A Gauss-Markov-Potts Prior model for images in Bayesian Computed Tomography

Ali Mohammad-Djafari

Groupe Problèmes Inverses
Laboratoire des Signaux et Systèmes
(UMR 8506 CNRS - SUPELEC - Univ Paris Sud 11)
Supélec, Plateau de Moulon, 91192 Gif-sur-Yvette, FRANCE.

djafari@lss.supelec.fr
http://djafari.free.fr
http://www.lss.supelec.fr

KTH, Dept. of Mathematics, Stockhol, Sweden
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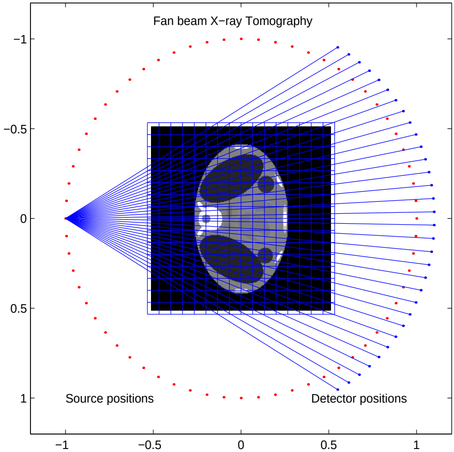
CT as a linear inverse problem



$$g(s_i) = \int_{L_i} f(r) \ \mathrm{d}l_i \longrightarrow \text{Discretization} \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

Classical methods in CT

$$g(\boldsymbol{s}_i) = \int_{L_i} f(\boldsymbol{r}) \; \mathrm{d}l_i \longrightarrow \text{Discretization} \longrightarrow \boldsymbol{g} = \boldsymbol{H} \boldsymbol{f} + \boldsymbol{\epsilon}$$

- $\boldsymbol{H}$ is a huge dimensional matrix of line integrals
- $\boldsymbol{H} \boldsymbol{f}$ is the forward or projection operation
- $\boldsymbol{H}^t \boldsymbol{g}$ is the backward or backprojection operation
- $(\boldsymbol{H}^t \boldsymbol{H})^{-1} \boldsymbol{H}^t \boldsymbol{g}$ is the filtered backprojection minimizing the LS criterion $Q(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H} \boldsymbol{f}\|^2$
- Iterative methods :
  $$\widehat{\boldsymbol{f}}^{(k+1)} = \widehat{\boldsymbol{f}}^{(k)} + \alpha^{(k)} \boldsymbol{H}^t \left(\boldsymbol{g} - \boldsymbol{H} \widehat{\boldsymbol{f}}^{(k)} \right)$$
  try to minimize the Least squares criterion
- Other criteria :
  - Robust criteria : $Q(\boldsymbol{f}) = \sum_i \phi(|g_i - [\boldsymbol{H} \boldsymbol{f}]_i|)$
  - Likelihood : $\mathcal{L}(\boldsymbol{f}) = p(\boldsymbol{g} | \boldsymbol{f})$
  - Regularization : $J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H} \boldsymbol{f}\|^2 + \lambda \|\boldsymbol{D} \boldsymbol{f}\|^2$.

Bayesian estimation approach

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Observation model $\mathcal{M}$ + Hypothesis on the noise $\boldsymbol{\epsilon} \longrightarrow$
  $p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M}) = p_{\boldsymbol{\epsilon}}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f})$
- A priori information $\quad p(\boldsymbol{f}|\mathcal{M})$
- Bayes : $\quad p(\boldsymbol{f}|\boldsymbol{g};\mathcal{M}) = \dfrac{p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M})\, p(\boldsymbol{f}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$

Link with regularisation :

Maximum A Posteriori (MAP) :

$$\begin{aligned}\widehat{\boldsymbol{f}} &= \arg\max_{\boldsymbol{f}}\{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\max_{\boldsymbol{f}}\{p(\boldsymbol{g}|\boldsymbol{f})\, p(\boldsymbol{f})\} \\ &= \arg\min_{\boldsymbol{f}}\{-\ln p(\boldsymbol{g}|\boldsymbol{f}) - \ln p(\boldsymbol{f})\}\end{aligned}$$

with $\quad Q(\boldsymbol{g}, \boldsymbol{H}\boldsymbol{f}) = -\ln p(\boldsymbol{g}|\boldsymbol{f}) \quad$ and $\quad \lambda\Phi(\boldsymbol{f}) = -\ln p(\boldsymbol{f})$

Two main steps in the Bayesian approach

- Prior modeling
  - Separable :
    Gaussian, Generalized Gaussian, Gamma,
    mixture of Gaussians, mixture of Gammas, ...
  - Markovian : Gauss-Markov, GGM, ...
  - Separable or Markovian with hidden variables
    (contours, region labels)
- Choice of the estimator and computational aspects
  - MAP, Posterior mean, Marginal MAP
  - MAP needs optimization algorithms
  - Posterior mean needs integration methods
  - Marginal MAP needs integration and optimization
  - Approximations :
    - Gaussian approximation (Laplace)
    - Numerical exploration MCMC
    - Variational Bayes (Separable approximation)

