Inverse Problems: from Regularization to Bayesian inference solutions

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   - Gauss-Markov, GG-Marvov
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1. Direct and indirect observation

- Direct observation of a few quantities are possible: length, time, electrical charge, number of particles
- For many others, we only can measure them by transforming them.
  Example: Thermometer transforms variation of temeprature f to variation of length g.

- Relating measurable quantity g to the desired quantity f is called Forward modeling: $g = \mathcal{H}(f)$.
- Predicting the measurements g if we knew the desired quantity f and the measurement system is called Forward problem.
- Infering on the desired quantity f from the measurement g is called Inverse problem.

2. Inverse problems examples

- Example 1:
  Measuring variation of temperature with a therometer
  - $f(t)$ variation of temperature over time
  - $g(t)$ variation of length of the liquid in thermometer
- Example 2: Seeing outside of a body: Making an image using a camera, a microscope or a telescope
  - $f(x, y)$ real scene
  - $g(x, y)$ observed image
- Example 3: Seeing inside of a body: Computed Tomography usng X rays, US, Microwave, etc.
  - $f(x, y)$ a section of a real 3D body $f(x, y, z)$
  - $g_\phi(r)$ a line of observed radiographe $g_\phi(r, z)$

- Example 1: Deconvolution
- Example 2: Image restoration
- Example 3: Image reconstruction

Measuring variation of temperature with a therometer

- $f(t)$ variation of temperature over time
- $g(t)$ variation of length of the liquid in thermometer

- Forward model: Convolution

$$g(t) = \int f(t')\, h(t-t')\, \mathrm{d}t' + \epsilon(t)$$

$h(t)$: impulse response of the measurement system

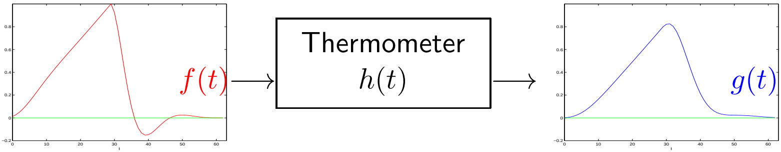
- Inverse problem: Deconvolution

Given the forward model $\mathcal{H}$ (impulse response $h(t)$))
and a set of data $g(t_i), i = 1, \cdots, M$
find $f(t)$

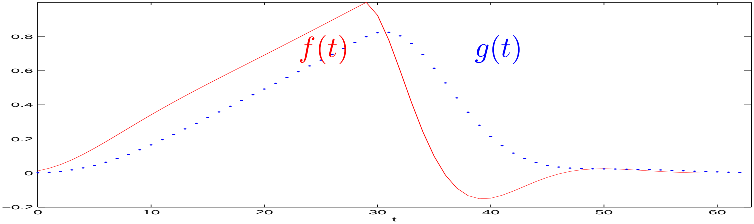
Measuring variation of temperature with a therometer

Forward model: Convolution

$$g(t) = \int f(t')\, h(t-t')\, \mathrm{d}t' + \epsilon(t)$$

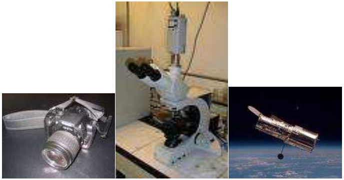


Inversion: Deconvolution



Seeing outside of a body: Making an image with a camera, a microscope or a telescope

- $f(x,y)$ real scene
- $g(x,y)$ observed image



- Forward model: Convolution

$$g(x,y) = \iint f(x',y')\, h(x-x',y-y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x,y)$$

$h(x,y)$: Point Spread Function (PSF) of the imaging system

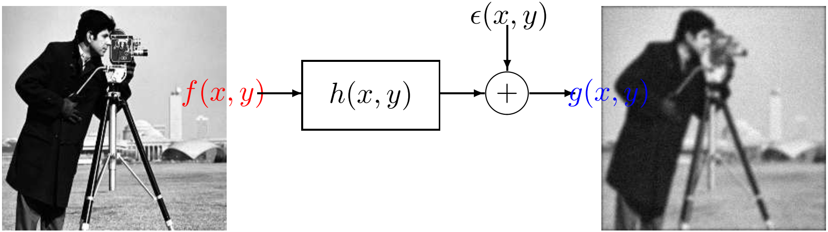
- Inverse problem: Image restoration

Given the forward model $\mathcal{H}$ (PSF $h(x,y)$))
and a set of data $g(x_i,y_i), i=1,\cdots,M$
find $f(x,y)$

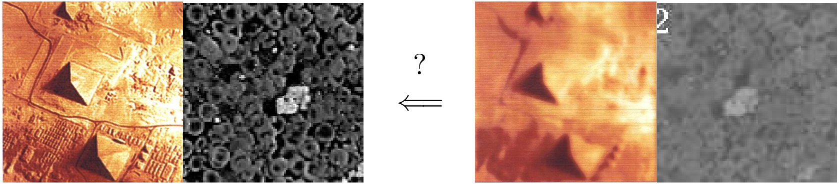
Making an image with an unfocused camera

Forward model: 2D Convolution

$$g(x,y) = \iint f(x',y')\, h(x-x',y-y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x,y)$$

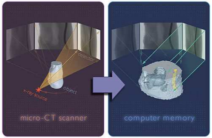


Inversion: Image Deconvolution or Restoration



Seeing inside of a body: Computed Tomography

- $f(x,y)$ a section of a real 3D body $f(x,y,z)$
- $g_\phi(r)$ a line of observed radiographe $g_\phi(r,z)$



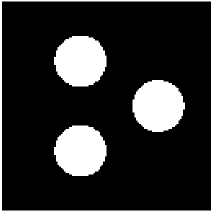
- Forward model:
  Line integrals or Radon Transform

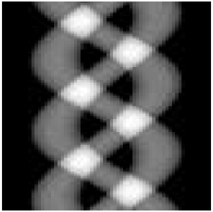
$$
\begin{aligned}
g_\phi(r) &= \int_{L_{r,\phi}} f(x,y)\, \mathrm{d}l + \epsilon_\phi(r) \\
&= \iint f(x,y)\, \delta(r - x\cos\phi - y\sin\phi)\, \mathrm{d}x\, \mathrm{d}y + \epsilon_\phi(r)
\end{aligned}
$$

