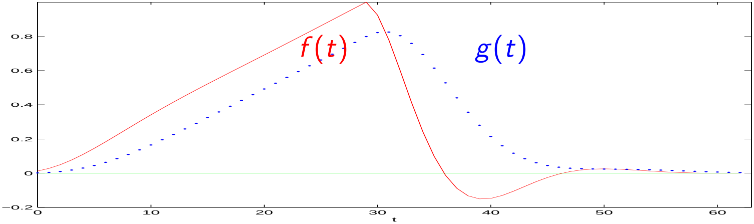
$$g(t) = \int f(t')\, h(t-t')\, \mathrm{d}t' + \epsilon(t)$$



$\longrightarrow$ **Thermometer $h(t)$** $\longrightarrow$



Inversion: Deconvolution



Making an image with a camera, a microscope or a telescope



- $f(x, y)$ real scene
- $g(x, y)$ observed image

- Forward model: Convolution

$$g(x, y) = \iint f(x', y')\, h(x - x', y - y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x, y)$$

  $h(x, y)$: Point Spread Function (PSF) of the imaging system

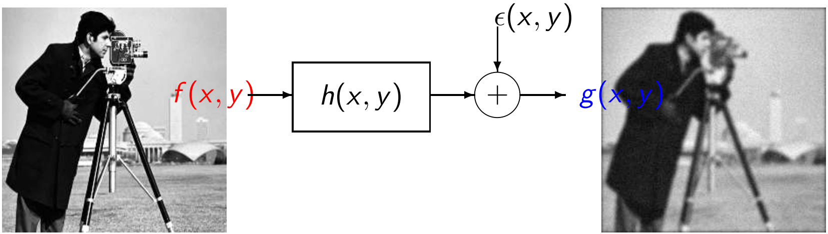
- Inverse problem: Image restoration

  Given the forward model $\mathcal{H}$ (PSF $h(x, y)$))
  and a set of data $g(x_i, y_i), i = 1, \cdots, M$
  find $f(x, y)$

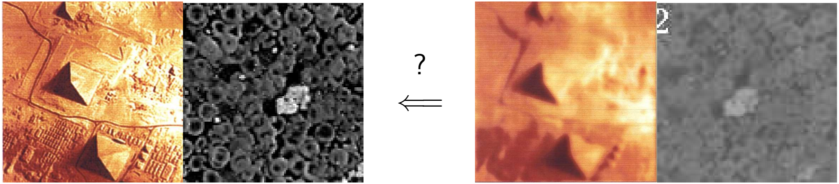
Making an image with an unfocused camera

Forward model: 2D Convolution

$$g(x,y) = \iint f(x',y')\, h(x-x', y-y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x,y)$$

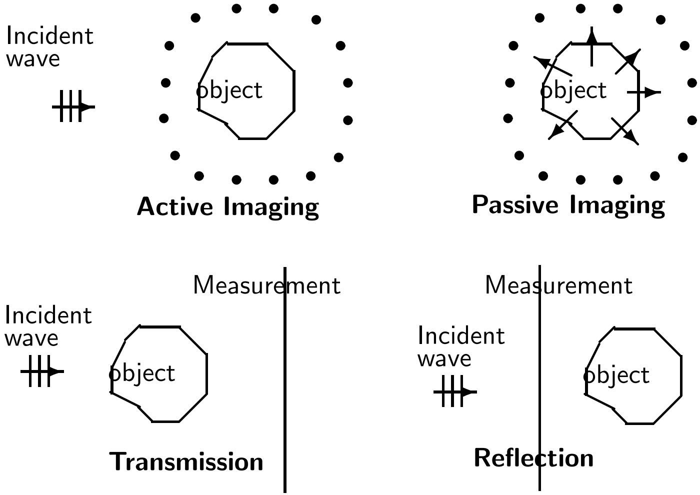


Inversion: Deconvolution



Making an image of the interior of a body

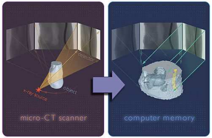
Different imaging systems:



Forward problem: Knowing the object predict the data

Inverse problem: From measured data find the object

Making an image of the interior of a body



- $f(x, y)$ a section of a real 3D body $f(x, y, z)$
- $g_\phi(r)$ a line of observed radiographe $g_\phi(r, z)$

- Forward model:
  Line integrals or Radon Transform

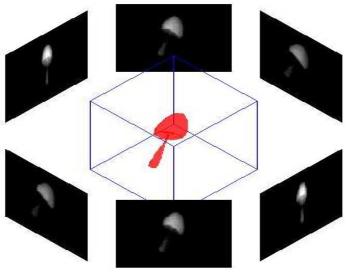
$$
\begin{aligned}
g_\phi(r) &= \int_{L_{r,\phi}} f(x, y) \, \mathrm{d}l + \epsilon_\phi(r) \\
&= \iint f(x, y) \, \delta(r - x\cos\phi - y\sin\phi) \, \mathrm{d}x \, \mathrm{d}y + \epsilon_\phi(r)
\end{aligned}
$$

- Inverse problem: Image reconstruction

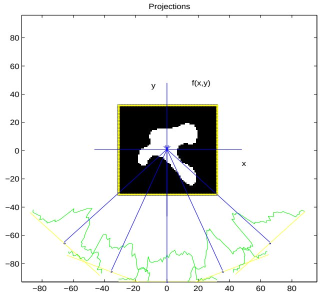
  Given the forward model $\mathcal{H}$ (Radon Transform) and
  a set of data $g_{\phi_i}(r), i = 1, \cdots, M$
  find $f(x, y)$

2D and 3D Computed Tomography

3D



2D



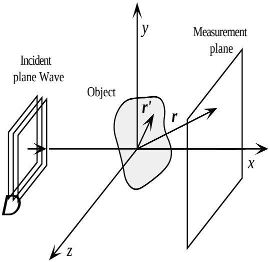
$$g_\phi(r_1,r_2) = \int_{\mathcal{L}_{r_1,r_2,\phi}} f(x,y,z)\,\mathrm{d}l \qquad g_\phi(r) = \int_{\mathcal{L}_{r,\phi}} f(x,y)\,\mathrm{d}l$$

Forward probelm: $f(x,y)$ or $f(x,y,z) \longrightarrow g_\phi(r)$ or $g_\phi(r_1,r_2)$

Inverse problem: $g_\phi(r)$ or $g_\phi(r_1,r_2) \longrightarrow f(x,y)$ or $f(x,y,z)$

Microwave or ultrasound imaging

Measurs: diffracted wave by the object $g(\boldsymbol{r}_i)$
Unknown quantity: $f(\boldsymbol{r}) = k_0^2(n^2(\boldsymbol{r}) - 1)$
Intermediate quantity : $\phi(\boldsymbol{r})$

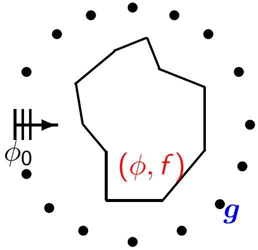


$$g(\boldsymbol{r}_i) = \iint_D G_m(\boldsymbol{r}_i, \boldsymbol{r}')\phi(\boldsymbol{r}')\, f(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}', \; \boldsymbol{r}_i \in S$$

$$\phi(\boldsymbol{r}) = \phi_0(\boldsymbol{r}) + \iint_D G_o(\boldsymbol{r}, \boldsymbol{r}')\phi(\boldsymbol{r}')\, f(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}', \; \boldsymbol{r} \in D$$

Born approximation $(\phi(\boldsymbol{r}') \simeq \phi_0(\boldsymbol{r}'))$ **):**

$$g(\boldsymbol{r}_i) = \iint_D G_m(\boldsymbol{r}_i, \boldsymbol{r}')\phi_0(\boldsymbol{r}')\, f(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}', \; \boldsymbol{r}_i \in S$$



Discretization :

$$\begin{cases} \boldsymbol{g} = \boldsymbol{G}_m \boldsymbol{F} \boldsymbol{\phi} \\ \boldsymbol{\phi} = \phi_0 + \boldsymbol{G}_o \boldsymbol{F} \boldsymbol{\phi} \end{cases} \longrightarrow \begin{cases} \boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) \\ \text{with} \quad \boldsymbol{F} = \text{diag}(\boldsymbol{f}) \\ \boldsymbol{H}(\boldsymbol{f}) = \boldsymbol{G}_m \boldsymbol{F} (\boldsymbol{I} - \boldsymbol{G}_o \boldsymbol{F})^{-1} \phi_0 \end{cases}$$

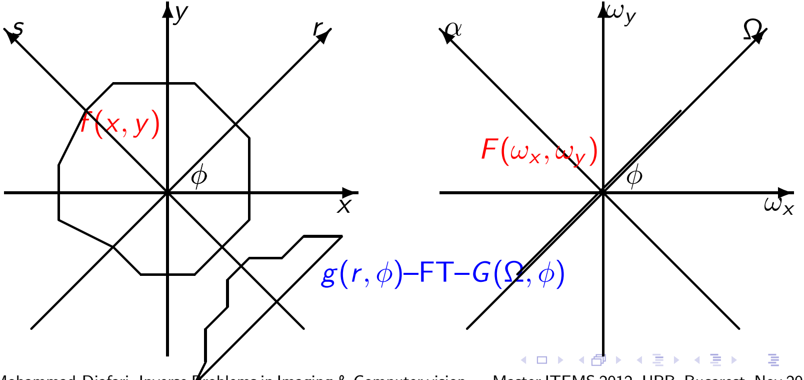
Fourier Synthesis in X ray Tomography

$$g(r,\phi) = \iint f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

$$G(\Omega,\phi) = \int g(r,\phi)\exp\{-j\Omega r\}\;\mathrm{d}r$$

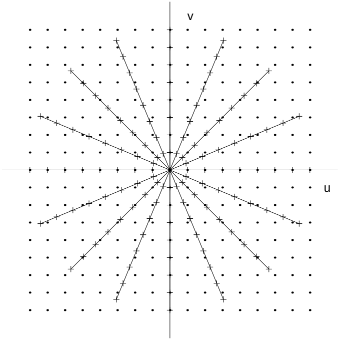
$$F(\omega_x,\omega_y) = \iint f(x,y)\exp\{-j\omega_x x,\omega_y y\}\;\mathrm{d}x\,\mathrm{d}y$$

$$F(\omega_x,\omega_y) = G(\Omega,\phi) \quad \text{for} \quad \omega_x = \Omega\cos\phi \quad \text{and} \quad \omega_y = \Omega\sin\phi$$