Full Bayesian approach

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Forward & errors model : $\longrightarrow p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M})$
- Prior models $\longrightarrow p(\boldsymbol{f}|\boldsymbol{\theta}; \mathcal{M})$ and $p(\boldsymbol{\theta}|\mathcal{M})$
- Bayes : $\longrightarrow p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\, p(\boldsymbol{f}|\boldsymbol{\theta};\mathcal{M})\, p(\boldsymbol{\theta}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$
- Joint MAP :

$$(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f},\boldsymbol{\theta})} \{p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\}$$

- Posterior means : $\left\{ \begin{array}{ll} \widehat{\boldsymbol{f}} & = \int \boldsymbol{f}\, p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta} \\ \widehat{\boldsymbol{\theta}} & = \int \boldsymbol{\theta}\, p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta} \end{array} \right.$
- Evidence of the model :

$$p(\boldsymbol{g}|\mathcal{M}) = \iint p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{f}|\boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{\theta}|\mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta}$$

Bayesian Computation

- Direct computation and use of $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})$ is too complex
- Approximations :
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)
- Main idea :
  Approximate $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})$ by $q(\boldsymbol{f}, \boldsymbol{\theta}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{\theta})$
  - Choice of approximation criterion :

$$\mathrm{KL}(q:p) = \int q \ln \frac{p}{q} = \int\int q_1(\boldsymbol{f})\, q_2(\boldsymbol{\theta}) \ln \frac{q_1(\boldsymbol{f})\, q_2(\boldsymbol{\theta})}{p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})}\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta}$$

  - Choice of appropriate families of probability laws for $q_1(\boldsymbol{f})$ and $q_2(\boldsymbol{\theta})$: Conjugate Exponential families

Approximation : Choice of criterion

$$
\begin{aligned}
\ln p(\boldsymbol{g}|\mathcal{M}) &= \ln \iint q(\boldsymbol{f},\boldsymbol{\theta}) \frac{p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta}|\mathcal{M})}{q(\boldsymbol{f},\boldsymbol{\theta})} \, \mathrm{d}\boldsymbol{f} \, \mathrm{d}\boldsymbol{\theta} \\
&\geq \iint q(\boldsymbol{f},\boldsymbol{\theta}) \ln \frac{p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta}|\mathcal{M})}{q(\boldsymbol{f},\boldsymbol{\theta})} \, \mathrm{d}\boldsymbol{f} \, \mathrm{d}\boldsymbol{\theta}.
\end{aligned}
$$

$$
\mathcal{F}(q) = \iint q(\boldsymbol{f},\boldsymbol{\theta}) \ln \frac{p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta}|\mathcal{M})}{q(\boldsymbol{f},\boldsymbol{\theta})} \, \mathrm{d}\boldsymbol{f} \, \mathrm{d}\boldsymbol{\theta}
$$

$$
\mathrm{KL}(q:p) = -\iint q(\boldsymbol{f},\boldsymbol{\theta}) \ln \frac{p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})}{q(\boldsymbol{f},\boldsymbol{\theta})} \, \mathrm{d}\boldsymbol{f} \, \mathrm{d}\boldsymbol{\theta}
$$

$$
\ln p(\boldsymbol{g}|\mathcal{M}) = \mathcal{F}(q) + \mathrm{KL}(q:p)
$$

Approximation : Separable Approximation

$$\ln p(\boldsymbol{g}|\mathcal{M}) = \mathcal{F}(q) + \mathrm{KL}(q:p)$$

$$q(\boldsymbol{f},\boldsymbol{\theta}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{\theta})$$

$$(\widehat{q}_1,\widehat{q}_2) = \arg\min_{(q_1,q_2)} \{\mathrm{KL}(q_1 q_2 : p)\} = \arg\max_{(q_1,q_2)} \{\mathcal{F}(q_1 q_2)\}$$

$\mathrm{KL}(q_1 q_2 : p)$ is convex in q_1 for a given q_2 and vise versa :

$$\left\{ \begin{array}{ll} \widehat{q}_1 & = \arg\min_{q_1} \left\{\mathrm{KL}(q_1 \widehat{q}_2 : p)\right\} = \arg\max_{q_1} \left\{\mathcal{F}(q_1 \widehat{q}_2)\right\} \\ \widehat{q}_2 & = \arg\min_{q_2} \left\{\mathrm{KL}(\widehat{q}_1 q_2 : p)\right\} = \arg\max_{q_2} \left\{\mathcal{F}(\widehat{q}_1 q_2)\right\} \end{array} \right.$$

$$\left\{ \begin{array}{ll} \widehat{q}_1(\boldsymbol{f}) & \propto \exp\left\{\langle \ln p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\rangle_{\widehat{q}_2(\boldsymbol{\theta})}\right\} \\ \widehat{q}_2(\boldsymbol{\theta}) & \propto \exp\left\{\langle \ln p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\rangle_{\widehat{q}_1(\boldsymbol{f})}\right\} \end{array} \right.$$