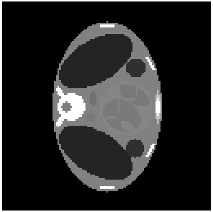
- Inverse problem: Image reconstruction

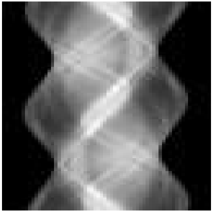
  Given the forward model $\mathcal{H}$ (Radon Transform) and
  a set of data $g_{\phi_i}(r), i = 1, \cdots, M$
  find $f(x,y)$

Computed Tomography: Radon Transform





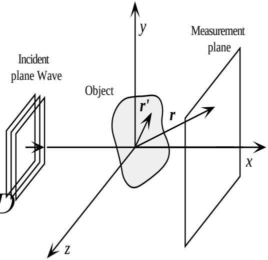




Forward:	$f(x,y)$	$\longrightarrow$	$g(r,\phi)$
Inverse:	$f(x,y)$	$\longleftarrow$	$g(r,\phi)$

Microwave or ultrasound imaging

Mesaurs: diffracted wave by the object $\phi_d(\boldsymbol{r}_i)$
Unknown quantity: $f(\boldsymbol{r}) = k_0^2(n^2(\boldsymbol{r}) - 1)$
Intermediate quantity : $\phi(\boldsymbol{r})$

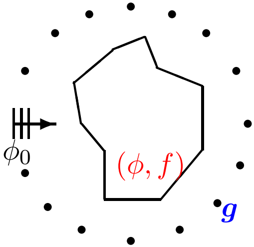


$$\phi_d(\boldsymbol{r}_i) = \iint_D G_m(\boldsymbol{r}_i, \boldsymbol{r}')\phi(\boldsymbol{r}')\, f(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}', \ \boldsymbol{r}_i \in S$$

$$\phi(\boldsymbol{r}) = \phi_0(\boldsymbol{r}) + \iint_D G_o(\boldsymbol{r}, \boldsymbol{r}')\phi(\boldsymbol{r}')\, f(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}', \ \boldsymbol{r} \in D$$

Born approximation $(\phi(\boldsymbol{r}') \simeq \phi_0(\boldsymbol{r}'))$ **):**

$$\phi_d(\boldsymbol{r}_i) = \iint_D G_m(\boldsymbol{r}_i, \boldsymbol{r}')\phi_0(\boldsymbol{r}')\, f(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}', \ \boldsymbol{r}_i \in S$$



Discretization :

$$\begin{cases} \boldsymbol{\phi}_d = \boldsymbol{G}_m \boldsymbol{F} \boldsymbol{\phi} \\ \boldsymbol{\phi} = \phi_0 + \boldsymbol{G}_o \boldsymbol{F} \boldsymbol{\phi} \end{cases} \longrightarrow \begin{cases} \boldsymbol{\phi}_d = \boldsymbol{H}(\boldsymbol{f}) \\ \text{with} \quad \boldsymbol{F} = \mathrm{diag}(\boldsymbol{f}) \\ \boldsymbol{H}(\boldsymbol{f}) = \boldsymbol{G}_m \boldsymbol{F}(\boldsymbol{I} - \boldsymbol{G}_o \boldsymbol{F})^{-1} \phi_0 \end{cases}$$

General formulation of inverse problems

- General non linear inverse problems:

$$g(\boldsymbol{s}) = [\mathcal{H} f(\boldsymbol{r})](\boldsymbol{s}) + \epsilon(\boldsymbol{s}), \quad \boldsymbol{r} \in \mathcal{R}, \quad \boldsymbol{s} \in \mathcal{S}$$

- Linear models:

$$g(\boldsymbol{s}) = \int f(\boldsymbol{r})\, h(\boldsymbol{r}, \boldsymbol{s})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s})$$

  If $h(\boldsymbol{r}, \boldsymbol{s}) = h(\boldsymbol{r} - \boldsymbol{s}) \longrightarrow$ Convolution.

- Discrete data:

$$g(\boldsymbol{s}_i) = \int h(\boldsymbol{s}_i, \boldsymbol{r})\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s}_i), \quad i = 1, \cdots, m$$

- Inversion: Given the forward model $\mathcal{H}$ and the data $\boldsymbol{g} = \{g(\boldsymbol{s}_i), i = 1, \cdots, m)\}$ estimate $f(\boldsymbol{r})$
- Well-posed and Ill-posed problems (Hadamard): existance, uniqueness and stability
- Need for prior information

3. Inverse problems: Analytical methods

Two communities working on Inverse problems:

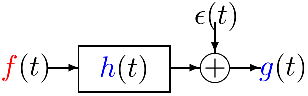
- Mathematical departments:
  Analytical methods: Existance and Uniqueness
  Differential equations, PDE
- Engineering and Computer sciences:
  Algebraic methods: Discretization, Uniqueness and Stability
  Integral equations, Discretization using Moments method, Galerkin, ...

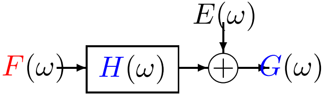
Two examples:

- Deconvolution: Inverse filtering and Wiener filtering
- X ray Computed Tomography: Radon transform:
  Direct Inversion or Filtered Backprojection methods

Deconvolution: Analytical methods

Time domain	Fourier domain
Forward model:	
$g(t) = h(t) * f(t) + \epsilon(t)$	$G(\omega) = H(\omega)\, F(\omega) + E(\omega)$





Deconvolution:

$$g(t) \rightarrow \boxed{w(t) = IFT\{\frac{1}{H(\omega)}\}} \rightarrow \widehat{f}(t)$$

Inverse filtering

$$G(\omega) \rightarrow \boxed{\frac{1}{H(\omega)}} \rightarrow \widehat{F}(\omega)$$