Fourier Synthesis in X ray tomography

$$G(\omega_x, \omega_y) = \iint f(x, y) \exp\{-j(\omega_x x + \omega_y y)\} \, dx \, dy$$



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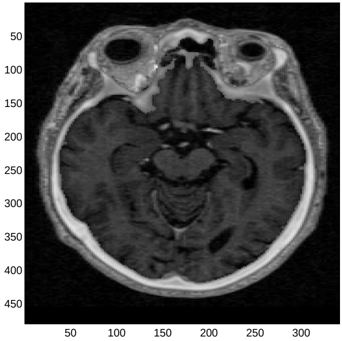
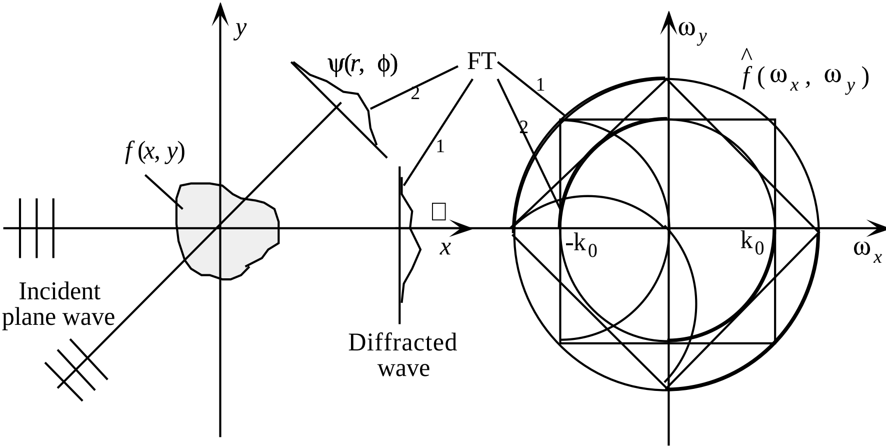


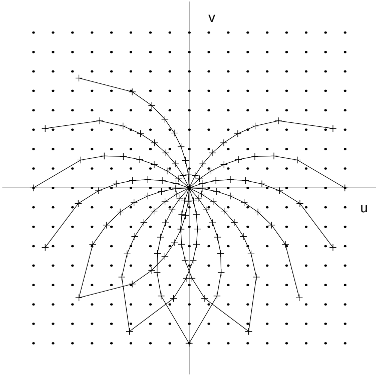
Forward problem: Given $f(x, y)$ compute $G(\omega_x, \omega_y)$
Inverse problem: Given $G(\omega_x, \omega_y)$ on those lines estimate $f(x, y)$

Fourier Synthesis in Diffraction tomography



Fourier Synthesis in Diffraction tomography

$$G(\omega_x, \omega_y) = \iint f(x, y) \exp\left\{-j\left(\omega_x x + \omega_y y\right)\right\} \, \mathrm{d}x \, \mathrm{d}y$$



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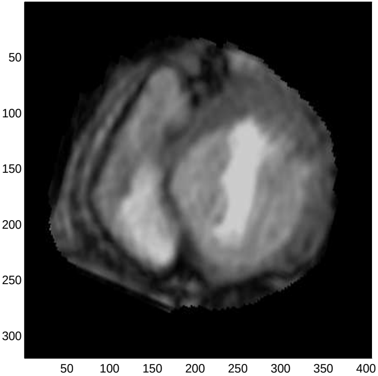
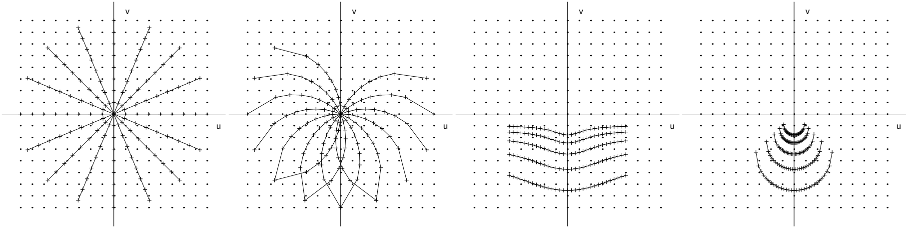


Forward problem: Given $f(x, y)$ compute $G(\omega_x, \omega_y)$
Inverse problem : Given $G(\omega_x, \omega_y)$ on those semi cercles estimate $f(x, y)$

Fourier Synthesis in different imaging systems

$$G(\omega_x, \omega_y) = \iint f(x, y) \exp\{-j(\omega_x x + \omega_y y)\} \, dx \, dy$$



X ray Tomography | Diffraction | Eddy current | SAR & Radar

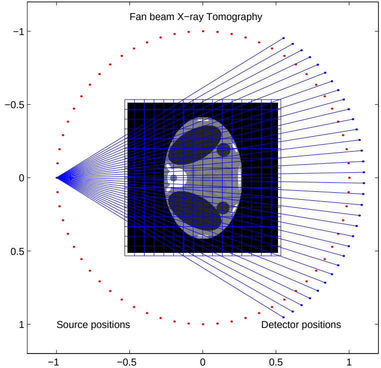
Forward problem: Given $f(x, y)$ compute $G(\omega_x, \omega_y)$
Inverse problem : Given $G(\omega_x, \omega_y)$ on those algebraic lines, cercles or curves, estimate $f(x, y)$

Invers Problems: other examples and applications

- X ray, Gamma ray Computed Tomography (CT)
- Microwave and ultrasound tomography
- Positron emission tomography (PET)
- Magnetic resonance imaging (MRI)
- Photoacoustic imaging
- Radio astronomy
- Geophysical imaging
- Non Destructive Evaluation (NDE) and Testing (NDT) techniques in industry
- Hyperspectral imaging
- Earth observation methods (Radar, SAR, IR, ...)
- Survey and tracking in security systems

Computed tomography (CT)

A Multislice CT Scanner



$$g(s_i) = \int_{L_i} f(\boldsymbol{r}) \, \mathrm{d}l_i + \epsilon(s_i)$$

Discretization

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \epsilon$$

Positron emission tomography (PET)

Magnetic resonance imaging (MRI)

Nuclear magnetic resonance imaging (NMRI), Para-sagittal MRI of the head

Radio astronomy (interferometry imaging systems)

The Very Large Array in New Mexico, an example of a radio telescope.

General formulation of inverse problems

- General non linear inverse problems:

$$g(\boldsymbol{s}) = [\mathcal{H} f(\boldsymbol{r})](\boldsymbol{s}) + \epsilon(\boldsymbol{s}), \quad \boldsymbol{r} \in \mathcal{R}, \quad \boldsymbol{s} \in \mathcal{S}$$

- Linear models:

$$g(\boldsymbol{s}) = \int f(\boldsymbol{r})\, h(\boldsymbol{r}, \boldsymbol{s})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s})$$

  If $h(\boldsymbol{r}, \boldsymbol{s}) = h(\boldsymbol{r} - \boldsymbol{s}) \longrightarrow$ Convolution.

- Discrete data:

$$g(\boldsymbol{s}_i) = \int h(\boldsymbol{s}_i, \boldsymbol{r})\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s}_i), \quad i = 1, \cdots, m$$

- Inversion: Given the forward model $\mathcal{H}$ and the data $\boldsymbol{g} = \{g(\boldsymbol{s}_i), i = 1, \cdots, m)\}$ estimate $f(\boldsymbol{r})$
- Well-posed and Ill-posed problems (Hadamard): existance, uniqueness and stability
- Need for prior information

Analytical methods (mathematical physics)

$$g(\boldsymbol{s}_i) = \int h(\boldsymbol{s}_i, \boldsymbol{r})\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s}_i), \quad i = 1, \cdots, m$$

$$g(\boldsymbol{s}) = \int h(\boldsymbol{s}, \boldsymbol{r})\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}$$

$$\widehat{f}(\boldsymbol{r}) = \int w(\boldsymbol{s}, \boldsymbol{r})\, g(\boldsymbol{s})\, \mathrm{d}\boldsymbol{s}$$

$w(\boldsymbol{s}, \boldsymbol{r})$ minimizing a criterion:

$$\begin{aligned}
Q(w(\boldsymbol{s},\boldsymbol{r})) &= \left\| g(\boldsymbol{s}) - [\mathcal{H}\widehat{f}(\boldsymbol{r})](\boldsymbol{s}) \right\|_2^2 = \int \left| g(\boldsymbol{s}) - [\mathcal{H}\widehat{f}(\boldsymbol{r})](\boldsymbol{s}) \right|^2 \mathrm{d}\boldsymbol{s} \\
&= \int \left| g(\boldsymbol{s}) - \int h(\boldsymbol{s},\boldsymbol{r})\, \widehat{f}(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} \right|^2 \mathrm{d}\boldsymbol{s} \\
&= \int \left| g(\boldsymbol{s}) - \int \int h(\boldsymbol{s},\boldsymbol{r}) w(\boldsymbol{s},\boldsymbol{r})\, g(\boldsymbol{s})\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{r} \right|^2 \mathrm{d}\boldsymbol{s}
\end{aligned}$$

Trivial solution: $h(\boldsymbol{s},\boldsymbol{r}) w(\boldsymbol{s},\boldsymbol{r}) = \delta(\boldsymbol{r})\delta(\boldsymbol{s})$

Analytical methods

- Trivial solution:

$$w(\boldsymbol{s},\boldsymbol{r}) = h^{-1}(\boldsymbol{s},\boldsymbol{r})$$

  Example: Fourier Transform:

$$g(\boldsymbol{s}) = \int f(\boldsymbol{r}) \exp\{-j\boldsymbol{s}.\boldsymbol{r}\} \, \mathrm{d}\boldsymbol{r}$$

$$h(\boldsymbol{s},\boldsymbol{r}) = \exp\{-j\boldsymbol{s}.\boldsymbol{r}\} \longrightarrow w(\boldsymbol{s},\boldsymbol{r}) = \exp\{+j\boldsymbol{s}.\boldsymbol{r}\}$$

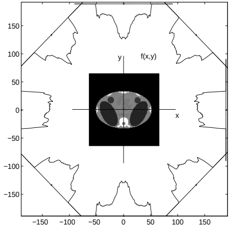
$$\hat{f}(\boldsymbol{r}) = \int g(\boldsymbol{s}) \exp\{+j\boldsymbol{s}.\boldsymbol{r}\} \, \mathrm{d}\boldsymbol{s}$$

- Known classical solutions for specific expressions of $h(\boldsymbol{s},\boldsymbol{r})$:
  - 1D cases: 1D Fourier, Hilbert, Weil, Melin, ...
  - 2D cases: 2D Fourier, Radon, ...

