Image Reconstruction in Computed Tomography
From deterministic to probabilistic methods

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Seeing outside of a body: Making an image with a camera, a microscope or a telescope

- $f(x, y)$ real scene
- $g(x, y)$ observed image

**Forward model: Convolution**

$$g(x, y) = \iint f(x', y') h(x - x', y - y') \, dx' \, dy' + \epsilon(x, y)$$

$h(x, y)$: Point Spread Function (PSF) of the imaging system

**Inverse problem: Image restoration**

Given the forward model $\mathcal{H}$ (PSF $h(x, y)$) and a set of data $g(x_i, y_i), i = 1, \cdots, M$ find $f(x, y)$
Making an image with an unfocused camera

Forward model: 2D Convolution

\[ g(x, y) = \int \int f(x', y') h(x - x', y - y') \, dx' \, dy' + \varepsilon(x, y) \]

Inversion: Deconvolution
Different ways to see inside of a body

Active Imaging

Passive Imaging

Transmission

Reflection

Incident wave

Object
Seeing inside of a body: Computed Tomography

- \( f(x, y) \) a section of a real 3D body \( f(x, y, z) \)
- \( g_\phi(r) \) a line of observed radiograph \( g_\phi(r, z) \)

Forward model:
Line integrals or Radon Transform

\[
g_\phi(r) = \int_{L_r,\phi} f(x, y) \, dl + \epsilon_\phi(r)
= \int \int f(x, y) \delta(r - x \cos \phi - y \sin \phi) \, dx \, dy + \epsilon_\phi(r)
\]

Inverse problem: Image reconstruction

Given the forward model \( \mathcal{H} \) (Radon Transform) and a set of data \( g_{\phi_i}(r), i = 1, \ldots, M \) find \( f(x, y) \)
2D and 3D Computed Tomography

\[
g_\phi(r_1, r_2) = \int_{\mathcal{L}_{r_1, r_2, \phi}} f(x, y, z) \, dl \quad g_\phi(r) = \int_{\mathcal{L}_{r, \phi}} f(x, y) \, dl
\]

Forward problem: \( f(x, y) \) or \( f(x, y, z) \rightarrow g_\phi(r) \) or \( g_\phi(r_1, r_2) \)

Inverse problem: \( g_\phi(r) \) or \( g_\phi(r_1, r_2) \rightarrow f(x, y) \) or \( f(x, y, z) \)
Microwave or ultrasound imaging

Mesaurs: diffracted wave by the object $\phi_d(r_i)$

Unknown quantity: $f(r) = k_0^2(n^2(r) - 1)$

Intermediate quantity: $\phi(r)$

$$\phi_d(r_i) = \int\int_D G_m(r_i, r') \phi(r') f(r') \, dr', \ r_i \in S$$

$$\phi(r) = \phi_0(r) + \int\int_D G_o(r, r') \phi(r') f(r') \, dr', \ r \in D$$

Born approximation ($\phi(r') \approx \phi_0(r')$):

$$\phi_d(r_i) = \int\int_D G_m(r_i, r') \phi_0(r') f(r') \, dr', \ r_i \in S$$

Discretization:

$$\begin{cases}
\phi_d = G_m F \phi \\
\phi = \phi_0 + G_o F \phi
\end{cases} \quad \begin{cases}
\phi_d = H(f) \\
\text{with} \quad F = \text{diag}(f) \\
H(f) = G_m F (I - G_o F)^{-1} \phi_0
\end{cases}$$
Fourier Synthesis in X ray Tomography

\[ g(r, \phi) = \int \int f(x, y) \delta(r - x \cos \phi - y \sin \phi) \, dx \, dy \]

\[ G(\Omega, \phi) = \int g(r, \phi) \exp \{-j\Omega r\} \, dr \]

\[ F(\omega_x, \omega_y) = \int \int f(x, y) \exp \{-j\omega_x x, \omega_y y\} \, dx \, dy \]

\[ F(\omega_x, \omega_y) = P(\Omega, \phi) \quad \text{for} \quad \omega_x = \Omega \cos \phi \quad \text{and} \quad \omega_y = \Omega \sin \phi \]
Fourier Synthesis in X-ray tomography

\[ F(\omega_x, \omega_y) = \int \int f(x, y) \exp \{-j\omega_x x, \omega_y y\} \, dx \, dy \]
Fourier Synthesis in Diffraction tomography

\[ f(x, y) \]