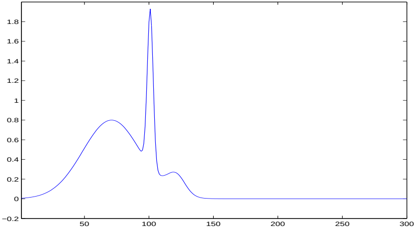
Deconvolution:

$$g(t) \rightarrow \boxed{W(\omega)} \rightarrow \widehat{f}(t)$$

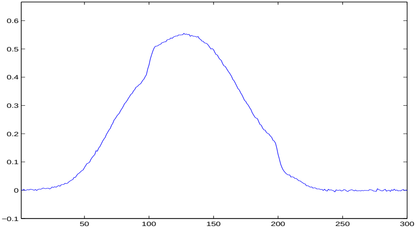
Wiener filtering

$$G(\omega) \rightarrow \boxed{\frac{H^*(\omega)}{|H(\omega)|^2 + \frac{S_\epsilon(\omega)}{S_f(\omega)}}} \rightarrow \widehat{F}(\omega)$$

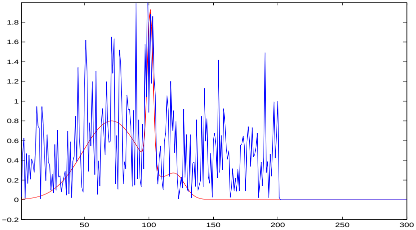
Deconvolution example



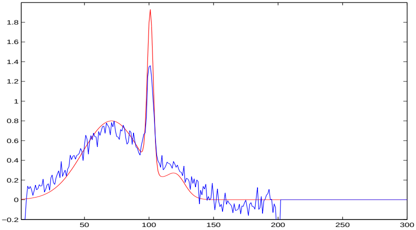
$f(t)$



$g(t)$

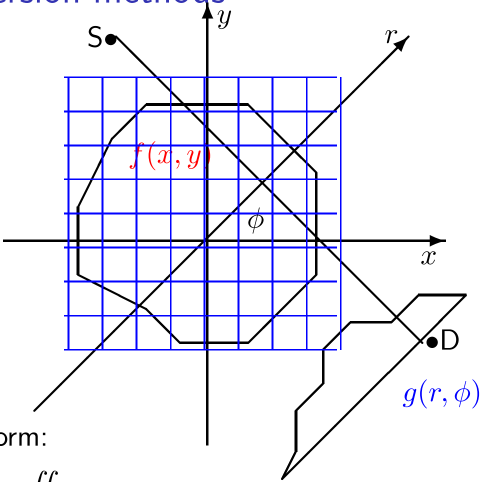


Inverse Filtering



Wiener Filtering

Analytical Inversion methods



Radon Transform:

$$g(r,\phi) = \iint_D f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right)\int_0^{\pi}\int_{-\infty}^{+\infty}\frac{\frac{\partial}{\partial r}g(r,\phi)}{(r - x\cos\phi - y\sin\phi)}\,\mathrm{d}r\,\mathrm{d}\phi$$

Filtered Backprojection method

$$\textcolor{red}{f(x,y)} = \left(-\frac{1}{2\pi^2}\right) \int_0^{\pi} \int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} \textcolor{blue}{g(r,\phi)}}{(r - x\cos\phi - y\sin\phi)} \, \mathrm{d}r \, \mathrm{d}\phi$$

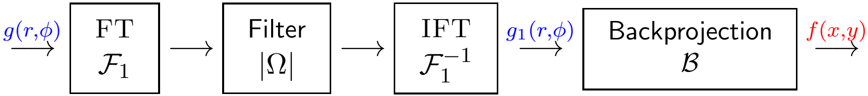
Derivation $\mathcal{D}$: $\quad \overline{g}(r,\phi) = \frac{\partial g(r,\phi)}{\partial r}$

Hilbert Transform $\mathcal{H}$: $\quad g_1(r',\phi) = \frac{1}{\pi} \int_0^{\infty} \frac{\overline{g}(r,\phi)}{(r - r')} \, \mathrm{d}r$

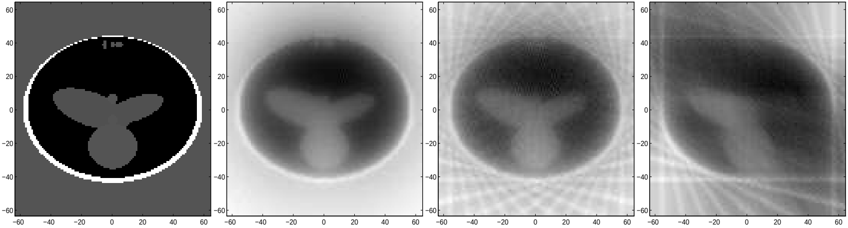
Backprojection $\mathcal{B}$: $\quad f(x,y) = \frac{1}{2\pi} \int_0^{\pi} g_1(r' = x\cos\phi + y\sin\phi, \phi) \, \mathrm{d}\phi$

$$\textcolor{red}{f(x,y)} = \mathcal{B}\,\mathcal{H}\,\mathcal{D}\,\textcolor{blue}{g(r,\phi)} = \mathcal{B}\,\mathcal{F}_1^{-1}\,|\Omega|\,\mathcal{F}_1\,\textcolor{blue}{g(r,\phi)}$$

- Backprojection of filtered projections:



Limitations : Limited angle or noisy data



Original — 64 proj. — 16 proj. — 8 proj. $[0, \pi/2]$

- Limited angle or noisy data
- Accounting for detector size
- Other measurement geometries: fan beam, ...

4. Inverse problems: Discretization and Algebraic methods

$$g(\boldsymbol{s}_i) = \int h(\boldsymbol{s}_i, \boldsymbol{r})\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s}_i), \quad i = 1, \cdots, M$$

- $f(\boldsymbol{r})$ is assumed to be well approximated by

$$f(\boldsymbol{r}) \simeq \sum_{j=1}^{N} f_j\, b_j(\boldsymbol{r})$$

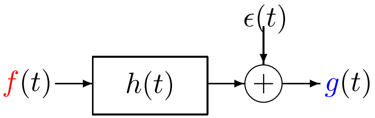
with $\{b_j(\boldsymbol{r})\}$ a basis or any other set of known functions

$$g(\boldsymbol{s}_i) = g_i \simeq \sum_{j=1}^{N} f_j \int h(\boldsymbol{s}_i, \boldsymbol{r})\, b_j(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}, \quad i = 1, \cdots, M$$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \;\text{ with }\; H_{ij} = \int h(\boldsymbol{s}_i, \boldsymbol{r})\, b_j(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}$$

- $\boldsymbol{H}$ is huge dimensional
- LS solution : $\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{Q(\boldsymbol{f})\}$ with
  $Q(\boldsymbol{f}) = \sum_i |g_i - [\boldsymbol{H}\boldsymbol{f}]_i|^2 = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2$
  does not give satisfactory result.