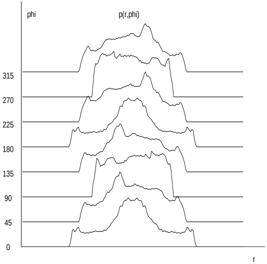
X ray Tomography

$$g(r,\phi) = -\ln\left(\frac{I}{I_0}\right) = \int_{L_{r,\phi}} f(x,y)\,\mathrm{d}l$$

$$g(r,\phi) = \iint_D f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$



$f(x,y)$ → RT → $g(r,\phi)$



IRT

?

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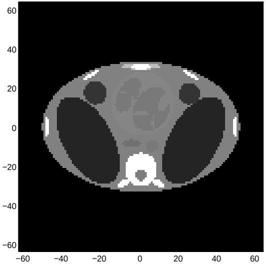
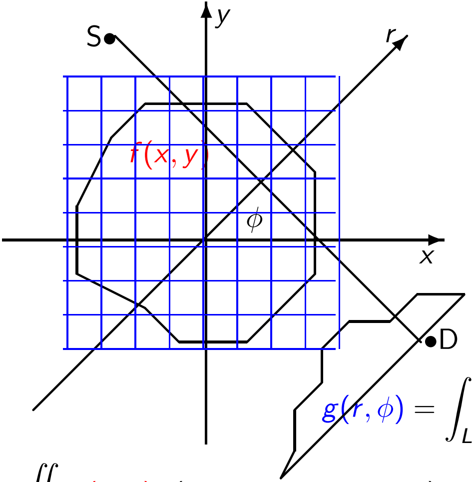


Analytical Inversion methods



$$g(r, \phi) = \int_L f(x, y) \, \mathrm{d}l$$

Radon:

$$g(r, \phi) = \iint_D f(x, y) \, \delta(r - x \cos\phi - y \sin\phi) \, \mathrm{d}x \, \mathrm{d}y$$

$$f(x, y) = \left(-\frac{1}{2\pi^2} \right) \int_0^{\pi} \int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} g(r, \phi)}{(r - x \cos\phi - y \sin\phi)} \, \mathrm{d}r \, \mathrm{d}\phi$$

Filtered Backprojection method

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right) \int_0^{\pi} \int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} g(r,\phi)}{(r - x\cos\phi - y\sin\phi)} \,\mathrm{d}r \,\mathrm{d}\phi$$

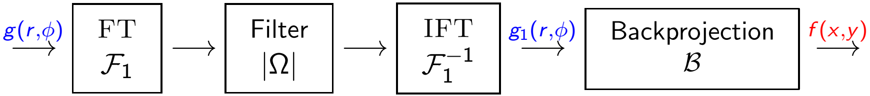
Derivation $\mathcal{D}$: $\quad \overline{g}(r,\phi) = \dfrac{\partial g(r,\phi)}{\partial r}$

Hilbert Transform $\mathcal{H}$: $\quad g_1(r',\phi) = \dfrac{1}{\pi} \displaystyle\int_0^{\infty} \frac{\overline{g}(r,\phi)}{(r - r')} \,\mathrm{d}r$

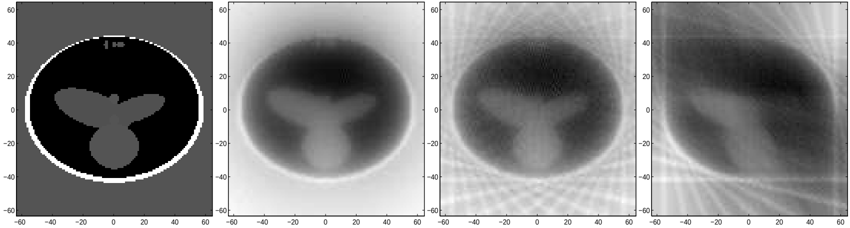
Backprojection $\mathcal{B}$: $\quad f(x,y) = \dfrac{1}{2\pi} \displaystyle\int_0^{\pi} g_1(r' = x\cos\phi + y\sin\phi, \phi) \,\mathrm{d}\phi$

$$f(x,y) = \mathcal{B}\,\mathcal{H}\,\mathcal{D}\, g(r,\phi) = \mathcal{B}\,\mathcal{F}_1^{-1}\,|\Omega|\,\mathcal{F}_1\, g(r,\phi)$$

- Backprojection of filtered projections:



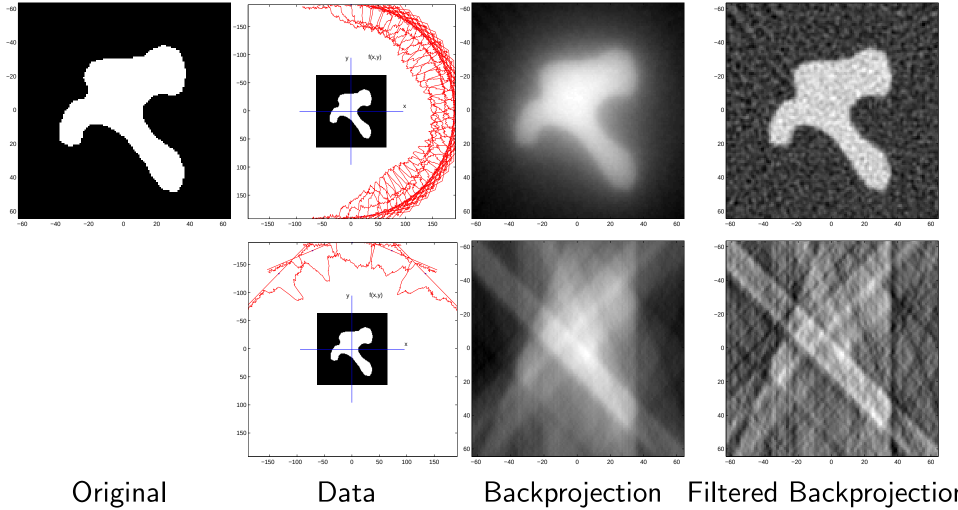
Limitations : Limited angle or noisy data



Original   64 proj.   16 proj.   8 proj. $[0, \pi/2]$

- Limited angle or noisy data
- Accounting for detector size
- Other measurement geometries: fan beam, ...

Limitations : Limited angle or noisy data



Parametric methods

- $f(\boldsymbol{r})$ is described in a parametric form with a very few number of parameters $\boldsymbol{\theta}$ and one searches $\widehat{\boldsymbol{\theta}}$ which minimizes a criterion such as:
- Least Squares (LS): $Q(\boldsymbol{\theta}) = \sum_i |g_i - [\mathcal{H} f(\boldsymbol{\theta})]_i|^2$
- Robust criteria : $Q(\boldsymbol{\theta}) = \sum_i \phi(|g_i - [\mathcal{H} f(\boldsymbol{\theta})]_i|)$
  with different functions ϕ (L_1, Hubert, ...).
- Likelihood : $\mathcal{L}(\boldsymbol{\theta}) = -\ln p(\boldsymbol{g}|\boldsymbol{\theta})$
- Penalized likelihood : $\mathcal{L}(\boldsymbol{\theta}) = -\ln p(\boldsymbol{g}|\boldsymbol{\theta}) + \lambda\Omega(\boldsymbol{\theta})$

Examples:

- Spectrometry: $f(t)$ modelled as a sum og gaussians
  $f(t) = \sum_{k=1}^{K} a_k \mathcal{N}(t|\mu_k, v_k)$ $\quad \boldsymbol{\theta} = \{a_k, \mu_k, v_k\}$
- Tomography in CND: $f(x,y)$ is modelled as a superposition of circular or elleiptical discs $\quad \boldsymbol{\theta} = \{a_k, \mu_k, r_k\}$

Non parametric methods

$$g(\boldsymbol{s}_i) = \int h(\boldsymbol{s}_i, \boldsymbol{r})\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s}_i), \quad i = 1, \cdots, M$$

- $f(\boldsymbol{r})$ is assumed to be well approximated by

$$f(\boldsymbol{r}) \simeq \sum_{j=1}^{N} f_j\, b_j(\boldsymbol{r})$$

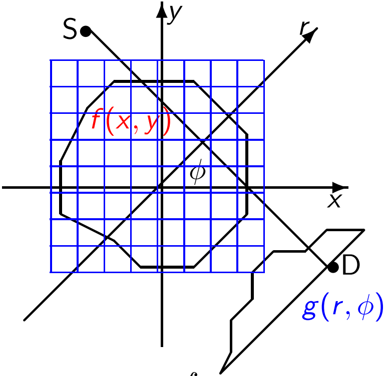
with $\{b_j(\boldsymbol{r})\}$ a basis or any other set of known functions

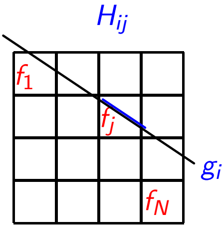
$$g(\boldsymbol{s}_i) = g_i \simeq \sum_{j=1}^{N} f_j \int h(\boldsymbol{s}_i, \boldsymbol{r})\, b_j(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}, \quad i = 1, \cdots, M$$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \quad \text{with} \quad H_{ij} = \int h(\boldsymbol{s}_i, \boldsymbol{r})\, b_j(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}$$

- $\boldsymbol{H}$ is huge dimensional
- LS solution : $\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{Q(\boldsymbol{f})\}$ with
  $Q(\boldsymbol{f}) = \sum_i |g_i - [\boldsymbol{H}\boldsymbol{f}]_i|^2 = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2$
  does not give satisfactory result.

Algebraic methods: Discretization





$$f(x,y) = \sum_j f_j \, b_j(x,y)$$

$$b_j(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \text{ pixel } j \\ 0 & \text{else} \end{cases}$$

$$g(r,\phi) = \int_L f(x,y) \, \mathrm{d}l$$

$$g_i = \sum_{j=1}^{N} H_{ij} \, f_j + \epsilon_i$$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

Inversion: Deterministic methods

Data matching

- Observation model
  $g_i = h_i(\boldsymbol{f}) + \epsilon_i, \quad i = 1, \ldots, M \longrightarrow \boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) + \boldsymbol{\epsilon}$
- Misatch between data and output of the model $\Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f}))$

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{\Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f}))\}$$

- Examples:

– LS $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \|\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})\|^2 = \sum_i |g_i - h_i(\boldsymbol{f})|^2$

– L_p $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \|\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})\|^p = \sum_i |g_i - h_i(\boldsymbol{f})|^p, \quad 1 < p < 2$

– KL $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \sum_i g_i \ln \frac{g_i}{h_i(\boldsymbol{f})}$

- In general, does not give satisfactory results for inverse problems.