Incident plane wave

Diffracted wave

\[ \psi(r, \phi) \]

\[ FT \]

\[ f(\omega_x, \omega_y) \]

Fourier Synthesis in Diffraction tomography

\[ F(\omega_x, \omega_y) = \int \int f(x, y) \exp \{-j\omega_x x, \omega_y y\} \, dx \, dy \]
Fourier Synthesis in different imaging systems

$$F(\omega_x, \omega_y) = \iiint f(x, y) \exp\{-j\omega_x x, \omega_y y\} \, dx \, dy$$
Invers Problems: other examples and applications

- X ray, Gamma ray Computed Tomography (CT)
- Microwave and ultrasound tomography
- Positron emission tomography (PET)
- Magnetic resonance imaging (MRI)
- Photoacoustic imaging
- Radio astronomy
- Geophysical imaging
- Non Destructive Evaluation (NDE) and Testing (NDT) techniques in industry
- Hyperspectral imaging
- Earth observation methods (Radar, SAR, IR, ...)
- Survey and tracking in security systems
Computed tomography (CT)

A Multislice CT Scanner

\[ g(s_i) = \int_{L_i} f(r) \, dl_i + \epsilon(s_i) \]

Discretization

\[ g = Hf + \epsilon \]
Magnetic resonance imaging (MRI)

Nuclear magnetic resonance imaging (NMRI), Para-sagittal MRI of the head
X ray Tomography

\[ g(r, \phi) = -\ln \left( \frac{l}{l_0} \right) = \int_{L_{r,\phi}} f(x, y) \, dl \]

\[ g(r, \phi) = \int \int_D f(x, y) \delta(r - x \cos \phi - y \sin \phi) \, dx \, dy \]

\[ f(x, y) \rightarrow RT \rightarrow g(r, \phi) \]

IRT

\[ \Rightarrow \]
Analytical Inversion methods

Radon:

\[ g(r, \phi) = \int \int_D f(x, y) \delta(r - x \cos \phi - y \sin \phi) \, dx \, dy \]

\[ f(x, y) = \left(-\frac{1}{2\pi^2}\right) \int_0^\pi \int_{-\infty}^{+\infty} \frac{\partial}{\partial r} g(r, \phi) \frac{1}{(r - x \cos \phi - y \sin \phi)} \, dr \, d\phi \]
Filtered Backprojection method

\[ f(x, y) = \left( -\frac{1}{2\pi^2} \right) \int_0^\pi \int_{-\infty}^{+\infty} \frac{\partial g(r, \phi)}{\partial r} \frac{dr}{(r - x \cos \phi - y \sin \phi)} d\phi \]

Derivation \( \mathcal{D} \):

\[ \overline{g}(r, \phi) = \frac{\partial g(r, \phi)}{\partial r} \]

Hilbert Transform \( \mathcal{H} \):

\[ g_1(r', \phi) = \frac{1}{\pi} \int_0^\infty \frac{\overline{g}(r, \phi)}{(r - r')} dr \]

Backprojection \( \mathcal{B} \):

\[ f(x, y) = \frac{1}{2\pi} \int_0^\pi g_1(r' = x \cos \phi + y \sin \phi, \phi) d\phi \]

\[ f(x, y) = \mathcal{B} \mathcal{H} \mathcal{D} g(r, \phi) = \mathcal{B} \mathcal{F}_1^{-1} |\Omega| \mathcal{F}_1 g(r, \phi) \]

- Backprojection of filtered projections:

\[ g(r, \phi) \rightarrow \mathcal{F}_1 \rightarrow \text{Filter} \rightarrow \mathcal{F}_1^{-1} \rightarrow g_1(r, \phi) \rightarrow \mathcal{B} \rightarrow f(x, y) \]
Limitations: Limited angle or noisy data

- Limited angle or noisy data
- Accounting for detector size
- Other measurement geometries: fan beam, ...