Convolution: Discretization



$$g(t) = \int f(t')\, h(t-t')\, \mathrm{d}t' + \epsilon(t) = \int h(t')\, f(t-t')\, \mathrm{d}t' + \epsilon(t)$$

- The signals $f(t)$, $g(t)$, $h(t)$ are discretized with the same sampling period $\Delta T = 1$,
- The impulse response is finite (FIR) : $h(t) = 0$, for t such that $t < -q\Delta T$ or $\forall t > p\Delta T$.

$$g(m) = \sum_{k=-q}^{p} h(k)\, f(m-k) + \epsilon(m), \quad m = 0, \cdots, M$$

$$\boldsymbol{g} = \boldsymbol{H}\, \boldsymbol{f} + \boldsymbol{\epsilon}$$

Convolution: Discretized matrix vector forms

$$
\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix}
=
\begin{bmatrix}
h(p) & \cdots & h(0) & \cdots & h(-q) & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & & \ddots & & \ddots & & & & \vdots \\
\vdots & & & & & & & & & \vdots \\
\vdots & & h(p) & \cdots & h(0) & \cdots & h(-q) & & & \vdots \\
\vdots & & & & & & & & & \vdots \\
\vdots & & & \ddots & & & \ddots & & & \vdots \\
\vdots & & & & \ddots & & & & & 0 \\
0 & \cdots & \cdots & & 0 & h(p) & \cdots & h(0) & \cdots & h(-q)
\end{bmatrix}
\begin{bmatrix} f(-p) \\ \vdots \\ f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f(M) \\ f(M+1) \\ \vdots \\ f(M+q) \end{bmatrix}
$$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- $\boldsymbol{g}$ is a $(M+1)$-dimensional vector,
- $\boldsymbol{f}$ has dimension $M+p+q+1$,
- $\boldsymbol{h} = [h(p), \cdots, h(0), \cdots, h(-q)]$ has dimension $(p+q+1)$
- $\boldsymbol{H}$ has dimensions $(M+1) \times (M+p+q+1)$.

Convolution: Discretized matrix vector form

- If system is causal ($q=0$) we obtain

$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix} = \begin{bmatrix} h(p) & \cdots & h(0) & 0 & \cdots & \cdots & 0 \\ 0 & & & & & & \vdots \\ \vdots & & & & & & \vdots \\ \vdots & & h(p) & \cdots & h(0) & & \vdots \\ \vdots & & & & & & \vdots \\ \vdots & & & & & & 0 \\ 0 & \cdots & \cdots & 0 & h(p) & \cdots & h(0) \end{bmatrix} \begin{bmatrix} f(-p) \\ \vdots \\ f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f(M) \end{bmatrix}$$

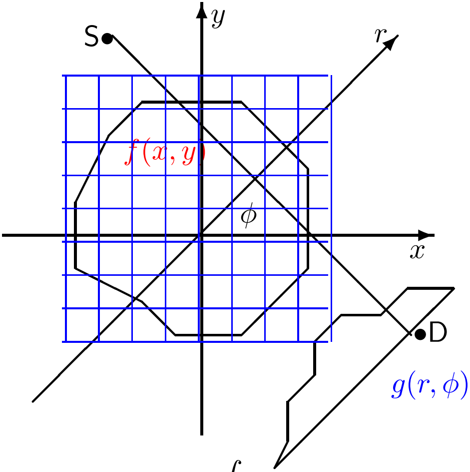
- $\boldsymbol{g}$ is a $(M+1)$-dimensional vector,
- $\boldsymbol{f}$ has dimension $M+p+1$,
- $\boldsymbol{h}=[h(p),\cdots,h(0)]$ has dimension $(p+1)$
- $\boldsymbol{H}$ has dimensions $(M+1)\times(M+p+1)$.

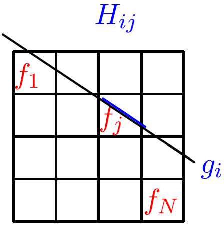
Convolution: Causal system and causal input

$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix} = \begin{bmatrix} h(0) & & & & & \\ h(1) & \ddots & & & & \\ \vdots & & & & & \\ h(p) & \cdots & & h(0) & & \\ 0 & \ddots & & & \ddots & \\ \vdots & & & & & \\ 0 & \cdots & 0 & h(p) & \cdots & h(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f(M) \end{bmatrix}$$

- $\boldsymbol{g}$ is a $(M+1)$-dimensional vector,
- $\boldsymbol{f}$ has dimension $M+1$,
- $\boldsymbol{h} = [h(p), \cdots, h(0)]$ has dimension $(p+1)$
- $\boldsymbol{H}$ has dimensions $(M+1) \times (M+1)$.

Discretization of Radon Transfrom in CT





$$f(x,y) = \sum_j f_j \, b_j(x,y)$$

$$b_j(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \text{ pixel } j \\ 0 & \text{else} \end{cases}$$

$$g(r,\phi) = \int_L f(x,y) \, \mathrm{d}l$$

$$g_i = \sum_{j=1}^{N} H_{ij} \, f_j + \epsilon_i$$

$$\boldsymbol{g} = \boldsymbol{H} \boldsymbol{f} + \boldsymbol{\epsilon}$$

Deterministic methods

$$\boldsymbol{g} = \boldsymbol{H}\,\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Backprojection: $\widehat{\boldsymbol{f}} = \boldsymbol{H}'\boldsymbol{g}$, but $\boldsymbol{H}'\boldsymbol{H} \neq \boldsymbol{I}$
- Minimum norm solution of $\boldsymbol{H}\boldsymbol{f} = \boldsymbol{g}$:

  $$\text{minimize } \|\boldsymbol{f}\|^2 \text{ s.t. } \boldsymbol{H}\boldsymbol{f} = \boldsymbol{g} \longrightarrow \widehat{\boldsymbol{f}} = \boldsymbol{H}'(\boldsymbol{H}\boldsymbol{H}')^{-1}\boldsymbol{g}$$