Separable Approximation : Choice of appropriate families for q_1 and q_2

- Degenerate case 1 : $\longrightarrow$ Joint MAP

$$\left\{\begin{array}{ll}\widehat{q}_1(\boldsymbol{f}|\widetilde{\boldsymbol{f}}) & = \delta(\boldsymbol{f}-\widetilde{\boldsymbol{f}}) \\ \widehat{q}_2(\boldsymbol{\theta}|\widetilde{\boldsymbol{\theta}}) & = \delta(\boldsymbol{\theta}-\widetilde{\boldsymbol{\theta}})\end{array}\right. \longrightarrow \left\{\begin{array}{l}\widetilde{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \left\{p(\boldsymbol{f},\widetilde{\boldsymbol{\theta}}|\boldsymbol{g};\mathcal{M})\right\} \\ \widetilde{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \left\{p(\widetilde{\boldsymbol{f}},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})\right\}\end{array}\right.$$

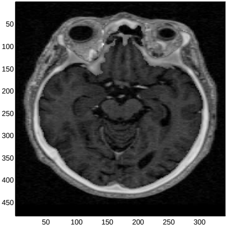
- Degenerate case 2 : $\longrightarrow$ Expectation-Maximisation

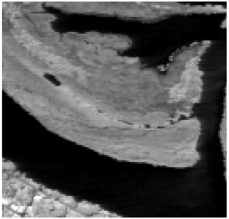
$$\left\{\begin{array}{ll}\widehat{q}_1(\boldsymbol{f}) & \propto p(\boldsymbol{f}|\boldsymbol{\theta},\boldsymbol{g}) \\ \widehat{q}_2(\boldsymbol{\theta}|\widetilde{\boldsymbol{\theta}}) & = \delta(\boldsymbol{\theta}-\widetilde{\boldsymbol{\theta}})\end{array}\right. \longrightarrow \left\{\begin{array}{ll}Q(\boldsymbol{\theta},\widetilde{\boldsymbol{\theta}}) = & \langle \ln p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})\rangle_{q_1(\boldsymbol{f}|\widetilde{\boldsymbol{\theta}})} \\ \widetilde{\boldsymbol{\theta}} & = \arg\max_{\boldsymbol{\theta}} \left\{Q(\boldsymbol{\theta},\widetilde{\boldsymbol{\theta}})\right\}\end{array}\right.$$

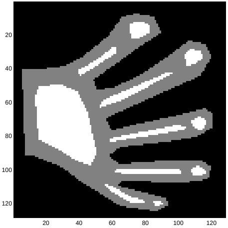
- Appropriate choice for inverse problems

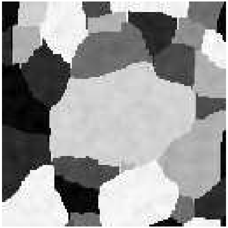
$$\left\{\begin{array}{ll}\widehat{q}_1(\boldsymbol{f}) & \propto p(\boldsymbol{f}|\widetilde{\boldsymbol{\theta}},\boldsymbol{g};\mathcal{M}) \\ \widehat{q}_2(\boldsymbol{\theta}) & \propto p(\boldsymbol{\theta}|\boldsymbol{f},\boldsymbol{g};\mathcal{M})\end{array}\right. \longrightarrow \left\{\begin{array}{l}\text{Accounting for uncertainties} \\ \text{of } \widehat{\boldsymbol{\theta}} \text{ in } \widehat{\boldsymbol{f}} \text{ and vise versa.}\end{array}\right.$$

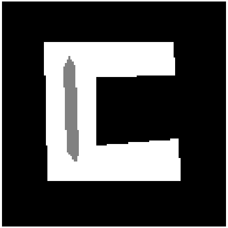
Which images I am looking for ?



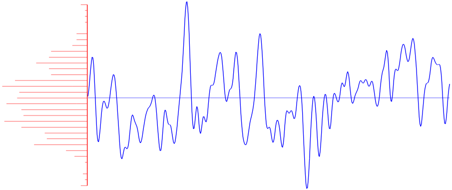




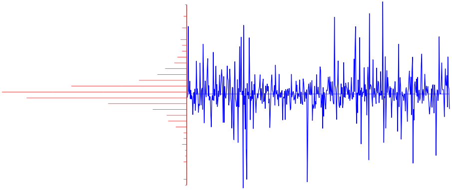




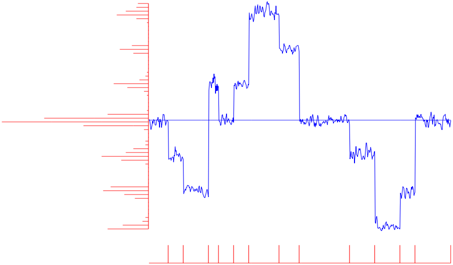
Which image I am looking for ?