Regularization theory

Inverse problems = Ill posed problems

$\longrightarrow$ Need for prior information

Functional space (Tikhonov):

$$g = \mathcal{H}(f) + \epsilon \longrightarrow J(f) = ||g - \mathcal{H}(f)||_2^2 + \lambda||\mathcal{D}f||_2^2$$

Finite dimensional space (Philips & Towmey): $\boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) + \boldsymbol{\epsilon}$

- Minimum norme LS (MNLS): $J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda||\boldsymbol{f}||^2$
- Classical regularization: $J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda||\boldsymbol{D}\boldsymbol{f}||^2$
- More general regularization:

$$J(\boldsymbol{f}) = \mathcal{Q}(\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})) + \lambda\Omega(\boldsymbol{D}\boldsymbol{f})$$

or

$$J(\boldsymbol{f}) = \Delta_1(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) + \lambda\Delta_2(\boldsymbol{f}, \boldsymbol{f}_\infty)$$

Limitations:

- Errors are considered implicitly white and Gaussian
- Limited prior information on the solution
- Lack of tools for the determination of the hyperparameters

Inversion: Probabilistic methods

Taking account of errors and uncertainties $\longrightarrow$ Probability theory

- Maximum Likelihood (ML)
- Minimum Inaccuracy (MI)
- Probability Distribution Matching (PDM)
- Maximum Entropy (ME) and Information Theory (IT)
- Bayesian Inference (BAYES)

Advantages:

- Explicit account of the errors and noise
- A large class of priors via explicit or implicit modeling
- A coherent approach to combine information content of the data and priors

Limitations:

- Practical implementation and cost of calculation

Bayesian estimation approach

$$\mathcal{M}: \qquad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Observation model $\mathcal{M}$ + Hypothesis on the noise $\boldsymbol{\epsilon} \longrightarrow$
$$p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M}) = p_{\epsilon}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f})$$
- A priori information $\quad p(\boldsymbol{f}|\mathcal{M})$
- Bayes : $$p(\boldsymbol{f}|\boldsymbol{g};\mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M})\, p(\boldsymbol{f}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$$

Link with regularization :

Maximum A Posteriori (MAP) :

$$\begin{aligned} \widehat{\boldsymbol{f}} &= \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{g}|\boldsymbol{f})\; p(\boldsymbol{f})\} \\ &= \arg\min_{\boldsymbol{f}} \{-\ln p(\boldsymbol{g}|\boldsymbol{f}) - \ln p(\boldsymbol{f})\} \end{aligned}$$

with $\quad Q(\boldsymbol{g}, \boldsymbol{H}\boldsymbol{f}) = -\ln p(\boldsymbol{g}|\boldsymbol{f}) \quad$ and $\quad \lambda\Omega(\boldsymbol{f}) = -\ln p(\boldsymbol{f})$

Case of linear models and Gaussian priors

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Hypothesis on the noise: $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma_\epsilon^2 \boldsymbol{I}) \longrightarrow$
  $p(\boldsymbol{g}|\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2\right\}$
- Hypothesis on $\boldsymbol{f}$: $\boldsymbol{f} \sim \mathcal{N}(0, \sigma_f^2(\boldsymbol{D}^t\boldsymbol{D})^{-1}) \longrightarrow$
  $p(\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_f^2}\|\boldsymbol{D}\boldsymbol{f}\|^2\right\}$
- A posteriori:
  $$p(\boldsymbol{f}|\boldsymbol{g}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 - \frac{1}{2\sigma_f^2}\|\boldsymbol{D}\boldsymbol{f}\|^2\right\}$$
- MAP : $\quad \widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$
  with $\quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2, \qquad \lambda = \frac{\sigma_\epsilon^2}{\sigma_f^2}$
- Advantage : characterization of the solution

$$\boldsymbol{f}|\boldsymbol{g} \sim \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{P}}) \text{ with } \widehat{\boldsymbol{f}} = \widehat{\boldsymbol{P}}\boldsymbol{H}^t\boldsymbol{g}, \quad \widehat{\boldsymbol{P}} = \left(\boldsymbol{H}^t\boldsymbol{H} + \lambda\boldsymbol{D}^t\boldsymbol{D}\right)^{-1}$$

MAP estimation with other priors:

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{with} \quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\Omega(\boldsymbol{f})$$

Separable priors:

- Gaussian: $p(f_j) \propto \exp\left\{-\alpha|f_j|^2\right\} \longrightarrow \Omega(\boldsymbol{f}) = \alpha \sum_j |f_j|^2$
- Gamma: $p(f_j) \propto f_j^{\alpha} \exp\{-\beta f_j\} \longrightarrow \Omega(\boldsymbol{f}) = \alpha \sum_j \ln f_j + \beta f_j$
- Beta:
  $p(f_j) \propto f_j^{\alpha}(1 - f_j)^{\beta} \longrightarrow \Omega(\boldsymbol{f}) = \alpha \sum_j \ln f_j + \beta \sum_j \ln(1 - f_j)$
- Generalized Gaussian:
  $p(f_j) \propto \exp\{-\alpha|f_j|^p\}, \quad 1 < p < 2 \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j |f_j|^p,$

Markovian models:

$$p(f_j|\boldsymbol{f}) \propto \exp\left\{-\alpha \sum_{i \in N_j} \phi(f_j, f_i)\right\} \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j \sum_{i \in N_j} \phi(f_j, f_i),$$

MAP estimation with markovien priors:

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{with} \quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\Omega(\boldsymbol{f})$$

$$\Omega(\boldsymbol{f}) = \sum_j \phi(f_j - f_{j-1})$$

with $\phi(t)$:

Convex functions:

$$|t|^\alpha, \; \sqrt{1+t^2} - 1, \; \log(\cosh(t)), \; \begin{cases} t^2 & |t| \leq T \\ 2T|t| - T^2 & |t| > T \end{cases}$$

or Non convex functions:

$$\log(1+t^2), \quad \frac{t^2}{1+t^2}, \quad \arctan(t^2), \quad \begin{cases} t^2 & |t| \leq T \\ T^2 & |t| > T \end{cases}$$

Main advantages of the Bayesian approach

- MAP = Regularization
- Posterior mean ? Marginal MAP ?
- More information in the posterior law than only its mode or its mean
- Meaning and tools for estimating hyper parameters
- Meaning and tools for model selection
- More specific and specialized priors, particularly through the hidden variables
- More computational tools:
  - Expectation-Maximization for computing the maximum likelihood parameters
  - MCMC for posterior exploration
  - Variational Bayes for analytical computation of the posterior marginals
  - ...

Full Bayesian approach

$$\mathcal{M}: \qquad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

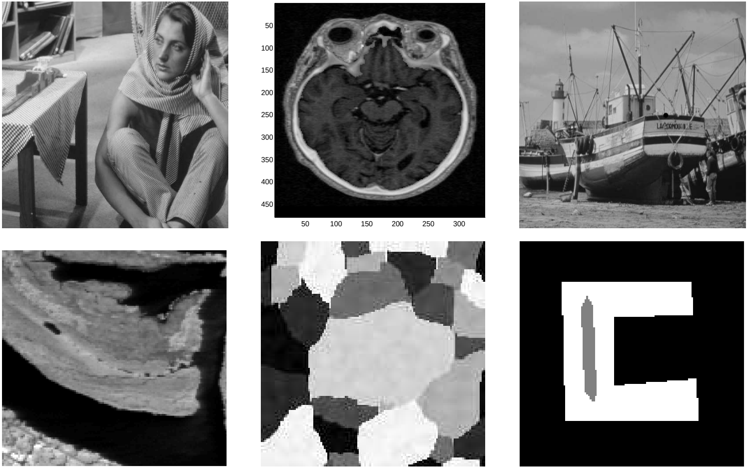
- Forward & errors model: $\longrightarrow p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1; \mathcal{M})$
- Prior models $\longrightarrow p(\boldsymbol{f}|\boldsymbol{\theta}_2; \mathcal{M})$
- Hyperparameters $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \longrightarrow p(\boldsymbol{\theta}|\mathcal{M})$
- Bayes: $\longrightarrow p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\; p(\boldsymbol{f}|\boldsymbol{\theta};\mathcal{M})\; p(\boldsymbol{\theta}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$
- Joint MAP: $\qquad (\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f},\boldsymbol{\theta})} \{p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\}$
- Marginalization: $\left\{\begin{array}{ll} p(\boldsymbol{f}|\boldsymbol{g}; \mathcal{M}) & = \int p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{\theta} \\ p(\boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M}) & = \int p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f} \end{array}\right.$
- Posterior means: $\left\{\begin{array}{ll} \widehat{\boldsymbol{f}} & = \int \boldsymbol{f}\; p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{\theta}\, \mathrm{d}\boldsymbol{f} \\ \widehat{\boldsymbol{\theta}} & = \int \boldsymbol{\theta}\; p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta} \end{array}\right.$
- Evidence of the model:

$$p(\boldsymbol{g}|\mathcal{M}) = \iint p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{f}|\boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{\theta}|\mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta}$$

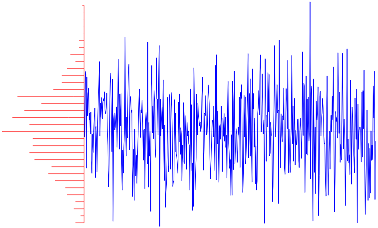
Two main steps in the Bayesian approach

- Prior modeling
  - Separable:
    Gaussian, Generalized Gaussian, Gamma,
    mixture of Gaussians, mixture of Gammas, ...
  - Markovian: Gauss-Markov, GGM, ...
  - Separable or Markovian with hidden variables
    (contours, region labels)
- Choice of the estimator and computational aspects
  - MAP, Posterior mean, Marginal MAP
  - MAP needs optimization algorithms
  - Posterior mean needs integration methods
  - Marginal MAP needs integration and optimization
  - Approximations:
    - Gaussian approximation (Laplace)
    - Numerical exploration MCMC
    - Variational Bayes (Separable approximation)

Which images I am looking for?

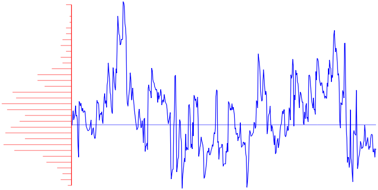


Which signals I am looking for?