  (Filtered Backprojection)
- Least Squares:

  $$\text{minimize } \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 \longrightarrow \widehat{\boldsymbol{f}} = (\boldsymbol{H}'\boldsymbol{H})^{-1}\boldsymbol{H}'\boldsymbol{g}$$

  (Backprojection followed by image deconvolution)
- Constraint optimization:

  $$\text{minimize } \|\boldsymbol{g}-\boldsymbol{H}\boldsymbol{f}\|^2 \text{ s.t. } \|\boldsymbol{f}\|^2 < c \longrightarrow \widehat{\boldsymbol{f}} = (\boldsymbol{H}'\boldsymbol{H}+\lambda\boldsymbol{I})^{-1}\boldsymbol{H}'\boldsymbol{g}$$

  $$\text{minimize } \|\boldsymbol{f}\|^2 \text{ s.t. } \|\boldsymbol{g}-\boldsymbol{H}\boldsymbol{f}\|^2 < c \longrightarrow \widehat{\boldsymbol{f}} = (\mu\boldsymbol{H}'\boldsymbol{H}+\boldsymbol{I})^{-1}\boldsymbol{H}'\boldsymbol{g}$$
- Minimum norme LS (MNLS):

  $$J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda||\boldsymbol{f}||^2$$

Deterministic methods: General Data matching methods

- Observation model: $\boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) + \boldsymbol{\epsilon}$
  $g_i = h_i(\boldsymbol{f}) + \epsilon_i, \quad i = 1, \ldots, M$
- Misatch between data and output of the model $\Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f}))$

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{\Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f}))\}$$

- Examples:

– LS $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \|\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})\|^2 = \sum_i |g_i - h_i(\boldsymbol{f})|^2$

– L_p $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \|\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})\|^p = \sum_i |g_i - h_i(\boldsymbol{f})|^p, \quad 1 < p < 2$

– KL $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \sum_i g_i \ln \frac{g_i}{h_i(\boldsymbol{f})}$

- In general, does not give satisfactory results for inverse problems.

5. Inverse problems: Regularization theory

Inverse problems = Ill posed problems

$\longrightarrow$ Need for prior information

- Functional space (Tikhonov):

$$g = \mathcal{H}(f) + \epsilon \longrightarrow J(f) = ||g - \mathcal{H}(f)||_2^2 + \lambda||\mathcal{D}f||_2^2$$

$$\widehat{f}(t) = (\mathcal{H}^*\mathcal{H} + \lambda\mathcal{D}^*\mathcal{D})^{-1}\mathcal{H}g(t)$$

- Finite dimensional space (Philips & Towmey):

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \longrightarrow J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda||\boldsymbol{D}\boldsymbol{f}||^2$$

$$\widehat{\boldsymbol{f}} = (\boldsymbol{H}^*\boldsymbol{H} + \lambda\boldsymbol{D}^*\boldsymbol{D})^{-1}\boldsymbol{H}\boldsymbol{g}$$

- Minimum norme LS (MNLS):

$$J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda||\boldsymbol{f}||^2 \longrightarrow \widehat{\boldsymbol{f}} = (\boldsymbol{H}^*\boldsymbol{H} + \lambda\boldsymbol{I})^{-1}\boldsymbol{H}\boldsymbol{g}$$

- More general regularization:

$$J(\boldsymbol{f}) = \mathcal{Q}(\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})) + \lambda\Omega(\boldsymbol{D}\boldsymbol{f})$$

or

$$J(\boldsymbol{f}) = \Delta_1(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) + \lambda\Delta_2(\boldsymbol{f}, \boldsymbol{f}_0)$$

Algebraic Regularization methods

$$J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda\Omega(\boldsymbol{f})$$

- Classical quadratic regularization:

$$\Omega(\boldsymbol{f}) = ||\boldsymbol{D}\boldsymbol{f}||^2 = \sum_j |f_j - f_{j-1}|^2$$

- Sparsity enforcing regularization:

$$\Omega(\boldsymbol{f}) = ||\boldsymbol{f}||_1 = \sum_j |f_j|$$

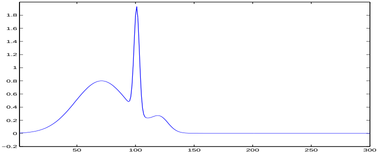
- Total Variation regularization:

$$\Omega(\boldsymbol{f}) = ||\boldsymbol{D}\boldsymbol{f}||_1 = \sum_j |f_j - f_{j-1}|$$

- Combined regularization:

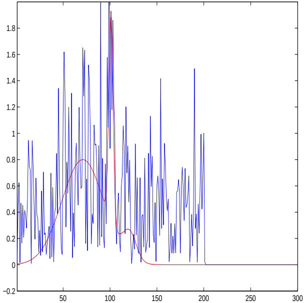
$$\Omega(\boldsymbol{f}) = \sum_j \lambda_1 |f_j|^2 + \lambda_1 |f_j - f_{j-1}|$$

Deconvolution example

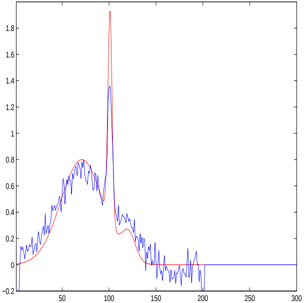


$f(t)$

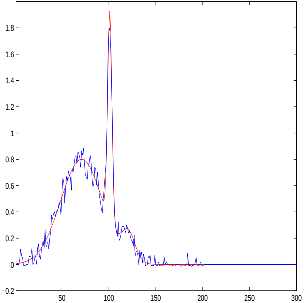
$g(t) = h(t) * f(t) + \epsilon(t)$



Inverse Filtering



Wiener



Regularization

6. Limitations of deterministic regularization methods

General regularization methods:

$$J(\boldsymbol{f}) = \mathcal{Q}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}) + \lambda\Omega(\boldsymbol{D}\boldsymbol{f})$$

or

$$J(\boldsymbol{f}) = \Delta_1(\boldsymbol{g}, \boldsymbol{H}\boldsymbol{f}) + \lambda\Delta_2(\boldsymbol{D}\boldsymbol{f}, \boldsymbol{D}\boldsymbol{f}_0)$$