CT as a linear inverse problem

\[ g(s_i) = \int_{L_i} f(r) \, dl_i + \epsilon(s_i) \rightarrow \text{Discretization} \rightarrow g = Hf + \epsilon \]

\( g, f \) and \( H \) are huge dimensional
Algebraic methods: Discretization

\[ f(x, y) = \sum_j f_j b_j(x, y) \]
\[ b_j(x, y) = \begin{cases} 
1 & \text{if } (x, y) \in \text{pixel } j \\
0 & \text{else} 
\end{cases} \]

\[ g(r, \phi) = \int_L f(x, y) \, dl \]

\[ g_i = \sum_{j=1}^N H_{ij} f_j + \epsilon_i \]

\[ \mathbf{g} = \mathbf{Hf} + \epsilon \]
Inversion: Deterministic methods

Data matching

- Observation model
  \[ g_i = h_i(f) + \epsilon_i, \quad i = 1, \ldots, M \rightarrow g = H(f) + \epsilon \]

- Misatch between data and output of the model \( \Delta(g, H(f)) \)
  \[ \hat{f} = \arg \min_f \{ \Delta(g, H(f)) \} \]

- Examples:
  - LS \( \Delta(g, H(f)) = \| g - H(f) \|_2^2 = \sum_i |g_i - h_i(f)|^2 \)
  - \( L_p \) \( \Delta(g, H(f)) = \| g - H(f) \|_p^p = \sum_i |g_i - h_i(f)|^p, \quad 1 < p < 2 \)
  - KL \( \Delta(g, H(f)) = \sum_i g_i \ln \frac{g_i}{h_i(f)} \)

- In general, does not give satisfactory results for inverse problems.
Deterministic Inversion Algorithms

Least Squares Based Methods

\[ \hat{f} = \arg \min_f \{ J(f) \} \quad \text{with} \quad J(f) = \| g - Hf \|^2 \]

\[ \nabla J(f) = -2H^t(g - Hf) \]

Gradient based algorithms:

- Initialize: \( f^{(0)} \)
- Iterate: \( f^{(k+1)} = f^{(k)} - \alpha \nabla J(f^{(k)}) \)

At each iteration: \( f^{(k+1)} = f^{(k)} + \alpha H^t(g - Hf^{(k)}) \)

we have to do the following operations:

- Compute \( \hat{g} = Hf \) (Forward projection)
- Compute \( \delta g = g - \hat{g} \) (Error or residual)
- Distribute \( \delta f = H^t \delta g \) (Backprojection of error)
- Update \( f^{(k+1)} = f^{(k)} + \delta f \)
Gradient based algorithms

Operations at each iteration: \( f^{(k+1)} = f^{(k)} + \alpha H^t \left( g - Hf^{(k)} \right) \)

- Compute \( \hat{g} = Hf \) (Forward projection)
- Compute \( \delta g = g - \hat{g} \) (Error or residual)
- Distribute \( \delta f = H^t \delta g \) (Backprojection of error)
- Update \( f^{(k+1)} = f^{(k)} + \delta f \)

Initial guess \( f^{(0)} \) → estimated image \( f^{(k)} \) → Forward projection \( H \) → projections of estimated image \( \hat{g} = Hf^{(k)} \) → \( \delta g = g - \hat{g} \) → Backprojection \( H^t \) → correction term in projection space \( \delta g = g - \hat{g} \)
Gradient based algorithms

- Fixed step gradient:
  \[ f^{(k+1)} = f^{(k)} + \alpha H^t \left( g - Hf^{(k)} \right) \]

- Steepest descent gradient:
  \[ f^{(k+1)} = f^{(k)} + \alpha^{(k)} H^t \left( g - Hf^{(k)} \right) \]

  with \( \alpha^{(k)} = \arg \min_{\alpha} \{ J(f + \alpha \delta f) \} \)

- Conjugate Gradient

  \[ f^{(k+1)} = f^{(k)} + \alpha^{(k)} d^{(k)} \]

  The successive directions \( d^{(k)} \) have to be conjugate to each other.
Main idea: Use the data as they arrive

\[ f^{(k+1)} = f^{(k)} + \alpha^{(k)} [H^t]_{i\star} \left( g_i - [Hf^{(k)}]_i \right) \]

which can also be written as:

\[ f^{(k+1)} = f^{(k)} + \frac{\left( g_i - [Hf^{(k)}]_i \right)}{h^t_{i\star} h_{i\star}} h^t_{i\star} \]

\[ = f^{(k)} + \sum_i \frac{\left( g_i - \sum_j H_{ij} f^{(k)}_j \right)}{\sum_i H_{ij}^2} H_{ij} \]
Use the data as they arrive

\[
f^{(k+1)} = f^{(k)} + \frac{\left( g_i - [Hf^{(k)}]_i \right)}{h^t_{i*} h_{i*}} h^t_{i*}
\]

\[
= f^{(k)} + \sum_i \frac{\left( g_i - \sum_j H_{ij} f^{(k)}_j \right)}{\sum_i H_{ij}^2} H_{ij}
\]