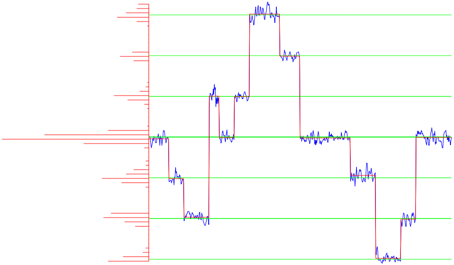
Gauss-Markov



Generalized GM

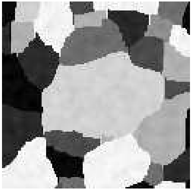


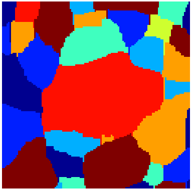
Piecewize Gaussian

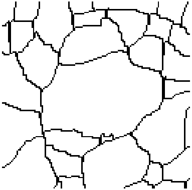


Mixture of GM

Gauss-Markov-Potts prior models for images







$f(\boldsymbol{r})$ $\qquad$ $z(\boldsymbol{r})$ $\qquad$ $c(\boldsymbol{r}) = 1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k, m_k, v_k) = \mathcal{N}(m_k, v_k)$$

$$p(f(\boldsymbol{r})) = \sum_k P(z(\boldsymbol{r}) = k)\, \mathcal{N}(m_k, v_k) \quad \text{Mixture of Gaussians}$$

- Separable iid hidden variables : $\quad p(\boldsymbol{z}) = \prod_{\boldsymbol{r}} p(z(\boldsymbol{r}))$
- Markovian hidden variables : $\quad p(\boldsymbol{z})$ Potts-Markov :

$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r} \in \mathcal{R}} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Four different cases

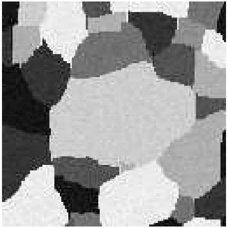
To each pixel of the image is associated 2 variables $f(\boldsymbol{r})$ and $z(\boldsymbol{r})$

- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ iid :
  Mixture of Gaussians

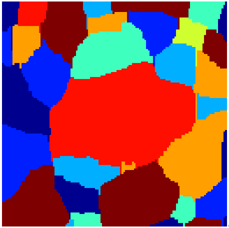
- $\boldsymbol{f}|\boldsymbol{z}$ Gauss-Markov, $\boldsymbol{z}$ iid :
  Mixture of Gauss-Markov

- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ Potts-Markov :
  Mixture of Independent Gaussians
  (MIG with Hidden Potts)

- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\boldsymbol{z}$ Potts-Markov :
  Mixture of Gauss-Markov
  (MGM with hidden Potts)



$f(\boldsymbol{r})$



$z(\boldsymbol{r})$

Case 1 : $f|z$ Gaussian iid, z iid

Independent Mixture of Independent Gaussiens (IMIG) :

$$p(f(\boldsymbol{r})|z(\boldsymbol{r})=k)=\mathcal{N}(m_k,v_k),\quad \forall \boldsymbol{r}\in\mathcal{R}$$
$$p(f(\boldsymbol{r}))=\sum_{k=1}^K \alpha_k \mathcal{N}(m_k,v_k), \text{with } \sum_k \alpha_k=1.$$

$$p(\boldsymbol{z})=\prod_{\boldsymbol{r}} p(z(\boldsymbol{r})=k)=\prod_{\boldsymbol{r}} \alpha_k=\prod_k \alpha_k^{n_k}$$

Noting

$$m_z(\boldsymbol{r})=m_k, v_z(\boldsymbol{r})=v_k, \alpha_z(\boldsymbol{r})=\alpha_k, \forall \boldsymbol{r}\in\mathcal{R}_k$$

we have :

$$\begin{aligned} p(\boldsymbol{f}|\boldsymbol{z}) &= \prod_{\boldsymbol{r}\in\mathcal{R}} \mathcal{N}(m_z(\boldsymbol{r}),v_z(\boldsymbol{r})) \\ p(\boldsymbol{z}) &= \prod_{\boldsymbol{r}} \alpha_z(\boldsymbol{r}) = \prod_k \alpha_k^{\sum_{\boldsymbol{r}\in\mathcal{R}} \delta(z(\boldsymbol{r})-k)} = \prod_k \alpha_k^{n_k} \end{aligned}$$

Case 2 : $f|z$ Gauss-Markov, z iid

Independent Mixture of Gauss-Markov (IMGM) :