Limitations:

- Errors are considered implicitly white and Gaussian
- Limited prior information on the solution
- Lack of tools for the determination of the regularization parameters, even if there are a few methods: L curve, Cross Validation
- Lack of tools for handeling uncertainties

$\Downarrow$

Probabilistic Methods and Bayesian inference

7. Probabilistic methods for inversion

Taking account of errors and uncertainties $\longrightarrow$ Probability theory

- Maximum Likelihood (ML)
- Minimum Inaccuracy (MI)
- Probability Distribution Matching (PDM)
- Maximum Entropy (ME) and Information Theory (IT)
- Bayesian Inference (BAYES)

Advantages:

- Explicit account of the errors and noise
- A large class of priors via explicit or implicit modeling
- A coherent approach to combine information content of the data and priors

Limitations:

- Practical implementation and cost of calculation

8. Bayesian inference for inverse problems

$$\mathcal{M}: \qquad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Observation model $\mathcal{M}$ + Hypothesis on the noise $\boldsymbol{\epsilon} \longrightarrow$
  $$p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M}) = p_{\boldsymbol{\epsilon}}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f})$$
- A priori information $\quad p(\boldsymbol{f}|\mathcal{M})$
- Bayes : $$p(\boldsymbol{f}|\boldsymbol{g};\mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M})\, p(\boldsymbol{f}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$$

Link with regularization :

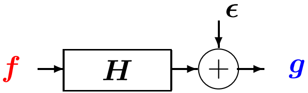
Maximum A Posteriori (MAP) :

$$\begin{aligned}
\widehat{\boldsymbol{f}} &= \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{g}|\boldsymbol{f})\; p(\boldsymbol{f})\} \\
&= \arg\min_{\boldsymbol{f}} \{-\ln p(\boldsymbol{g}|\boldsymbol{f}) - \ln p(\boldsymbol{f})\}
\end{aligned}$$

with $\quad Q(\boldsymbol{g}, \boldsymbol{H}\boldsymbol{f}) = -\ln p(\boldsymbol{g}|\boldsymbol{f}) \quad$ and $\quad \lambda\Omega(\boldsymbol{f}) = -\ln p(\boldsymbol{f})$

Bayesian inference for inverse problems

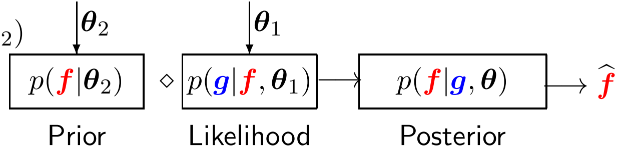
- Linear Inverse problems: $\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$



- Bayesian inference:

$$p(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{\theta}) = \frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)}{p(\boldsymbol{g}|\boldsymbol{\theta})}$$

with $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$



- Point estimators:
  - Maximum A Posteriori (MAP): $\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{\theta})\}$
  - Posterior Mean (PM): $\widehat{\boldsymbol{f}} = \mathsf{E}_{p(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{\theta})}\{\boldsymbol{f}\} = \int \boldsymbol{f}\, p(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{f}$

Case of linear models and Gaussian priors

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Hypothesis on the noise: $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma_\epsilon^2 \boldsymbol{I})$

$$p(\boldsymbol{g}|\boldsymbol{f}) \propto \exp\left[-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2\right]$$

- Hypothesis on $\boldsymbol{f}$: $\boldsymbol{f} \sim \mathcal{N}(0, \sigma_f^2 \boldsymbol{I})$

$$p(\boldsymbol{f}) \propto \exp\left[-\frac{1}{2\sigma_f^2}\|\boldsymbol{f}\|^2\right]$$

- A posteriori:

$$p(\boldsymbol{f}|\boldsymbol{g}) \propto \exp\left[-\frac{1}{2\sigma_\epsilon^2}\left[\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \frac{\sigma_\epsilon^2}{\sigma_f^2}\|\boldsymbol{f}\|^2\right]\right]$$

- MAP : $\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$

  with $J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{f}\|^2, \qquad \lambda = \frac{\sigma_\epsilon^2}{\sigma_f^2}$

- Advantage : characterization of the solution

$\boldsymbol{f}|\boldsymbol{g} \sim \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{P}})$ with $\widehat{\boldsymbol{f}} = \left(\boldsymbol{H}^t\boldsymbol{H} + \lambda\boldsymbol{I}\right)^{-1}\boldsymbol{H}^t\boldsymbol{g}$, $\widehat{\boldsymbol{P}} = \sigma_\epsilon^2\left(\boldsymbol{H}^t\boldsymbol{H} + \lambda\boldsymbol{I}\right)^{-1}$

MAP estimation with other priors:

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{with} \quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\Omega(\boldsymbol{f})$$

Separable priors:

- Gaussian:
  $p(f_j) \propto \exp\left[-\alpha|f_j|^2\right] \longrightarrow \Omega(\boldsymbol{f}) = \|\boldsymbol{f}\|^2 = \alpha\sum_j |f_j|^2$
- Gamma: $p(f_j) \propto f_j^\alpha \exp\left[-\beta f_j\right] \longrightarrow \Omega(\boldsymbol{f}) = \alpha\sum_j \ln f_j + \beta f_j$
- Beta:
  $p(f_j) \propto f_j^\alpha (1 - f_j)^\beta \longrightarrow \Omega(\boldsymbol{f}) = \alpha\sum_j \ln f_j + \beta\sum_j \ln(1 - f_j)$
- Generalized Gaussian: $p(f_j) \propto \exp\left[-\alpha|f_j|^p\right], \quad 1 < p < 2 \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha\sum_j |f_j|^p,$

Markovian models:

$$p(f_j|\boldsymbol{f}) \propto \exp\left[-\alpha\sum_{i\in N_j} \phi(f_j, f_i)\right] \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha\sum_j \sum_{i\in N_j} \phi(f_j, f_i),$$

Main advantages of the Bayesian approach

- MAP = Regularization
- Posterior mean ? Marginal MAP ?
- More information in the posterior law than only its mode or its mean
- Meaning and tools for estimating hyper parameters
- Meaning and tools for model selection
- More specific and specialized priors, particularly through the hidden variables
- More computational tools:
  - Expectation-Maximization for computing the maximum likelihood parameters
  - MCMC for posterior exploration
  - Variational Bayes for analytical computation of the posterior marginals
  - ...