Update each pixel at each time

\[
f^{(k+1)}_j = f^{(k)}_j + \frac{\left( g_i - \sum_j H_{ij} f^{(k)}_j \right)}{\sum_i H_{ij}^2} H_{ij}
\]
Algebraic Reconstruction Techniques (ART)

\[ f^{(k+1)} = f^{(k)} + \sum_i \left( \frac{g_i - \sum_j H_{ij} f_j^{(k)}}{\sum_j H_{ij}^2} \right) H_{ij} \]

or

\[ f_j^{(k+1)} = f_j^{(k)} + \left( \frac{g_i - \sum_j H_{ij} f_j^{(k)}}{\sum_j H_{ij}^2} \right) H_{ij} \]

Initial guess \( f^{(0)} \) → estimated image \( f^{(k)} \) → Forward projection \( \mathbf{H} \) → projections of estimated image \( \hat{g}_i = \sum_j H_{ij} f_j^{(k)} \) → compare → Measured projections \( g_i \)

\[ \delta f_j = \sum_i H_{ij} \delta g_i \sum_j H_{ij} \]

Backprojection \( \mathbf{H}^t \) → correction term in projection space \( \delta g_i = g_i - \hat{g}_i \)
Algebraic Reconstruction using KL distance

\[ \hat{f} = \arg \min_f \{ J(f) \} \quad \text{with} \quad J(f) = \sum_i g_i \ln \frac{g_i}{\sum_j H_{ij} f_j} \]

\[ f_j^{(k+1)} = \frac{f_j^{(k)}}{\sum_i H_{ij}} \sum_i H_{ij} \frac{g_i}{\sum_j H_{ij} f_j^{(k)}} \]

Interestingly, this is the OSEM (Ordered subset Expectation-Maximization) algorithm which is based on Maximum Likelihood and proposed first by Shepp & Vardi.
Inversion: Regularization theory

Inverse problems = Ill posed problems

→ Need for prior information

Functional space (Tikhonov):

\[ g = \mathcal{H}(f) + \epsilon \rightarrow J(f) = \|g - \mathcal{H}(f)\|^2 + \lambda \|Df\|^2 \]

Finite dimensional space (Philips & Towmey):

\[ g = \mathcal{H}(f) + \epsilon \]

- Minimum norme LS (MNLS):
  \[ J(f) = \|g - \mathcal{H}(f)\|^2 + \lambda \|f\|^2 \]

- Classical regularization:
  \[ J(f) = \|g - \mathcal{H}(f)\|^2 + \lambda \|Df\|^2 \]

- More general regularization:

\[ J(f) = Q(g - \mathcal{H}(f)) + \lambda \Omega(Df) \]

or

\[ J(f) = \Delta_1(g, \mathcal{H}(f)) + \lambda \Delta_2(f, f_0) \]

Limitations:

- Errors are considered implicitly white and Gaussian
- Limited prior information on the solution
- Lack of tools for the determination of the hyperparameters
Bayesian estimation approach

\[ \mathcal{M} : \quad \mathbf{g} = \mathbf{Hf} + \mathbf{\epsilon} \]

- Observation model \( \mathcal{M} \) + Hypothesis on the noise \( \mathbf{\epsilon} \)
  \[ p(\mathbf{g}|\mathbf{f}; \mathcal{M}) = p\mathbf{\epsilon}(\mathbf{g} - \mathbf{Hf}) \]
- A priori information \( p(\mathbf{f}|\mathcal{M}) \)
- Bayes:
  \[ p(\mathbf{f}|\mathbf{g}; \mathcal{M}) = \frac{p(\mathbf{g}|\mathbf{f}; \mathcal{M}) p(\mathbf{f}|\mathcal{M})}{p(\mathbf{g}|\mathcal{M})} \]

**Link with regularization:**

Maximum A Posteriori (MAP):

\[ \hat{\mathbf{f}} = \arg \max_{\mathbf{f}} \{ p(\mathbf{f}|\mathbf{g}) \} = \arg \max_{\mathbf{f}} \{ p(\mathbf{g}|\mathbf{f}) p(\mathbf{f}) \} \]
\[ = \arg \min_{\mathbf{f}} \{ -\ln p(\mathbf{g}|\mathbf{f}) - \ln p(\mathbf{f}) \} \]

with \( Q(\mathbf{g}, \mathbf{Hf}) = -\ln p(\mathbf{g}|\mathbf{f}) \) and \( \lambda \Omega(\mathbf{f}) = -\ln p(\mathbf{f}) \)

