Inverse problems in imaging science: from classical regularization methods to state of the art Bayesian methods

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***Abstract*—Inverses problems arise in almost all the engineering and applied sciences where we have indirect measurement. Many classical signal and image processing research subjects are directly expressed as inverse problems: signal deconvolution, image restoration, image reconstruction in many imaging systems such as X ray Tomography, Microwave and Ultrasound imaging, Synthetic aperture radar (SAR), etc. In this tutorial, first we express in a unifying approach all these applications in a common mathematical framework. Then, mentioning the ill-posed nature of these inverse problems, we describe the regularization methods which were very successful during 1960-2000. Mentioning the limitations of these methods, we see how the Bayesian approach can give tools to go beyond these difficulties. In particular, we will see how this approach can be useful to account for many different a priori knowledges: smoothness, positivity, piecewise continuity, sparsity, finite number of materials (compact homogeneous regions), etc. We also discuss the computational aspects of the Bayesian approach and the practical implementations of the proposed algorithms.**

Keywords*—Inverse problems, Deconvolution, Image restoration, Computed Tomography, Regularization, Bayesian approach, Prior models, sparsity, Markov models, Hierarchical models, MCMC, Variational Bayesian Approximation.

I. Introduction

Inverse problems arise in many imaging systems such as: i) medical imaging such as X-ray Compted Tomography (CT), Untrasound and microwave tomography; ii) Non Destructive Evalaution (NDE) in industrial imaging such as gammagraphy, ultrasound or Eddy current tomography. The main idea in all these imaging systems is to relate an internal property of the object under the test (human body or an industrial object) to the observed data via a forward model. The objective of the inverse problem is then to reconstruct an image from the observed data [1]. In general the inverse problems are ill-posed in the sense defined by Hadamard: A problem is said to be well-posed if the solution exists, is unique and stable. When one of these conditions is not satisfied, the problem is said ill-posed.

II. Regularization methods

Noting the ill-posedness of the inverse problems, many Regularization methods have been proposed and applied successfully. One of the main regularization methods is defining the solution as the minimizer of a criterion which has two parts: a data-model matching part and a regularizer or a priori part. The data-model matching part is in general a distance measure between the observed data and the output of the forward model and the the regularization part is often a smoothness measure or a distance measure of the desired solution to a prior one [2]. Recently, sparsity measures have also had great success in many applications [3].

However, deterministic regularization methods have a few limitations: the selection of the criteria or distances of the both data-model matching and a priori and the determination of the regularization parameter. Another main limitation is that in these methods only a solution is computed without having any tool for quantifying the uncertainty of the proposed solution.

To push farther these limitations, the Bayesian inference has become the main approach. The main idea is first to use the forward model and some knowledge about the errors (modelling and the measurement noise) to define what is called the likelihood which gives the data-model matching part of the regularization methods. The next step is to translate the prior knowledge or the desired property of the solution into a prior probability law which gives the regularizing part of the regularization methods. The third step is to use the Bayes or Laplace rule to combine the likelihood and the prior to obtain the posterior probability law of the unknown quantity from which we can deduce any information about the solution. The classical point estimators are the Maximum A Posteriori (MAP) and the Expected A Posteriori (EAP). There is a very direct link between the MAP solution and the regularization methods: In both methods, the solution is obtained via the optimization of a two part criterion.

III. Basics of the Bayesian approach

Between the advantages of the Bayesian approach to deterministic regularization methods, we may mention: more tools for and probabilistic arguments for selection the likelihood part and the a priori part; more tools for interpreting the proposed solution and in particular a natural way to quantify the uncertainty of the solution via the a posteriori law; more tools for the determination of the regularization parameter and much more. In particular, we may mention the possibility of proposing unsupervised methods and effective solutions in many real applications.

The Bayesian approach with simple prior models such as Gaussian or Gauss-Markov (to account for smoothness for example), Gamma (to account for positivity), Laplace, Generalized Gaussian, Student-t or Cauchy (to account for sparsity) have been proposed and used in many applications successfully. Nowadays, the state of the art Bayesian methods use more sophisticated hierarchical models such as mixture models or Gauss-Markov-Potts models in different applications of imaging systems with great success. These methods push much farther the limitations of the regularization methods and very often there are no more equivalence between them. However, Bayesian computation, excepted for the linear and Gaussian models, still is too costly. For non linear or non-Gaussian models, there are three main computational methods: the Laplace or Gaussian approximation, the classical and more general Markov Chain Monte Carlo (MCMC) methods and the Bayesian Variational Approximation (BVA) methods which have recently became a standard.

IV. State of the art Bayesian methods

This tutorial gives an overview of these methods. The Target audience of this tutorial is all those who have been face to inverse problems and have heard about the Bayesian inference and Bayesian estimation will be interested to this topic. They will learn about the state of the art prior modelling and the corresponding Bayesian computation algorithms.

First many examples of inverse problems are exposed. In fact, many classical signal and image processing subjects are directly expressed as inverse problems: signal deconvolution, image restoration, image reconstruction in many imaging systems such as X ray Tomography, Microwave, Ultrasound, Synthetic aperture radar (SAR), etc. Many other data, signal and image processing subjects which are not directly expressed as such, can also be presented as inverse problems. Just to mention a few: Factor analysis, Blind sources separation, Antenna array processing, even classification and clustering, etc.

Then, the basics of regularization methods, their associated algorithms and their limitations are exposed and the need to push further these limitations brings us to the Bayesian approach. The main step in the Bayesian approach is the prior modelling. Simple prior laws (Gaussian, Generalized Gaussian, Gauss-Markov and more general Markovian priors) are nowadays commonly used. But, we need still more appropriate prior models which can account for the presence of the contours and homogeneous regions. Recently, we proposed a family of hierarchical prior models, called Gauss-Markov-Potts, which seems to be more appropriate for many applications in Imaging systems such as X ray Computed Tomography (CT) or Microwave imaging in Non Destructive Testing (NDT). Finally, the usefulness of these prior models and appropriate corresponding Bayesian computation in many practical CT or other imaging systems in 1D, 2D and 3D cases will be shown.

V. Conclusion

In this tutorial, first a unifying presentation of the inverse problems which commonly arise in signal and image processing and imaging systems is presented. Then, the deterministic regularization methods and their limitations are presented. The main part of this tutorial is the presentation of the Bayesian approach with different prior models to account for different prior knowledge or desired property of the solution: regularity and smoothness, sparsity, piecewise continuity, finite number of homogeneous materials, etc. As the computational aspects are very important in real applications, the Approximate Bayesian Computation (ABC) are presented.

Here are a set of main references related to this tutorial: [4], [5], [6], [7], [8], [9], [10], [11], [12]

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