MAP estimation and Compressed Sensing

$$\begin{cases} \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \\ \boldsymbol{f} = \boldsymbol{W}\boldsymbol{z} \end{cases}$$

- $\boldsymbol{W}$ a code book matrix, $\boldsymbol{z}$ coefficients
- Gaussian:

$$p(\boldsymbol{z}) = \mathcal{N}(0, \sigma_z^2 \boldsymbol{I}) \propto \exp\left[-\frac{1}{2\sigma_z^2}\sum_j |z_j|^2\right]$$
$$J(\boldsymbol{z}) = -\ln p(\boldsymbol{z}|\boldsymbol{g}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{W}\boldsymbol{z}\|^2 + \lambda\sum_j |z_j|^2$$

- Generalized Gaussian (sparsity, $\beta = 1$):

$$p(\boldsymbol{z}) \propto \exp\left[-\lambda\sum_j |z_j|^\beta\right]$$
$$J(\boldsymbol{z}) = -\ln p(\boldsymbol{z}|\boldsymbol{g}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{W}\boldsymbol{z}\|^2 + \lambda\sum_j |z_j|^\beta$$

- $\boldsymbol{z} = \arg\min_{\boldsymbol{z}} \{J(\boldsymbol{z})\} \longrightarrow \widehat{\boldsymbol{f}} = \boldsymbol{W}\widehat{\boldsymbol{z}}$

Bayesian Estimation: Two simple priors

- Example 1: Linear Gaussian case:

$$\left\{\begin{array}{l} p(\boldsymbol{g}|\boldsymbol{f},\theta_1)=\mathcal{N}(\boldsymbol{H}\boldsymbol{f},\theta_1\boldsymbol{I}) \\ p(\boldsymbol{f}|\theta_2)=\mathcal{N}(0,\theta_2\boldsymbol{I}) \end{array}\right. \longrightarrow p(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{\theta})=\mathcal{N}(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{P}})$$

with

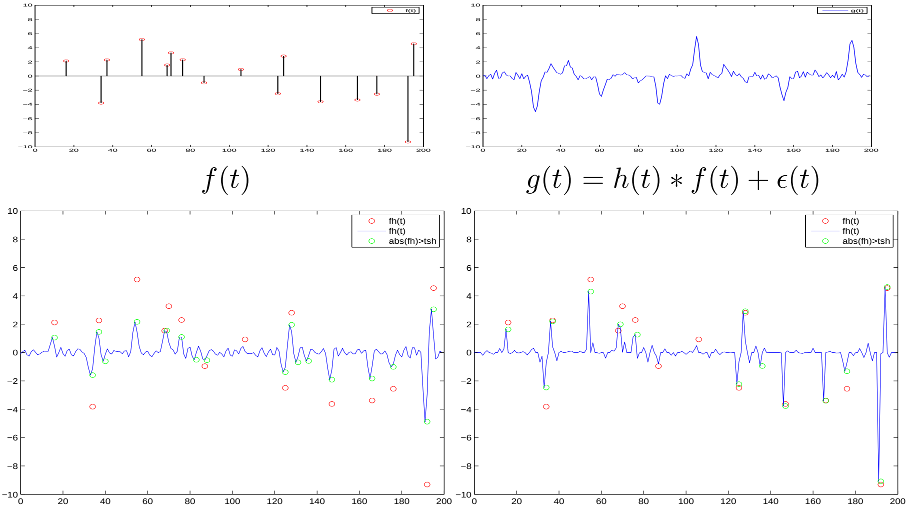
$$\left\{\begin{array}{l} \widehat{\boldsymbol{f}}=(\boldsymbol{H}'\boldsymbol{H}+\lambda\boldsymbol{I})^{-1}\boldsymbol{H}'\boldsymbol{g} \\ \widehat{\boldsymbol{P}}=\theta_1(\boldsymbol{H}'\boldsymbol{H}+\lambda\boldsymbol{I})^{-1}, \quad \lambda=\frac{\theta_1}{\theta_2} \end{array}\right.$$

$$\widehat{\boldsymbol{f}}=\arg\min_{\boldsymbol{f}}\{J(\boldsymbol{f})\} \text{ with } J(\boldsymbol{f})=\|\boldsymbol{g}-\boldsymbol{H}\boldsymbol{f}\|_2^2+\lambda\|\boldsymbol{f}\|_2^2$$

- Example 2: Double Exponential prior & MAP:

$$\widehat{\boldsymbol{f}}=\arg\min_{\boldsymbol{f}}\{J(\boldsymbol{f})\} \text{ with } J(\boldsymbol{f})=\|\boldsymbol{g}-\boldsymbol{H}\boldsymbol{f}\|_2^2+\lambda\|\boldsymbol{f}\|_1$$

Deconvolution example



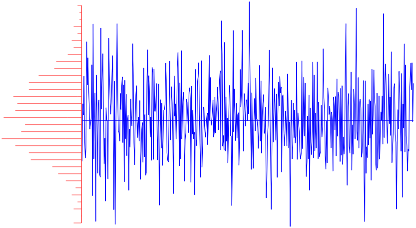
$f(t)$ $\quad$ $g(t) = h(t) * f(t) + \epsilon(t)$

Quadratic Reg. (Gaussian) $\quad$ L_1 Reg. (Laplace)

Two main steps in the Bayesian approach

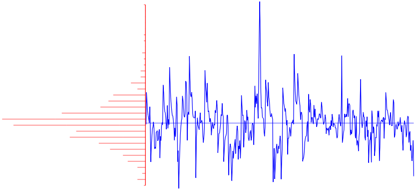
- Prior modeling
  - Separable:
    Gaussian, Gamma,
    Sparsity enforcing: Generalized Gaussian, mixture of Gaussians, mixture of Gammas, ...
  - Markovian:
    Gauss-Markov, GGM, ...
  - Markovian with hidden variables
    (contours, region labels)
- Choice of the estimator and computational aspects
  - MAP, Posterior mean, Marginal MAP
  - MAP needs optimization algorithms
  - Posterior mean needs integration methods
  - Marginal MAP and Hyperparameter estimation need integration and optimization
  - Approximations:
    - Gaussian approximation (Laplace)
    - Numerical exploration MCMC
    - Variational Bayes (Separable approximation)