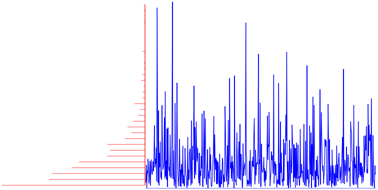
Gaussian

$p(f_j) \propto \exp\left\{-\alpha|f_j|^2\right\}$



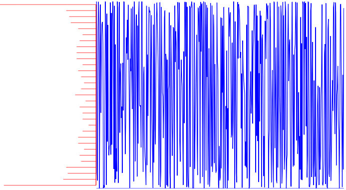
Generalized Gaussian

$p(f_j) \propto \exp\left\{-\alpha|f_j|^p\right\}, \quad 1 \leq p \leq 2$



Gamma

$p(f_j) \propto f_j^\alpha \exp\left\{-\beta f_j\right\}$



Beta

$p(f_j) \propto f_j^\alpha (1 - f_j)^\beta$

Different prior models for signals and images

- Separable $p(\boldsymbol{f}) = \prod_j p_j(f_j) \propto \exp\left\{-\beta \sum_j \phi(f_j)\right\}$

$$p(\boldsymbol{f}) \propto \exp\left\{-\beta \sum_{\boldsymbol{r}\in\mathcal{R}} \phi(f(\boldsymbol{r}))\right\}$$

- Markoviens (simple) $p(f_j|f_{j-1}) \propto \exp\left\{-\beta\phi(f_j - f_{j-1})\right\}$

$$p(\boldsymbol{f}) \propto \exp\left\{-\beta \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \phi(f(\boldsymbol{r}), f(\boldsymbol{r}'))\right\}$$

- Markovien with hidden variables
  $z(\boldsymbol{r})$ (lines, contours, regions)

$$p(\boldsymbol{f}|\boldsymbol{z}) \propto \exp\left\{-\beta \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \phi(f(\boldsymbol{r}), f(\boldsymbol{r}'), z(\boldsymbol{r}), z(\boldsymbol{r}'))\right\}$$

Different prior models for images: Separable

- Gaussian:

$$p(f_j) \propto \exp\left\{-\alpha|f_j|^2\right\} \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j |f_j|^2$$

- Generalized Gaussian (GG):

$$p(f_j) \propto \exp\left\{-\alpha|f_j|^p\right\}, \quad 1 \leq p \leq 2 \longrightarrow \quad \Phi(\boldsymbol{f}) = \alpha \sum_j |f_j|^p,$$

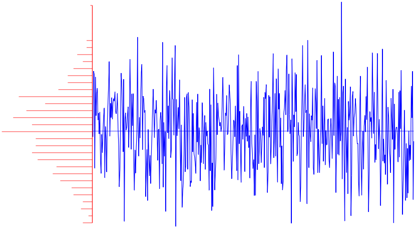
- Gamma: $f_j > 0$

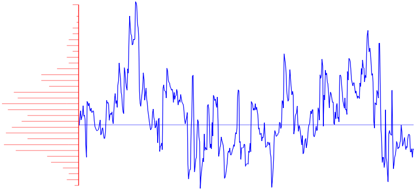
$$p(f_j) \propto f_j^\alpha \exp\left\{-\beta f_j\right\} \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j \ln f_j + \beta \sum_j f_j,$$

- Beta: $1 > f_j > 0$

$$p(f_j) \propto f_j^\alpha (1-f_j)^\beta \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j \ln f_j + \beta \sum_j \ln(1-f_j),$$

Different prior models for images: Separable



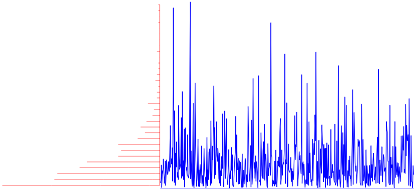


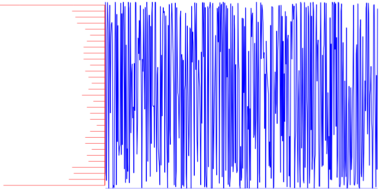
Gaussian

$$p(f_j) \propto \exp\left\{-\alpha|f_j|^2\right\}$$

Generalized Gaussian

$$p(f_j) \propto \exp\left\{-\alpha|f_j|^p\right\}, \quad 1 \leq p \leq 2$$





Gamma

$$p(f_j) \propto f_j^\alpha \exp\left\{-\beta f_j\right\}$$

Beta

$$p(f_j) \propto f_j^\alpha (1-f_j)^\beta$$

Different prior models: Simple Markovian

$$p(f_j|\boldsymbol{f}) \propto \exp\left\{-\alpha \sum_{i \in v_j} \phi(f_j, f_i)\right\} \longrightarrow \quad \Phi(\boldsymbol{f}) = \alpha \sum_j \sum_{i \in V_j} \phi(f_j, f_i)$$

- 1D case and one neigbor $V_j = j - 1$:

$$\Phi(\boldsymbol{f}) = \alpha \sum_j \phi(f_j - f_{j-1})$$

- 1D Case and two neighbors $V_j = \{j - 1, j + 1\}$:

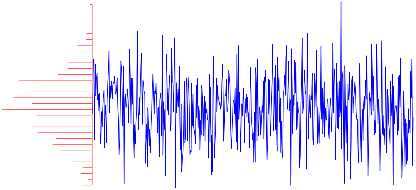
$$\Phi(\boldsymbol{f}) = \alpha \sum_j \phi\left(f_j - \beta(f_{j-1} + f_{j-1})\right)$$

- 2D case with 4 neighbors:

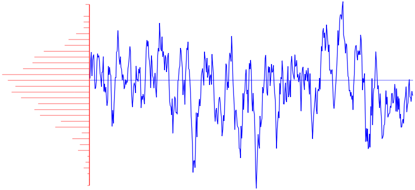
$$\Phi(\boldsymbol{f}) = \alpha \sum_{\boldsymbol{r} \in \mathcal{R}} \phi\left(f(\boldsymbol{r}) - \beta \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} f(\boldsymbol{r}')\right)$$

- $\phi(t) = |t|^\gamma$: Generalized Gaussian

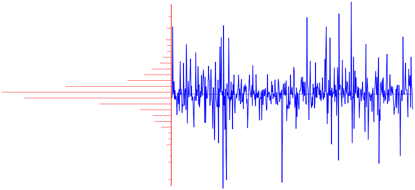
Different prior models: Simple Markovian



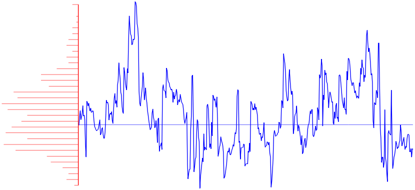
IID Gaussian
$p(f_j) \propto \exp\left\{-\alpha |f_j|^2\right\}$



Gauss-Markov
$p(f_j | f_{j-1}) \propto \exp\left\{-\alpha |f_j - f_{j-1}|^2\right\}$

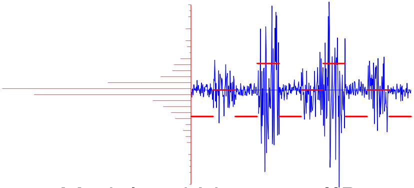


IID GG
$p(f_j) \propto \exp\left\{-\alpha |f_j|^p\right\}$



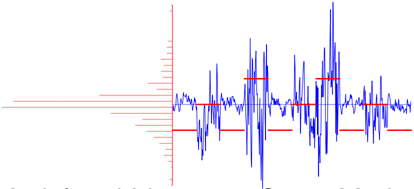
Markovian GG
$p(f_j | f_{j-1}) \propto \exp\left\{-\alpha |f_j - f_{j-1}|^p\right\}$

Different prior models: Non-stationnary signals



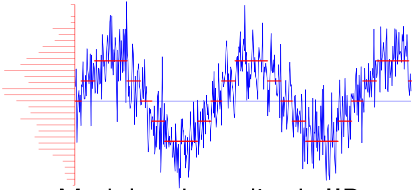
Modulated Variances IID

$$p(f_j|z_j) = \mathcal{N}(0, v(z_j))$$



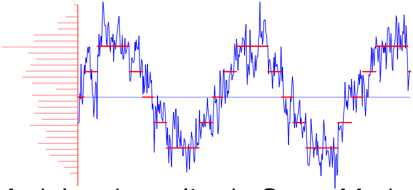
Modulated Variances Gauss-Markov

$$p(f_j|f_{j-1}, z_j) = \mathcal{N}(f_{j-1}, v(z_j))$$



Modulated amplituds IID

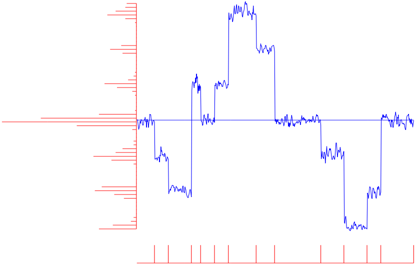
$$p(f_j|z_j) = \mathcal{N}(a(z_j), 1)$$

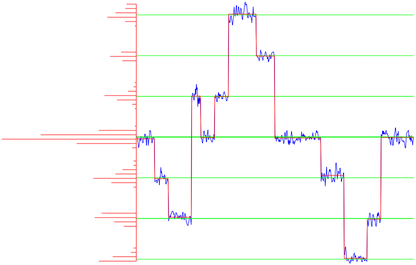


Modulated amplituds Gauss-Markov

$$p(f_j|f_{j-1}, z_j) = \mathcal{N}(a(f_{j-1}, z_j), 1)$$

Different prior models: Markovian with hidden variables





Piecewise Gaussians

(contours hidden variables)

$$p(f_j|q_j, f_{j-1}) = \mathcal{N}\left((1-q_j)f_{j-1}, \sigma_f^2\right)$$

$$p(\boldsymbol{f}|\boldsymbol{q}) \propto \exp\left\{-\alpha \sum_j \left|f_j - (1-q_j)f_{j-1}\right|^2\right\}$$

Mixture of Gaussians (MoG)

(regions labels hidden variables)

$$p(f_j|z_j = k) = \mathcal{N}\left(m_k, \sigma_k^2\right) \text{ \& } z_j \text{ markovian}$$

$$p(\boldsymbol{f}|\boldsymbol{z}) \propto \exp\left\{-\alpha \sum_k \sum_{j\in\mathcal{R}_k} \left(\frac{f_j - m_k}{\sigma_k}\right)^2\right\}$$