**But, Bayesian inference is not only limited to MAP**
Case of linear models and Gaussian priors

\[ g = Hf + \epsilon \]

- Hypothesis on the noise: \( \epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2 I) \)

\[ p(g|f) \propto \exp \left\{ -\frac{1}{2\sigma_\epsilon^2} \| g - Hf \|^2 \right\} \]

- Hypothesis on \( f \): \( f \sim \mathcal{N}(0, \sigma_f^2 I) \)

\[ p(f) \propto \exp \left\{ -\frac{1}{2\sigma_f^2} \| f \|^2 \right\} \]

- A posteriori:

\[ p(f|g) \propto \exp \left\{ -\frac{1}{2\sigma_\epsilon^2} \| g - Hf \|^2 - \frac{1}{2\sigma_f^2} \| f \|^2 \right\} \]

- MAP: \( \hat{f} = \arg \max_f \{ p(f|g) \} = \arg \min_f \{ J(f) \} \)

\[ J(f) = \| g - Hf \|^2 + \lambda \| f \|^2, \quad \lambda = \frac{\sigma_\epsilon^2}{\sigma_f^2} \]

- Advantage: characterization of the solution

\[ f|g \sim \mathcal{N}(\hat{f}, \hat{P}) \quad \text{with} \quad \hat{f} = \hat{P}H^t g, \quad \hat{P} = (H^tH + \lambda I)^{-1} \]
MAP estimation with other priors:

\[ \hat{f} = \arg \min_{f} \{ J(f) \} \quad \text{with} \quad J(f) = \| g - Hf \|^2 + \lambda \Omega(f) \]

Separable priors:

- **Gaussian:** \( p(f_j) \propto \exp \{ -\alpha |f_j|^2 \} \rightarrow \Omega(f) = \| f \|^2 = \alpha \sum_j |f_j|^2 \)
- **Gamma:** \( p(f_j) \propto f_j^\alpha \exp \{ -\beta f_j \} \rightarrow \Omega(f) = \alpha \sum_j \ln f_j + \beta f_j \)
- **Beta:**
  \[ p(f_j) \propto f_j^\alpha (1 - f_j)^\beta \rightarrow \Omega(f) = \alpha \sum_j \ln f_j + \beta \sum_j \ln(1 - f_j) \]
- **Generalized Gaussian:**
  \[ p(f_j) \propto \exp \{ -\alpha |f_j|^p \}, \quad 1 < p < 2 \rightarrow \Omega(f) = \alpha \sum_j |f_j|^p, \]

Markovian models:

\[ p(f_j | f) \propto \exp \left\{ -\alpha \sum_{i \in N_j} \phi(f_j, f_i) \right\} \rightarrow \Omega(f) = \alpha \sum_j \sum_{i \in N_j} \phi(f_j, f_i), \]
Main advantages of the Bayesian approach

- MAP = Regularization
- Posterior mean ≠ Marginal MAP ≠
- More information in the posterior law than only its mode or its mean
- Meaning and tools for estimating hyper parameters
- Meaning and tools for model selection
- More specific and specialized priors, particularly through the hidden variables
- More computational tools:
  - Expectation-Maximization for computing the maximum likelihood parameters
  - MCMC for posterior exploration
  - Variational Bayes for analytical computation of the posterior marginals
  - ...
MAP estimation and Compressed Sensing

\[
\begin{align*}
\{ \quad & g = Hf + \epsilon \\
& f = Wz
\end{align*}
\]

- **W** a code book matrix, **z** coefficients
- Gaussian:

  \[
  p(z) = \mathcal{N}(0, \sigma^2_z I) \propto \exp \left\{ -\frac{1}{2\sigma^2_z} \sum_j |z_j|^2 \right\}
  \]

  \[
  J(z) = -\ln p(z|g) = \|g - HWz\|^2 + \lambda \sum_j |z_j|^2
  \]

- Generalized Gaussian (sparsity, \( \beta = 1 \)):

  \[
  p(z) \propto \exp \left\{ -\lambda \sum_j |z_j|^{\beta} \right\}
  \]

  \[
  J(z) = -\ln p(z|g) = \|g - HWz\|^2 + \lambda \sum_j |z_j|^{\beta}
  \]