9. Prior modeling of signals



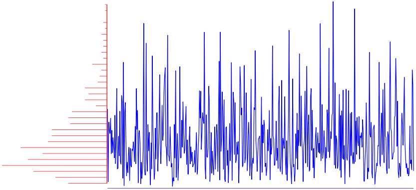
Gaussian

$$p(f_j) \propto \exp\left[-\alpha |f_j|^2\right]$$



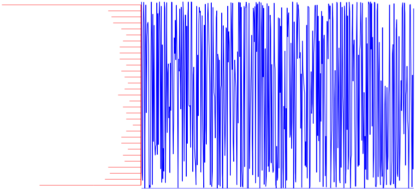
Generalized Gaussian

$$p(f_j) \propto \exp\left[-\alpha |f_j|^p\right], \quad 1 \leq p \leq 2$$



Gamma

$$p(f_j) \propto f_j^{\alpha} \exp\left[-\beta f_j\right]$$

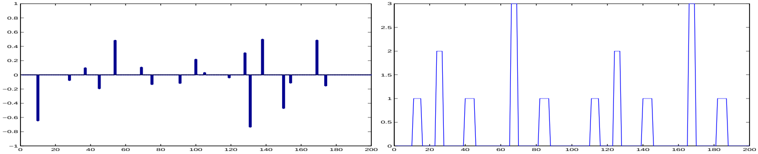


Beta

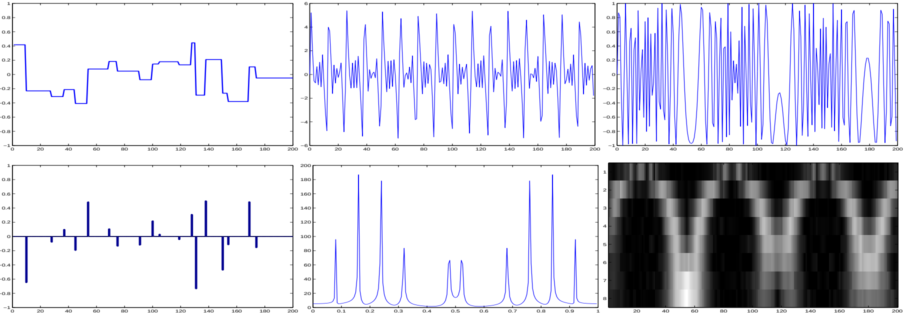
$$p(f_j) \propto f_j^{\alpha} (1 - f_j)^{\beta}$$

Sparsity enforcing prior models

- Sparse signals: Direct sparsity



- Sparse signals: Sparsity in a Transform domaine



Sparsity enforcing prior models

- Simple heavy tailed models:
  - Generalized Gaussian, Double Exponential
  - Student-t, Cauchy

  - Elastic net
  - Symmetric Weibull, Symmetric Rayleigh
  - Generalized hyperbolic

- Hierarchical mixture models:
  - Mixture of Gaussians
  - Bernoulli-Gaussian

  - Mixture of Gammas
  - Bernoulli-Gamma
  - Mixture of Dirichlet
  - Bernoulli-Multinomial

Simple heavy tailed models

- Generalized Gaussian, Double Exponential

$$p(\boldsymbol{f}|\gamma,\beta) = \prod_j \mathcal{GG}(f_j|\gamma,\beta) \propto \exp\left[-\gamma \sum_j |f_j|^\beta\right]$$

$\beta = 1$ Double exponential or Laplace.
$0 < \beta \leq 1$ are of great interest for sparsity enforcing.

- Student-t and Cauchy models

$$p(\boldsymbol{f}|\nu) = \prod_j \mathcal{S}t(f_j|\nu) \propto \exp\left[-\frac{\nu+1}{2} \sum_j \log\left(1 + f_j^2/\nu\right)\right]$$

Cauchy model is obtained when $\nu = 1$.

Mixture models

- Mixture of two Gaussians (MoG2) model

$$p(\boldsymbol{f}|\lambda, v_1, v_0) = \prod_j \left(\lambda \mathcal{N}(f_j|0, v_1) + (1-\lambda)\mathcal{N}(f_j|0, v_0)\right)$$

- Bernoulli-Gaussian (BG) model

$$p(\boldsymbol{f}|\lambda, v) = \prod_j p(f_j) = \prod_j \left(\lambda \mathcal{N}(f_j|0, v) + (1-\lambda)\delta(f_j)\right)$$

- Mixture of Gammas

$$p(\boldsymbol{f}|\lambda, v_1, v_0) = \prod_j \left(\lambda \mathcal{G}(f_j|\alpha_1, \beta_1) + (1-\lambda)\mathcal{G}(f_j|\alpha_2, \beta_2)\right)$$

- Bernoulli-Gamma model

$$p(\boldsymbol{f}|\lambda, \alpha, \beta) = \prod_j \left[\lambda \mathcal{G}(f_j|\alpha, \beta) + (1-\lambda)\delta(f_j)\right]$$

10. Full Bayesian: Estimation of Hyper Parameters

- Inverse problems: $\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$
- Posterior law:

$$p(\boldsymbol{f}|\boldsymbol{\theta}, \boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)$$

- Examples:
  Gaussian noise, Gaussian prior and MAP:

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \text{ with } J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|_2^2 + \lambda \|\boldsymbol{f}\|_2^2$$

  Gaussian noise, Double Exponential prior and MAP:

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \text{ with } J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|_2^2 + \lambda \|\boldsymbol{f}\|_1$$

- Full Bayesian: Joint Posterior:

$$p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)\, p(\boldsymbol{\theta})$$

- Joint MAP:

$$(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f}, \boldsymbol{\theta})} \{p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g})\}$$

Main conclusions

- Determinsitic Regularization methods have been very successful.
- Bayesian approach is an appropriate tool with much greater possibilities
- More details tomorrow with advanced Bayesian approach