Particular case of Gauss-Markov models

$$\left\{\begin{array}{l} \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \quad \text{with} \\ \boldsymbol{f} \sim \mathcal{N}\left(0, \sigma_f^2 (\boldsymbol{D}^t \boldsymbol{D})^{-1}\right) \end{array}\right. = \left\{\begin{array}{l} \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \\ \boldsymbol{f} = \boldsymbol{C}\boldsymbol{f} + \boldsymbol{z} \text{ with } \boldsymbol{z} \sim \mathcal{N}(0, \sigma_f^2 \boldsymbol{I}) \\ \text{and} \quad \boldsymbol{D} = (\boldsymbol{I} - \boldsymbol{C}) \end{array}\right.$$

$$\boldsymbol{f}|\boldsymbol{g} \sim \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{P}}) \quad \text{with} \quad \widehat{\boldsymbol{f}} = \widehat{\boldsymbol{P}}\boldsymbol{H}^t\boldsymbol{g}, \quad \widehat{\boldsymbol{P}} = \left(\boldsymbol{H}^t\boldsymbol{H} + \lambda\boldsymbol{D}^t\boldsymbol{D}\right)^{-1}$$

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \left\{J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2\right\}$$

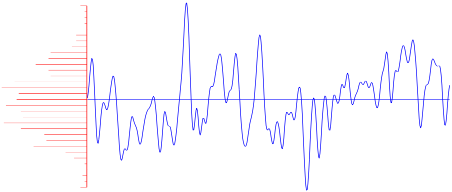
$$\left\{\begin{array}{l} \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \\ \text{with} \quad \boldsymbol{f} \sim \mathcal{N}\left(0, \sigma_f^2 (\boldsymbol{D}\boldsymbol{D}^t)\right) \end{array}\right. = \left\{\begin{array}{l} \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \\ \boldsymbol{f} = \boldsymbol{D}\boldsymbol{z} \quad \text{with} \quad \boldsymbol{z} \sim \mathcal{N}(0, \sigma_f^2 \boldsymbol{I}) \end{array}\right.$$

$$\boldsymbol{z}|\boldsymbol{g} \sim \mathcal{N}(\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{P}}) \quad \text{with} \quad \widehat{\boldsymbol{z}} = \widehat{\boldsymbol{P}}\boldsymbol{D}^t\boldsymbol{H}^t\boldsymbol{g}, \quad \widehat{\boldsymbol{P}} = \left(\boldsymbol{D}^t\boldsymbol{H}^t\boldsymbol{H}\boldsymbol{D} + \lambda\boldsymbol{I}\right)^{-1}$$

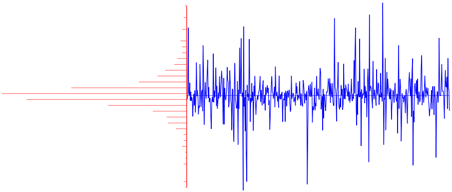
$$\widehat{\boldsymbol{z}} = \arg\min_{\boldsymbol{z}} \left\{J(\boldsymbol{z}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{D}\boldsymbol{z}\|^2 + \lambda\|\boldsymbol{z}\|^2\right\} \longrightarrow \widehat{\boldsymbol{f}} = \boldsymbol{D}\widehat{\boldsymbol{z}}$$

$\boldsymbol{z}$ Decomposition coeff on a basis (column of $\boldsymbol{D}$)

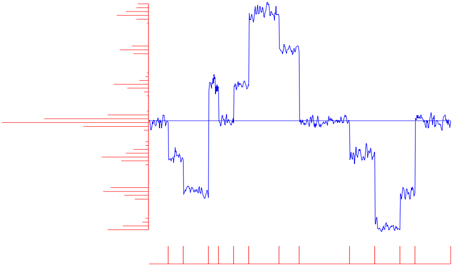
Which images I am looking for?



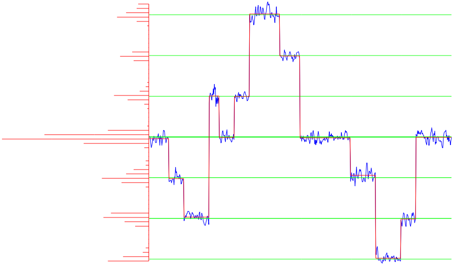
Gauss-Markov



Generalized GM



Piecewize Gaussian



Mixture of GM

Markovien prior models for images

$$\Omega(\boldsymbol{f}) = \sum_j \phi(f_j - f_{j-1})$$

- Gauss-Markov : $\phi(t) = |t|^2$
- Generalized Gauss-Markov : $\phi(t) = |t|^\alpha$
- Picewize Gauss-Markov or GGM : $\phi(t) = \begin{cases} t^2 & |t| \leq T \\ T^2 & |t| > T \end{cases}$

  or equivalently :

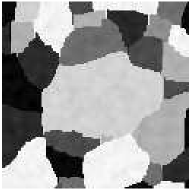
$$\Omega(\boldsymbol{f}|\boldsymbol{q}) = \sum_j (1 - q_j)\phi(f_j - f_{j-1})$$

  $\boldsymbol{q}$ line process (contours)
- Mixture of Gaussians :

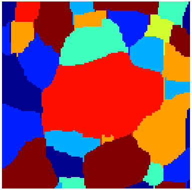
$$\Omega(\boldsymbol{f}|\boldsymbol{z}) = \sum_k \sum_{\{j:z_j=k\}} \left(\frac{f_j - m_k}{v_k}\right)^2$$

  $\boldsymbol{z}$ region labels process.

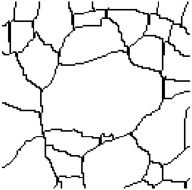
Gauss-Markov-Potts prior models for images



$f(\boldsymbol{r})$



$z(\boldsymbol{r})$



$c(\boldsymbol{r}) = 1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k, m_k, v_k) = \mathcal{N}(m_k, v_k)$$

$$p(f(\boldsymbol{r})) = \sum_k P(z(\boldsymbol{r}) = k)\, \mathcal{N}(m_k, v_k) \quad \text{Mixture of Gaussians}$$

- Separable iid hidden variables: $p(\boldsymbol{z}) = \prod_{\boldsymbol{r}} p(z(\boldsymbol{r}))$
- Markovian hidden variables: $p(\boldsymbol{z})$ Potts-Markov:

$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r} \in \mathcal{R}} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Four different cases

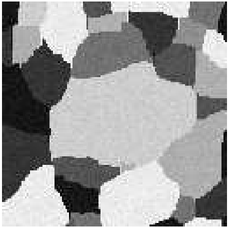
To each pixel of the image is associated 2 variables $f(\boldsymbol{r})$ and $z(\boldsymbol{r})$

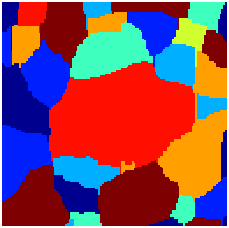
- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\quad \boldsymbol{z}$ iid :
  Mixture of Gaussians

- $\boldsymbol{f}|\boldsymbol{z}$ Gauss-Markov, $\quad \boldsymbol{z}$ iid :
  Mixture of Gauss-Markov

- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\quad \boldsymbol{z}$ Potts-Markov :
  Mixture of Independent Gaussians
  (MIG with Hidden Potts)

- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\quad \boldsymbol{z}$ Potts-Markov :
  Mixture of Gauss-Markov
  (MGM with hidden Potts)


$f(\boldsymbol{r})$


$z(\boldsymbol{r})$

Case 1: $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ iid

Independent Mixture of Independent Gaussiens (IMIG):

$$p(f(\boldsymbol{r})|z(\boldsymbol{r})=k)=\mathcal{N}(m_k,v_k), \quad \forall \boldsymbol{r}\in\mathcal{R}$$
$$p(f(\boldsymbol{r}))=\sum_{k=1}^{K}\alpha_k\mathcal{N}(m_k,v_k), \text{with } \sum_k \alpha_k=1.$$

$$p(\boldsymbol{z})=\prod_{\boldsymbol{r}} p(z(\boldsymbol{r})=k)=\prod_{\boldsymbol{r}}\alpha_k=\prod_k \alpha_k^{n_k}$$

Noting

$$m_z(\boldsymbol{r})=m_k, v_z(\boldsymbol{r})=v_k, \alpha_z(\boldsymbol{r})=\alpha_k, \forall \boldsymbol{r}\in\mathcal{R}_k$$

we have:

$$\begin{aligned} p(\boldsymbol{f}|\boldsymbol{z}) &= \prod_{\boldsymbol{r}\in\mathcal{R}}\mathcal{N}(m_z(\boldsymbol{r}),v_z(\boldsymbol{r})) \\ p(\boldsymbol{z}) &= \prod_{\boldsymbol{r}}\alpha_z(\boldsymbol{r})=\prod_k \alpha_k^{\sum_{\boldsymbol{r}\in\mathcal{R}}\delta(z(\boldsymbol{r})-k)}=\prod_k \alpha_k^{n_k} \end{aligned}$$

Case 2: $\boldsymbol{f}|\boldsymbol{z}$ Gauss-Markov, $\boldsymbol{z}$ iid

Independent Mixture of Gauss-Markov (IMGM):

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}), z(\boldsymbol{r}'), f(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) = \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})), \forall \boldsymbol{r} \in \mathcal{R}$$

$$\begin{array}{ll} \mu_z(\boldsymbol{r}) & = \frac{1}{|\mathcal{V}(\boldsymbol{r})|} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \mu_z^*(\boldsymbol{r}') \\ \mu_z^*(\boldsymbol{r}') & = \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, f(\boldsymbol{r}') + (1 - \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, m_z(\boldsymbol{r}') \\ & = \quad (1 - c(\boldsymbol{r}')) \quad f(\boldsymbol{r}') \ + \qquad c(\boldsymbol{r}') \qquad m_z(\boldsymbol{r}') \end{array}$$

$$\begin{array}{lll} p(\boldsymbol{f}|\boldsymbol{z}) & \propto \prod_{\boldsymbol{r}} \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})) & \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \boldsymbol{\Sigma}_k) \\ p(\boldsymbol{z}) & = \prod_{\boldsymbol{r}} v_z(\boldsymbol{r}) & = \prod_k \alpha_k^{n_k} \end{array}$$

with $1_k = 1, \forall \boldsymbol{r} \in \mathcal{R}_k$ and $\boldsymbol{\Sigma}_k$ a covariance matrix ($n_k \times n_k$).