- **z** = arg min \( J(z) \) \( \rightarrow \) \( \hat{f} = W\hat{z} \)
Full Bayesian approach

\[ M : \quad g = Hf + \epsilon \]

- Forward & errors model: \( p(g|f, \theta_1; M) \)
- Prior models \( p(f|\theta_2; M) \)
- Hyperparameters \( \theta = (\theta_1, \theta_2) \rightarrow p(\theta|M) \)
- Bayes: \( p(f, \theta|g; M) = \frac{p(g|f, \theta; M) p(f|\theta; M) p(\theta|M)}{p(g|M)} \)
- Joint MAP: \( (\hat{f}, \hat{\theta}) = \text{arg max } \{ p(f, \theta|g; M) \} \)
- Marginalization:
  \[ \begin{align*}
  p(f|g; M) &= \int p(f, \theta|g; M) df \\
  p(\theta|g; M) &= \int p(f, \theta|g; M) d\theta
  \end{align*} \]
- Posterior means:
  \[ \begin{align*}
  \hat{f} &= \int f p(f, \theta|g; M) df d\theta \\
  \hat{\theta} &= \int \theta p(f, \theta|g; M) df d\theta
  \end{align*} \]
- Evidence of the model:
  \[ p(g|M) = \int \int p(g|f, \theta; M)p(f|\theta; M)p(\theta|M) df d\theta \]
Two main steps in the Bayesian approach

- Prior modeling
  - Separable:
    Gaussian, Generalized Gaussian, Gamma, mixture of Gaussians, mixture of Gammas, ...
  - Markovian: Gauss-Markov, GGM, ...
  - Separable or Markovian with hidden variables (contours, region labels)

- Choice of the estimator and computational aspects
  - MAP, Posterior mean, Marginal MAP
  - MAP needs optimization algorithms
  - Posterior mean needs integration methods
  - Marginal MAP needs integration and optimization
  - Approximations:
    - Gaussian approximation (Laplace)
    - Numerical exploration MCMC
    - Variational Bayes (Separable approximation)
Which images I am looking for?
Which image I am looking for?

Gaussian
\[ p(f_j|f_{j-1}) \propto \exp\left\{ -\alpha |f_j - f_{j-1}|^2 \right\} \]

Generalized Gaussian
\[ p(f_j|f_{j-1}) \propto \exp\left\{ -\alpha |f_j - f_{j-1}|^p \right\} \]

Piecewise Gaussian
\[ p(f_j|q_j, f_{j-1}) = \mathcal{N}\left((1 - q_j)f_{j-1}, \sigma_f^2\right) \]

Mixture of GM
\[ p(f_j|z_j = k) = \mathcal{N}\left(m_k, \sigma_k^2\right) \]
Gauss-Markov-Potts prior models for images

"In NDT applications of CT, the objects are, in general, composed of a finite number of materials, and the voxels corresponding to each materials are grouped in compact regions”

How to model this prior information?

\[
p(f(r)|z(r) = k, m_k, v_k) = \mathcal{N}(m_k, v_k)
\]

\[
p(f(r)) = \sum P(z(r) = k) \mathcal{N}(m_k, v_k)
\]

Mixture of Gaussians

\[
p(z(r)|z(r^k), r' \in \mathcal{V}(r)) \propto \exp \left\{ \gamma \sum_{r' \in \mathcal{V}(r)} \delta(z(r) - z(r')) \right\}
\]
Four different cases

To each pixel of the image is associated 2 variables \( f(r) \) and \( z(r) \)

- \( f|z \) Gaussian iid, \( z \) iid : Mixture of Gaussians
- \( f|z \) Gauss-Markov, \( z \) iid : Mixture of Gauss-Markov
- \( f|z \) Gaussian iid, \( z \) Potts-Markov : Mixture of Independent Gaussians (MIG with Hidden Potts)
- \( f|z \) Markov, \( z \) Potts-Markov : Mixture of Gauss-Markov (MGM with hidden Potts)
Four different cases

Case 1: Mixture of Gaussians

Case 2: Mixture of Gauss-Markov

Case 3: MIG with Hidden Potts

Case 4: MGM with hidden Potts
Summary of the two proposed models

\[ f \mid z \text{ Gaussian iid} \]
\[ z \text{ Potts-Markov} \]