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}), z(\boldsymbol{r}'), f(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) = \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})), \forall \boldsymbol{r} \in \mathcal{R}$$

$$\begin{array}{ll}
\mu_z(\boldsymbol{r}) & = \frac{1}{|\mathcal{V}(\boldsymbol{r})|} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \mu_z^*(\boldsymbol{r}') \\
\mu_z^*(\boldsymbol{r}') & = \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, f(\boldsymbol{r}') + (1 - \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, m_z(\boldsymbol{r}') \\
 & = \quad (1 - c(\boldsymbol{r}')) \quad f(\boldsymbol{r}') \; + \qquad c(\boldsymbol{r}') \qquad m_z(\boldsymbol{r}')
\end{array}$$

$$\begin{array}{lll}
p(\boldsymbol{f}|\boldsymbol{z}) & \propto \prod_{\boldsymbol{r}} \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})) & \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \boldsymbol{\Sigma}_k) \\
p(\boldsymbol{z}) & = \prod_{\boldsymbol{r}} v_z(\boldsymbol{r}) & = \prod_k \alpha_k^{n_k}
\end{array}$$

with $1_k = 1, \forall \boldsymbol{r} \in \mathcal{R}_k$ and $\boldsymbol{\Sigma}_k$ a covariance matrix ($n_k \times n_k$).

Case 3 : $f|z$ Gauss iid, z Potts

Gauss iid as in Case 1 :

$$p(\boldsymbol{f}|\boldsymbol{z}) = \prod_{\boldsymbol{r}\in\mathcal{R}} \mathcal{N}(m_z(\boldsymbol{r}), v_z(\boldsymbol{r})) = \prod_k \prod_{\boldsymbol{r}\in\mathcal{R}_k} \mathcal{N}(m_k, v_k)$$

Potts-Markov

$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Case 4 : $f|z$ Gauss-Markov, z Potts

Gauss-Markov as in Case 2 :

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}), z(\boldsymbol{r}'), f(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) = \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})), \forall \boldsymbol{r} \in \mathcal{R}$$

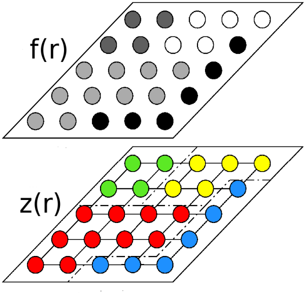
$$\begin{aligned} \mu_z(\boldsymbol{r}) &= \frac{1}{|\mathcal{V}(\boldsymbol{r})|} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \mu_z^*(\boldsymbol{r}') \\ \mu_z^*(\boldsymbol{r}') &= \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, f(\boldsymbol{r}') + (1 - \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, m_z(\boldsymbol{r}') \end{aligned}$$

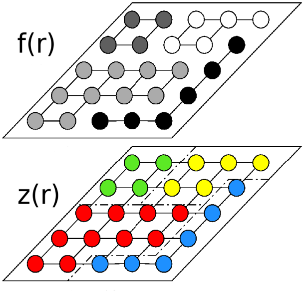
$$p(\boldsymbol{f}|\boldsymbol{z}) \propto \prod_{\boldsymbol{r}} \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})) \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \boldsymbol{\Sigma}_k)$$

Potts-Markov as in Case 3 :

$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r} \in \mathcal{R}} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Summary of the two proposed models





$\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid
$\boldsymbol{z}$ Potts-Markov

(MIG with Hidden Potts)

$\boldsymbol{f}|\boldsymbol{z}$ Markov
$\boldsymbol{z}$ Potts-Markov

(MGM with hidden Potts)

Bayesian Computation

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{z}, v_\epsilon)\, p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{m}, \boldsymbol{v})\, p(\boldsymbol{z}|\gamma, \boldsymbol{\alpha})\, p(\boldsymbol{\theta})$$

$$\boldsymbol{\theta} = \{v_\epsilon, (\alpha_k, m_k, v_k), k = 1, \cdot, K\} \qquad p(\boldsymbol{\theta}) \quad \text{Conjugate priors}$$

- Direct computation and use of $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})$ is too complex
- Possible approximations :
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)
- Main idea in Variational Bayesian methods : Approximate

  $$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M}) \quad \text{by} \quad q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta})$$

  - Choice of approximation criterion : $KL(q : p)$
  - Choice of appropriate families of probability laws for $q_1(\boldsymbol{f})$, $q_2(\boldsymbol{z})$ and $q_3(\boldsymbol{\theta})$

MCMC based algorithm

$$p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta})\, p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta})\, p(\boldsymbol{z})\, p(\boldsymbol{\theta})$$

General scheme :

$$\widehat{\boldsymbol{f}} \sim p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{z}} \sim p(\boldsymbol{z}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\boldsymbol{g})$$

- Estimate $\boldsymbol{f}$ using $p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta})\, p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}})$
  Needs optimisation of a quadratic criterion.
- Estimate $\boldsymbol{z}$ using $p(\boldsymbol{z}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}})\, p(\boldsymbol{z})$
  Needs sampling of a Potts Markov field.
- Estimate $\boldsymbol{\theta}$ using
  $p(\boldsymbol{\theta}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}},\sigma_\epsilon^2\boldsymbol{I})\, p(\widehat{\boldsymbol{f}}|\widehat{\boldsymbol{z}},(m_k,v_k))\, p(\boldsymbol{\theta})$
  Conjugate priors $\longrightarrow$ analytical expressions.