Case 3: $\boldsymbol{f}|\boldsymbol{z}$ Gauss iid, $\boldsymbol{z}$ Potts

Gauss iid as in Case 1:

$$p(\boldsymbol{f}|\boldsymbol{z}) = \prod_{\boldsymbol{r}\in\mathcal{R}} \mathcal{N}(m_z(\boldsymbol{r}), v_z(\boldsymbol{r})) = \prod_k \prod_{\boldsymbol{r}\in\mathcal{R}_k} \mathcal{N}(m_k, v_k)$$

Potts-Markov

$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Case 4: $\boldsymbol{f}|\boldsymbol{z}$ Gauss-Markov, $\boldsymbol{z}$ Potts

Gauss-Markov as in Case 2:

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}), z(\boldsymbol{r}'), f(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) = \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})), \forall \boldsymbol{r} \in \mathcal{R}$$

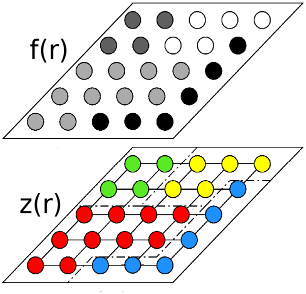
$$\begin{array}{ll} \mu_z(\boldsymbol{r}) & = \frac{1}{|\mathcal{V}(\boldsymbol{r})|} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \mu_z^*(\boldsymbol{r}') \\ \mu_z^*(\boldsymbol{r}') & = \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\; f(\boldsymbol{r}') + (1 - \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\; m_z(\boldsymbol{r}') \end{array}$$

$$p(\boldsymbol{f}|\boldsymbol{z}) \quad \propto \prod_{\boldsymbol{r}} \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})) \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \boldsymbol{\Sigma}_k)$$

Potts-Markov as in Case 3:

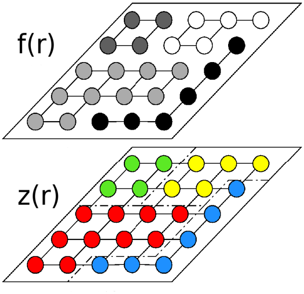
$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Summary of the two proposed models



$\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid
$\boldsymbol{z}$ Potts-Markov

(MIG with Hidden Potts)



$\boldsymbol{f}|\boldsymbol{z}$ Markov
$\boldsymbol{z}$ Potts-Markov

(MGM with hidden Potts)

Bayesian Computation

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{z}, v_\epsilon)\, p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{m}, \boldsymbol{v})\, p(\boldsymbol{z}|\gamma, \boldsymbol{\alpha})\, p(\boldsymbol{\theta})$$

$\boldsymbol{\theta} = \{v_\epsilon, (\alpha_k, m_k, v_k), k = 1, \cdot, K\}$ $\quad p(\boldsymbol{\theta})$ Conjugate priors

- Direct computation and use of $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})$ is too complex
- Possible approximations :
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)
- Main idea in Variational Bayesian methods:
  Approximate

  $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})$ by $q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta})$
  - Choice of approximation criterion : $KL(q : p)$
  - Choice of appropriate families of probability laws for $q_1(\boldsymbol{f})$, $q_2(\boldsymbol{z})$ and $q_3(\boldsymbol{\theta})$

MCMC based algorithm

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{z})\, p(\boldsymbol{\theta})$$

General scheme:

$$\widehat{\boldsymbol{f}} \sim p(\boldsymbol{f}|\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{z}} \sim p(\boldsymbol{z}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \boldsymbol{g})$$

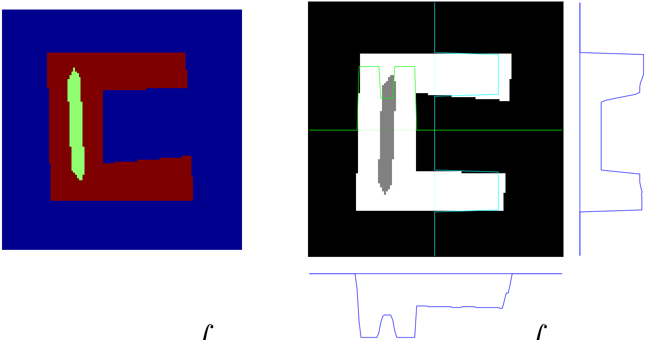
- Estimate $\boldsymbol{f}$ using $p(\boldsymbol{f}|\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta})\, p(\boldsymbol{f}|\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}})$
  Needs optimisation of a quadratic criterion.

- Estimate $\boldsymbol{z}$ using $p(\boldsymbol{z}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}})\, p(\boldsymbol{z})$
  Needs sampling of a Potts Markov field.

- Estimate $\boldsymbol{\theta}$ using
  $p(\boldsymbol{\theta}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}}, \sigma_\epsilon^2 \boldsymbol{I})\, p(\widehat{\boldsymbol{f}}|\widehat{\boldsymbol{z}}, (m_k, v_k))\, p(\boldsymbol{\theta})$
  Conjugate priors $\longrightarrow$ analytical expressions.

Application of CT in NDT

Reconstruction from only 2 projections

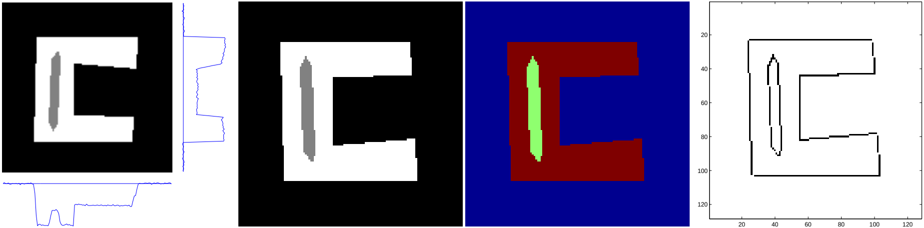


$$g_1(x) = \int f(x,y)\,\mathrm{d}y, \qquad g_2(y) = \int f(x,y)\,\mathrm{d}x$$

- Given the marginals $g_1(x)$ and $g_2(y)$ find the joint distribution $f(x,y)$.
- Infinite number of solutions : $f(x,y) = g_1(x)\,g_2(y)\,\Omega(x,y)$ $\Omega(x,y)$ is a Copula:

$$\int \Omega(x,y)\,\mathrm{d}x = 1 \quad \text{and} \quad \int \Omega(x,y)\,\mathrm{d}y = 1$$

Application in CT



$\boldsymbol{g}\|\boldsymbol{f}$	$\boldsymbol{f}\|\boldsymbol{z}$	$\boldsymbol{z}$	$\boldsymbol{c}$
$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$	iid Gaussian	iid	$c(\boldsymbol{r}) \in \{0, 1\}$
$\boldsymbol{g}\|\boldsymbol{f} \sim \mathcal{N}(\boldsymbol{H}\boldsymbol{f}, \sigma_\epsilon^2 \boldsymbol{I})$	or	or	$1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$
Gaussian	Gauss-Markov	Potts	binary

Proposed algorithm

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta} | \boldsymbol{g}) \propto p(\boldsymbol{g} | \boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}) \, p(\boldsymbol{f} | \boldsymbol{z}, \boldsymbol{\theta}) \, p(\boldsymbol{\theta})$$

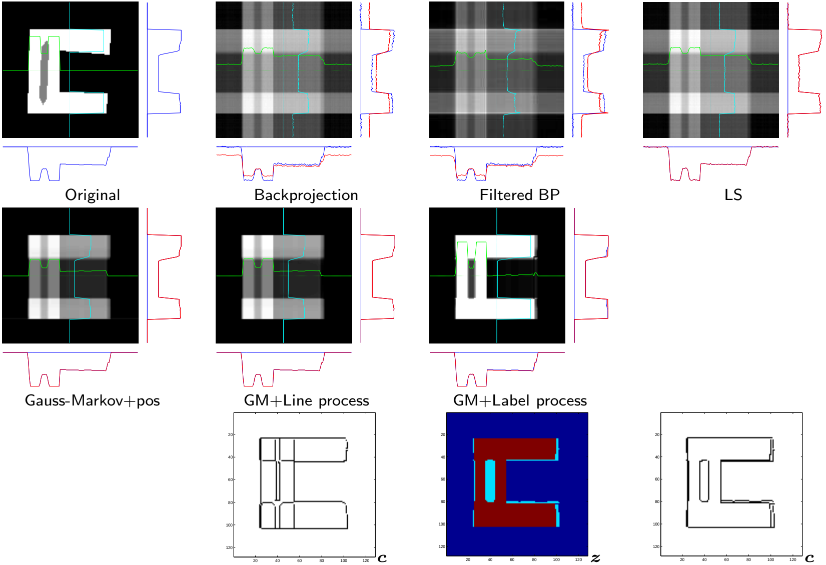
General scheme:

$$\widehat{\boldsymbol{f}} \sim p(\boldsymbol{f} | \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{z}} \sim p(\boldsymbol{z} | \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta} | \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \boldsymbol{g})$$

Iterative algorithme:

- Estimate $\boldsymbol{f}$ using $p(\boldsymbol{f} | \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \propto p(\boldsymbol{g} | \boldsymbol{f}, \boldsymbol{\theta}) \, p(\boldsymbol{f} | \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}})$
  Needs optimisation of a quadratic criterion.
- Estimate $\boldsymbol{z}$ using $p(\boldsymbol{z} | \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \propto p(\boldsymbol{g} | \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}}) \, p(\boldsymbol{z})$
  Needs sampling of a Potts Markov field.
- Estimate $\boldsymbol{\theta}$ using
  $p(\boldsymbol{\theta} | \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \boldsymbol{g}) \propto p(\boldsymbol{g} | \widehat{\boldsymbol{f}}, \sigma_\epsilon^2 \boldsymbol{I}) \, p(\widehat{\boldsymbol{f}} | \widehat{\boldsymbol{z}}, (m_k, v_k)) \, p(\boldsymbol{\theta})$
  Conjugate priors $\longrightarrow$ analytical expressions.