(MIG with Hidden Potts)

\[ f \mid z \text{ Markov} \]
\[ z \text{ Potts-Markov} \]

(MGM with hidden Potts)
Bayesian Computation

\[ p(f, z, \theta | g) \propto p(g | f, z, \nu_\epsilon) \ p(f | z, m, v) \ p(z | \gamma, \alpha) \ p(\theta) \]

\[ \theta = \{ \nu_\epsilon, (\alpha_k, m_k, \nu_k), k = 1, \cdots, K \} \quad p(\theta) \quad \text{Conjugate priors} \]

- Direct computation and use of \( p(f, z, \theta | g; \mathcal{M}) \) is too complex
- Possible approximations:
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)

- Main idea in Variational Bayesian methods:
  Approximate
  \[ p(f, z, \theta | g; \mathcal{M}) \quad \text{by} \quad q(f, z, \theta) = q_1(f) \ q_2(z) \ q_3(\theta) \]
  - Choice of approximation criterion: \( KL(q : p) \)
  - Choice of appropriate families of probability laws for \( q_1(f), q_2(z) \) and \( q_3(\theta) \)
MCMC based algorithm

\[ p(f, z, \theta | g) \propto p(g | f, z, \theta) p(f | z, \theta) p(z) p(\theta) \]

General scheme:

\[ \hat{f} \sim p(f | \hat{z}, \hat{\theta}, g) \rightarrow \hat{z} \sim p(z | \hat{f}, \hat{\theta}, g) \rightarrow \hat{\theta} \sim (\theta | \hat{f}, \hat{z}, g) \]

- Sample \( f \) from \( p(f | \hat{z}, \hat{\theta}, g) \propto p(g | f, \theta) p(f | \hat{z}, \hat{\theta}) \)
  Needs optimisation of a quadratic criterion.

- Sample \( z \) from \( p(z | \hat{f}, \hat{\theta}, g) \propto p(g | \hat{f}, \hat{z}, \hat{\theta}) p(z) \)
  Needs sampling of a Potts Markov field.

- Sample \( \theta \) from \( p(\theta | \hat{f}, \hat{z}, g) \propto p(g | \hat{f}, \sigma^2 \epsilon I) p(f | \hat{z}, (m_k, v_k)) p(\theta) \)
  Conjugate priors \( \rightarrow \) analytical expressions.
Application of CT in NDT

Reconstruction from only 2 projections

Given the marginals $g_1(x)$ and $g_2(y)$ find the joint distribution $f(x, y)$.

Infinite number of solutions: 

$$f(x, y) = g_1(x) g_2(y) \Omega(x, y)$$

$\Omega(x, y)$ is a Copula:

$$\int \Omega(x, y) \, dx = 1 \quad \text{and} \quad \int \Omega(x, y) \, dy = 1$$
Application in CT

\[
g|f = Hf + \epsilon
\]

\[
g|f \sim N(Hf, \sigma^2 I)
\]

Gaussian

Forward model

\[
f|z \sim \text{iid Gaussian}
\]

Gauss-Markov

Gauss-Markov-Potts Prior Model

\[
z \sim \text{iid}
\]

Potts

Auxiliary

Unsupervised Bayesian estimation:

\[
p(f, z, \theta | g) \propto p(g|f, z, \theta) p(f|z, \theta) p(\theta)
\]
Results: 2D case

Original

Backprojection

Filtered BP

LS

Gauss-Markov+pos

GM+Line process

GM+Label process
Some results in 3D case

(Results obtained with collaboration with CEA)

M. Defrise

Phantom

FeldKamp

Proposed method
Some results in 3D case

FeldKamp

Proposed method
Some results in 3D case

A photography of metalique esponge

Reconstruction by proposed method
Application: liquid evaporation in metallic esponge

Time 0  |  Time 1  |  Time 2
Conclusions

- Gauss-Markov-Potts are useful prior models for images incorporating regions and contours
- Bayesian computation needs often approximations (Laplace, MCMC, Variational Bayes)
- Application in different CT systems (X ray, Ultrasound, Microwave, PET, SPECT) as well as other inverse problems

Work in Progress and Perspectives:

- Efficient implementation in 2D and 3D cases using GPU
- Evaluation of performances and comparison with MCMC methods
- Application to other linear and non linear inverse problems: (PET, SPECT or ultrasound and microwave imaging)
Some references

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Questions and Discussions