Variational Bayes for Separable Approximation

- Objective : Approximate
  $p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})$ by $q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta})$
- Criterion :

$$\mathrm{KL}(q:p) = \sum_{\boldsymbol{z}} \iint q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}) \ln \frac{p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})}{q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta})} \,\mathrm{d}\boldsymbol{f}\,\mathrm{d}\boldsymbol{\theta}$$

- Algorithme : Iterate :

$$(\widehat{q}_1,\widehat{q}_2,\widehat{q}_3) = \arg\min_{(q_1,q_2,q_3)} \{\mathrm{KL}(q_1q_2q_3:p)\}$$

$$\begin{cases} \widehat{q}_1(\boldsymbol{f}) & \propto \exp\left\{\langle \ln p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\rangle_{\widehat{q}_2(\boldsymbol{z})\widehat{q}_3(\boldsymbol{\theta})}\right\} \\ \widehat{q}_2(\boldsymbol{z}) & \propto \exp\left\{\langle \ln p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\rangle_{\widehat{q}_1(\boldsymbol{f})\widehat{q}_3(\boldsymbol{\theta})}\right\} \\ \widehat{q}_3(\boldsymbol{\theta}) & \propto \exp\left\{\langle \ln p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\rangle_{\widehat{q}_1(\boldsymbol{f})\widehat{q}_2(\boldsymbol{z})}\right\} \end{cases}$$

Bayesian computation with Gauss-Markov-Potts prior models

$$p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta})\, p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta})\, p(\boldsymbol{z})\, p(\boldsymbol{\theta})$$

Approximations :

- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ iid :

$$\begin{aligned} q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g}) &= q_1(\boldsymbol{f}|\boldsymbol{z})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta}) \\ &= \sum_{\boldsymbol{r}} q_1(f(\boldsymbol{r})|z(\boldsymbol{r})) \sum_{\boldsymbol{r}} q_2(z(\boldsymbol{r})|z(\boldsymbol{r}')) \sum_{l} q_3(\theta_l) \end{aligned}$$

- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ Markov :

$$\begin{aligned} q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g}) &= q_1(\boldsymbol{f}|\boldsymbol{z})\, q_{2w}(\boldsymbol{z}_w)\, q_{2b}(\boldsymbol{z}_b)\, q_3(\boldsymbol{\theta}) \\ &= \sum_{\boldsymbol{r}} q_1(f(\boldsymbol{r})|z(\boldsymbol{r})) \\ & \sum_{\boldsymbol{r}\in\mathcal{R}_W} q_{2w}(z(\boldsymbol{r})|z(\boldsymbol{r}')) \sum_{\boldsymbol{r}\in\mathcal{R}_B} q_{2b}(z(\boldsymbol{r})|z(\boldsymbol{r}')) \\ & \sum_{l} q_3(\theta_l) \end{aligned}$$

Bayesian computation with Gauss-Markov-Potts prior models

In all cases :

$$
\begin{array}{lcl}
q(\boldsymbol{f}|\boldsymbol{z}) & = & \prod_{\boldsymbol{r}} \mathcal{N}(\tilde{\mu}_z(\boldsymbol{r}), \tilde{v}_z(\boldsymbol{r})) \\
p(\boldsymbol{z}) = \prod_{\boldsymbol{r}} \hat{\alpha}_k & = & \prod_{\boldsymbol{r}} p(z(\boldsymbol{r})|\tilde{z}(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \\
p(z(\boldsymbol{r})|\tilde{z}(\boldsymbol{r}')) & \propto & \tilde{c}\, \tilde{d}_1\, \tilde{d}_2(\boldsymbol{r})\, \tilde{e}(\boldsymbol{r}) \\
q(\theta_e|\tilde{\alpha}_e, \tilde{\beta}_e) & = & \mathcal{G}(\tilde{\alpha}_e, \tilde{\beta}_e) \\
q(m_k|\tilde{m}_k, \tilde{v}_k) & = & \mathcal{N}(\tilde{m}_k, \tilde{v}_k), \ \forall k \\
q(v_k^{-1}|\tilde{a}_k, \tilde{b}_k) & = & \mathcal{G}(\tilde{a}_k, \tilde{b}_k), \ \forall k \\
q(\boldsymbol{\alpha}) & \propto & \mathcal{D}(\tilde{\alpha}_1, \cdots, \tilde{\alpha}_K)
\end{array}
$$

The expressions of right hand sides depends on the case and are given in the paper.