Results

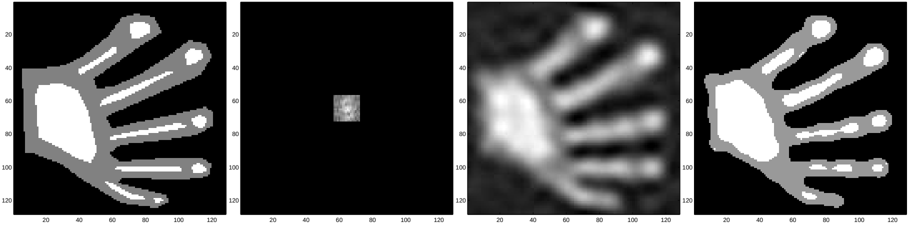

Original
Backprojection
Filtered BP
LS
Gauss-Markov+pos
GM+Line process
GM+Label process
c
z
c

Application in Microwave imaging

$$g(\boldsymbol{\omega}) = \int f(\boldsymbol{r}) \exp\{-j(\boldsymbol{\omega}.\boldsymbol{r})\} \, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{\omega})$$

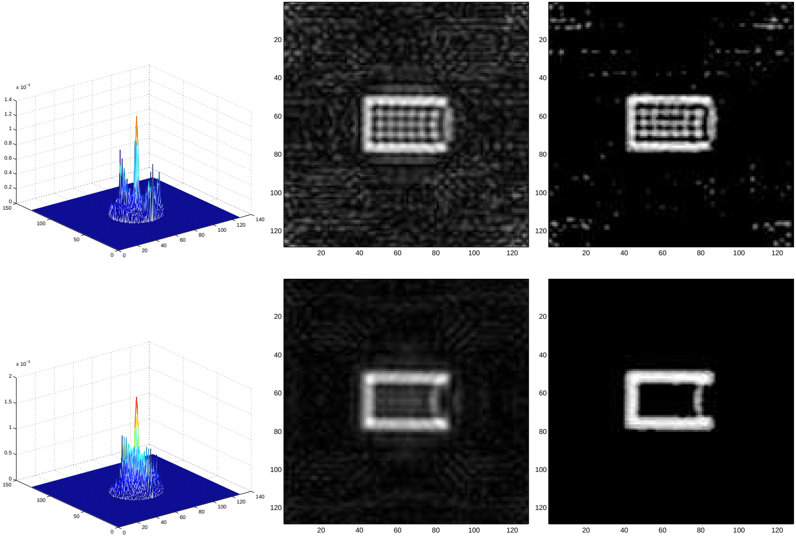
$$g(u, v) = \iint f(x, y) \exp\{-j(ux + vy)\} \, \mathrm{d}x \, \mathrm{d}y + \epsilon(u, v)$$

$$\boldsymbol{g} = \boldsymbol{H} \boldsymbol{f} + \boldsymbol{\epsilon}$$



$f(x, y)$ $g(u, v)$ $\widehat{f}$ IFT $\widehat{f}$ Proposed method

Application in Microwave imaging



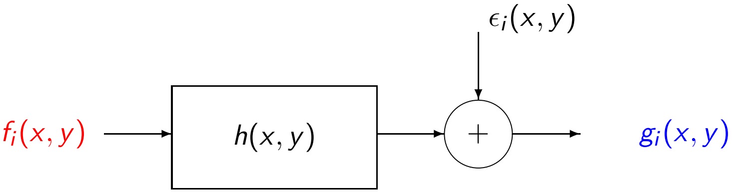
Conclusions

- Bayesian Inference for inverse problems
- Approximations (Laplace, MCMC, Variational)
- Gauss-Markov-Potts are useful prior models for images incorporating regions and contours
- Separable approximations for Joint posterior with Gauss-Markov-Potts priors
- Application in different CT (X ray, US, Microwaves, PET, SPECT)

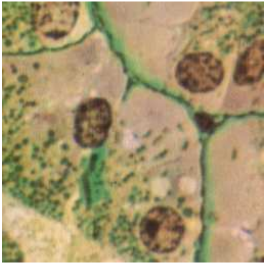
Perspectives :

- Efficient implementation in 2D and 3D cases
- Evaluation of performances and comparison with MCMC methods
- Application to other linear and non linear inverse problems: (PET, SPECT or ultrasound and microwave imaging)

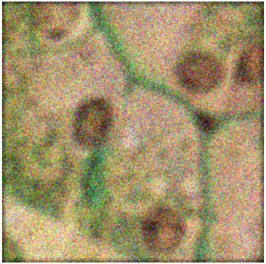
Color (Multi-spectral) image deconvolution



Observation model : $\boldsymbol{g}_i = \boldsymbol{H}\boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1,2,3$



? $\Longleftarrow$

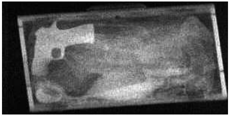


Images fusion and joint segmentation

(with O. Féron)

$$\left\{\begin{array}{l} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma^2_{i\,k}) \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}) \end{array}\right.$$

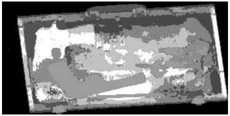
$\boldsymbol{g}_1$



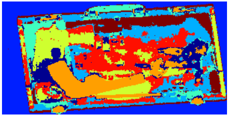
$\boldsymbol{g}_2$

$\longrightarrow$

$\widehat{\boldsymbol{f}}_1$



$\widehat{\boldsymbol{f}}_2$

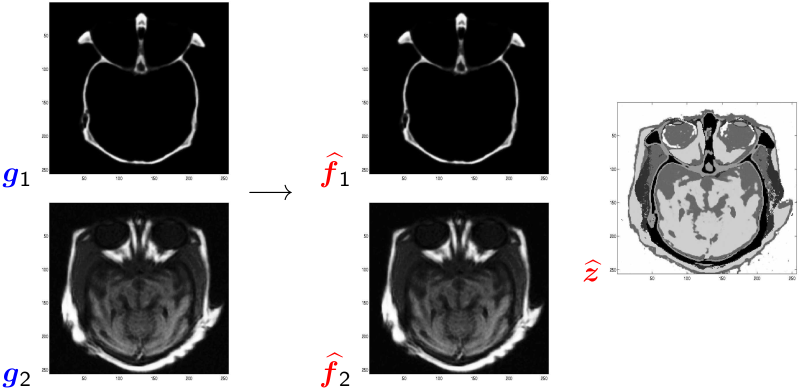


$\widehat{\boldsymbol{z}}$

Data fusion in medical imaging

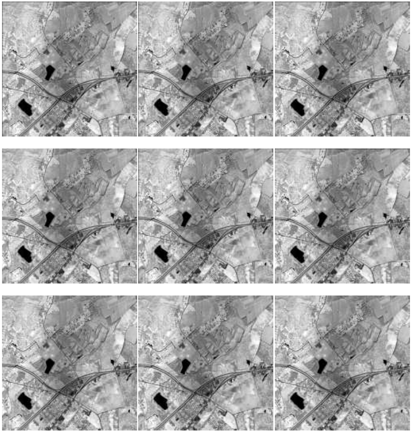
(with O. Féron)

$$\begin{cases} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma^2_{i\,k}) \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}) \end{cases}$$



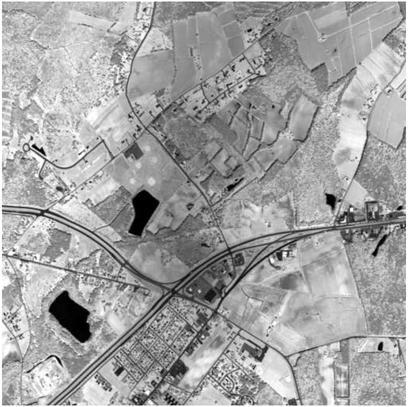
Super-Resolution

(with F. Humblot)



Low Resolution images

? ⟹

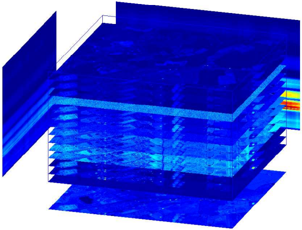
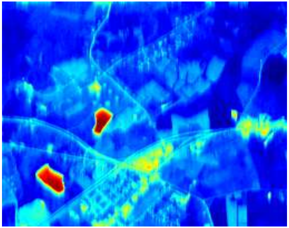
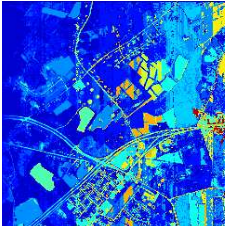


High Resolution image

Joint segmentation of hyper-spectral images

(with N. Bali & A. Mohammadpour)

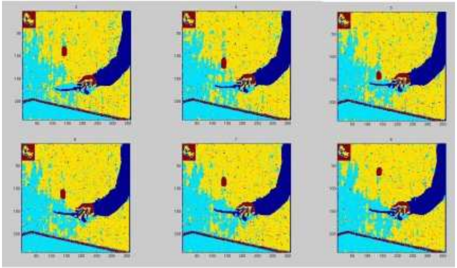
$$\begin{cases} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma^2_{i\,k}), \quad k = 1, \cdots, K \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}) \\ m_{i\,k} \text{ follow a Markovian model along the index } i \end{cases}$$

Segmentation of a video sequence of images

(with P. Brault)

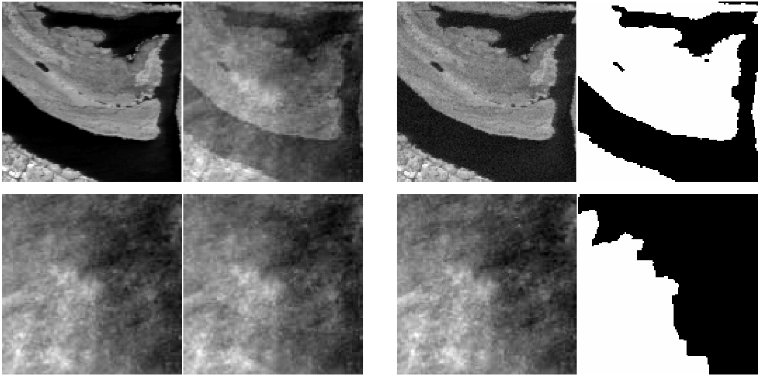
$$\left\{\begin{array}{l} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z_i(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma^2_{i\,k}), \quad k = 1, \cdots, K \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}_i) \\ z_i(\boldsymbol{r}) \ \text{ follow a Markovian model along the index } \ i \end{array}\right.$$



Source separation

(with H. Snoussi & M. Ichir)

$$
\begin{cases}
g_i(\boldsymbol{r}) = \sum_{j=1}^{N} A_{ij} f_j(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\
p(f_j(\boldsymbol{r}) | z_j(\boldsymbol{r}) = k) = \mathcal{N}(m_{j\,k}, \sigma^2_{j\,k}) \\
p(A_{ij}) = \mathcal{N}(A_{0\,ij}, \sigma^2_{0\,ij})
\end{cases}
$$



$\boldsymbol{f}$ $\boldsymbol{g}$ $\widehat{\boldsymbol{f}}$ $\widehat{\boldsymbol{z}}$

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Questions and Discussions