Bayesian computation with Gauss-Markov-Potts prior models

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = \frac{p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta})\, p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{z})}{p(\boldsymbol{g}|\boldsymbol{\theta})}$$

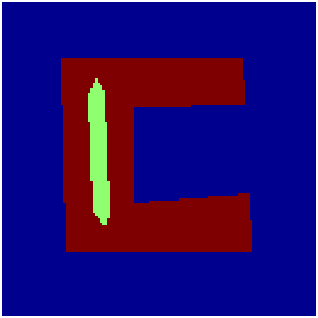
Approximations :

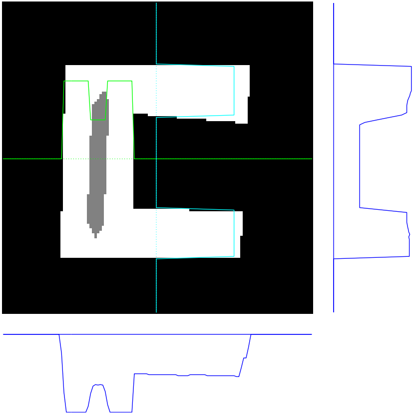
- $\boldsymbol{f}|\boldsymbol{z}$ iid, $\boldsymbol{z}$ iid :
  $q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f}|\boldsymbol{z})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta})$.
- $\boldsymbol{f}|\boldsymbol{z}$ iid, $\boldsymbol{z}$ Markov :
  $q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f}|\boldsymbol{z})\, q_{2w}(\boldsymbol{z}_w)\, q_{2b}(\boldsymbol{z}_b)\, q_3(\boldsymbol{\theta})$.
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\boldsymbol{z}$ iid :
  $q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = q_{1w}(\boldsymbol{f}_w|\boldsymbol{z})\, q_{1b}(\boldsymbol{f}_b|\boldsymbol{z})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta})$.
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\boldsymbol{z}$ iid :

$$q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = q_{1w}(\boldsymbol{f}_w|\boldsymbol{z})\, q_{1b}(\boldsymbol{f}_b|\boldsymbol{z})\, q_{2w}(\boldsymbol{z}_w)\, q_{2b}(\boldsymbol{z}_b)\, q_3(\boldsymbol{\theta})$$

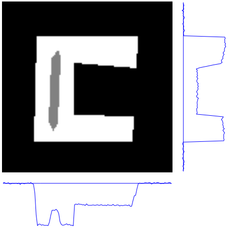
Application of CT in NDT

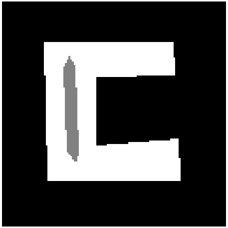
Reconstruction from only 2 projections

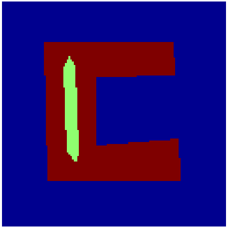


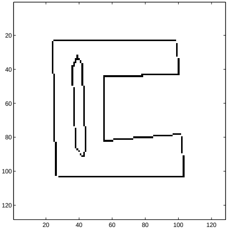


Application in CT









$\boldsymbol{g}|\boldsymbol{f}$

$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$

$\boldsymbol{g}|\boldsymbol{f} \sim \mathcal{N}(\boldsymbol{H}\boldsymbol{f}, v_\epsilon \boldsymbol{I})$

Gaussian

$\boldsymbol{f}|\boldsymbol{z}$

iid Gaussian

or

Gauss-Markov

$\boldsymbol{z}$

iid

or

Potts

$\boldsymbol{c}$

$c(\boldsymbol{r}) \in \{0, 1\}$

$1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$

binary

Proposed algorithm

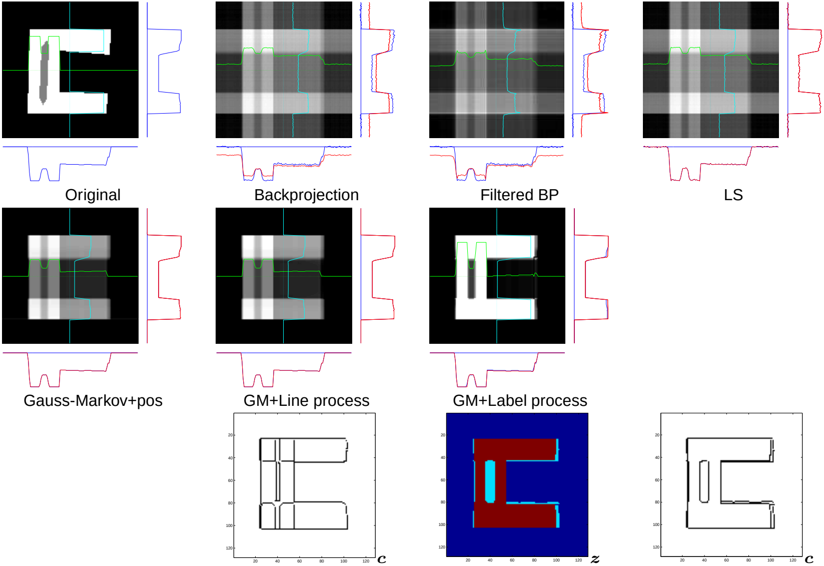
$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{\theta})$$

General scheme :

$$\widehat{\boldsymbol{f}} \sim p(\boldsymbol{f}|\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{z}} \sim p(\boldsymbol{z}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \boldsymbol{g})$$

- Estimate $\boldsymbol{f}$ using $p(\boldsymbol{f}|\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta})\, p(\boldsymbol{f}|\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}})$
  Needs optimisation of a quadratic criterion.
- Estimate $\boldsymbol{z}$ using $p(\boldsymbol{z}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}})\, p(\boldsymbol{z})$
  Needs sampling of a Potts Markov field.
- Estimate $\boldsymbol{\theta}$ using
  $p(\boldsymbol{\theta}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}}, \sigma_\epsilon^2 \boldsymbol{I})\, p(\widehat{\boldsymbol{f}}|\widehat{\boldsymbol{z}}, (m_k, v_k))\, p(\boldsymbol{\theta})$
  Conjugate priors $\longrightarrow$ analytical expressions.

Results

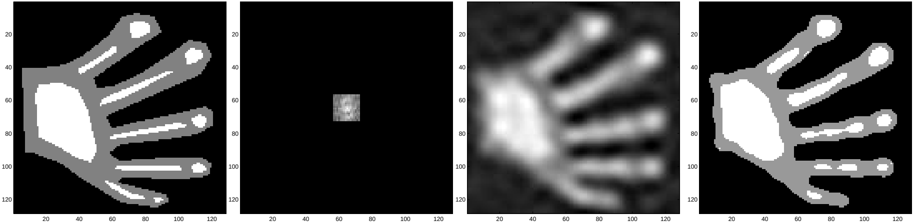

Original
Backprojection
Filtered BP
LS
Gauss-Markov+pos
GM+Line process
GM+Label process
c
z
c

Application in Microwave imaging

$$g(\boldsymbol{\omega}) = \int f(\boldsymbol{r}) \exp\{-j(\boldsymbol{\omega}.\boldsymbol{r})\} \; \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{\omega})$$

$$g(u,v) = \int f(x,y) \exp\{-j(ux+vy)\} \; \mathrm{d}x \; \mathrm{d}y + \epsilon(u,v)$$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$



$f(x,y)$ $\quad$ $g(u,v)$ $\quad$ $\widehat{f}$ IFT $\quad$ $\widehat{f}$ Proposed method

Conclusions

- Bayesian Inference for inverse problems
- Approximations (Laplace, MCMC, Variational)
- Gauss-Markov-Potts are useful prior models for images incorporating regions and contours
- Separable approximations for Joint posterior with Gauss-Markov-Potts priors
- Application in different CT (X ray, US, Microwaves, PET, SPECT)

Perspectives :

- Efficient implementation in 2D and 3D cases
- Evaluation of performances and comparison with MCMC methods
- Application to other linear and non linear inverse problems :
  (PET, SPECT or ultrasound and microwave imaging)

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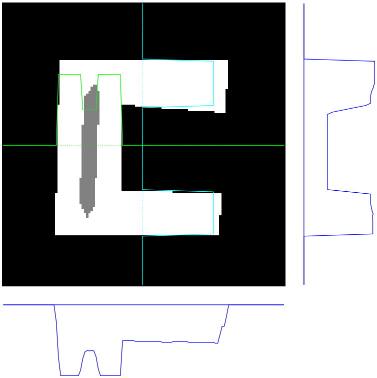
Questions and Discussions

- Thanks for your attentions
- ...
- ...
- Questions ?
- Discussions ?
- ...
- ...

CT from two projections = Joint distribution froms its marginals



$$g_1(x) = \int f(x,y)\, \mathrm{d}y$$
$$g_2(y) = \int f(x,y)\, \mathrm{d}x$$

Given the marginals $g_1(x)$ and $g_2(y)$ find the joint distribution $f(x,y)$

Infinite number of solutions

$$f(x,y) = g_1(x)\, g_2(y)\, \Omega(G_1(x), G_2(y))$$

$\Omega(u,v)$ is a Copula :

$\Omega(u,0) = 0,\ \Omega(u,1) = 1,$

$\Omega(0,u) = 0,\ \Omega(1,u) = 1$

$(x,y) \in [0,1]^2$, $G_1(x)$ and $G_2(y)$ are CDFs of $g_1(x)$ and $g_2(y)$

Example : $\Omega(u,v) = uv$

Any link between geometrical structure of f and Copula functions ?