A Gauss-Markov-Potts Prior model for images in Bayesian Computed Tomography

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Computed Tomography (CT) as an Invers Problem example
Classical methods: analytical and algebraic method
Probabilistic methods
Bayesian inference approach
Gauss-Markov-Potts prior models for images
Bayesian computation
VB with Gauss-Markov-Potts prior models
Application in Computed Tomography
Conclusions
Questions and Discussion
CT as a linear inverse problem

\[ g(s_i) = \int_{L_i} f(r) \, dl_i \quad \rightarrow \quad \text{Discretization} \quad \rightarrow \quad g = Hf + \epsilon \]
Classical methods in CT

\[ g(s_i) = \int_{L_i} f(r) \, dl_i \quad \longrightarrow \text{Discretization} \quad \longrightarrow \quad g = Hf + \epsilon \]

- \(H\) is a huge dimensional matrix of line integrals
- \(Hf\) is the forward or projection operation
- \(H^t g\) is the backward or backprojection operation
- \((H^t H)^{-1} H^t g\) is the filtered backprojection minimizing the LS criterion \(Q(f) = \|g - Hf\|^2\)
- Iterative methods:
  \[ \hat{f}^{(k+1)} = \hat{f}^{(k)} + \alpha^{(k)} H^t (g - H\hat{f}^{(k)}) \]
  try to minimize the Least squares criterion
- Other criteria:
  - Robust criteria: \(Q(f) = \sum_i \phi(\|g_i - [Hf]_i\|)\)
  - Likelihood: \(\mathcal{L}(f) = p(g|f)\)
  - Regularization: \(J(f) = \|g - Hf\|^2 + \lambda\|Df\|^2\).
Bayesian estimation approach

\[ g = H f + \epsilon \]

- **Observation model** \( M + \) Hypothesis on the noise \( \epsilon \)
  \[ p(g|f; M) = p_\epsilon(g - Hf) \]

- **A priori information**
  \[ p(f|M) \]

- **Bayes**
  \[ p(f|g; M) = \frac{p(g|f; M) p(f|M)}{p(g|M)} \]

**Link with regularisation**:

Maximum A Posteriori (MAP):

\[ \hat{f} = \arg \max_f \{p(f|g)\} = \arg \max_f \{p(g|f) \ p(f)\} \]

\[ = \arg \min_f \{-\ln p(g|f) - \ln p(f)\} \]

with \[ Q(g, Hf) = -\ln p(g|f) \] and \[ \lambda \Phi(f) = -\ln p(f) \]
Two main steps in the Bayesian approach

- Prior modeling
  - Separable:
    - Gaussian, Generalized Gaussian, Gamma, mixture of Gaussians, mixture of Gammas, ...
  - Markovian:
    - Gauss-Markov, GGM, ...
  - Separable or Markovian with hidden variables (contours, region labels)

- Choice of the estimator and computational aspects
  - MAP, Posterior mean, Marginal MAP
  - MAP needs optimization algorithms
  - Posterior mean needs integration methods
  - Marginal MAP needs integration and optimization
  - Approximations:
    - Gaussian approximation (Laplace)
    - Numerical exploration MCMC
    - Variational Bayes (Separable approximation)
Full Bayesian approach

\[ g = Hf + \epsilon \]

- Forward & errors model: \( p(g|f, \theta; M) \)
- Prior models: \( p(f|\theta; M) \) and \( p(\theta|M) \)
- Bayes: \( p(f, \theta|g; M) = \frac{p(g|f, \theta; M) p(f|\theta; M) p(\theta|M)}{p(g|M)} \)
- Joint MAP:
  \[
  (\hat{f}, \hat{\theta}) = \arg \max_{(f, \theta)} \{ p(f, \theta|g; M) \}
  \]
- Posterior means:
  \[
  \hat{f} = \int f \ p(f, \theta|g; M) \ df \ d\theta \\
  \hat{\theta} = \int \theta \ p(f, \theta|g; M) \ df \ d\theta
  \]
- Evidence of the model:
  \[
  p(g|M) = \int \int p(g|f, \theta; M)p(f|\theta; M)p(\theta|M) \ df \ d\theta
  \]
Bayesian Computation

- Direct computation and use of $p(f, \theta|g; \mathcal{M})$ is too complex

- Approximations:
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)

- Main idea:
  Approximate $p(f, \theta|g; \mathcal{M})$ by $q(f, \theta) = q_1(f) q_2(\theta)$

- Choice of approximation criterion:

  $$KL(q : p) = \int q \ln \frac{p}{q} = \int \int q_1(f) q_2(\theta) \ln \frac{q_1(f) q_2(\theta)}{p(f, \theta|g; \mathcal{M})} d.f\ d\theta$$

- Choice of appropriate families of probability laws for $q_1(f)$ and $q_2(\theta)$: Conjugate Exponential families
Approximation: Choice of criterion

$$\ln \rho(g|M) = \ln \int \int q(f, \theta) \frac{p(g, f, \theta|M)}{q(f, \theta)} \, df \, d\theta$$

$$\geq \int \int q(f, \theta) \ln \frac{p(g, f, \theta|M)}{q(f, \theta)} \, df \, d\theta.$$ 

$$\mathcal{F}(q) = \int \int q(f, \theta) \ln \frac{p(g, f, \theta|M)}{q(f, \theta)} \, df \, d\theta$$

$$\text{KL}(q : p) = - \int \int q(f, \theta) \ln \frac{p(f, \theta|g; M)}{q(f, \theta)} \, df \, d\theta$$

$$\ln \rho(g|M) = \mathcal{F}(q) + \text{KL}(q : p)$$
Approximation : Separable Approximation

\[ \ln p(g|\mathcal{M}) = \mathcal{F}(q) + \text{KL}(q : p) \]

\[ q(f, \theta) = q_1(f) q_2(\theta) \]

\[ (\hat{q}_1, \hat{q}_2) = \arg \min_{(q_1, q_2)} \{ \text{KL}(q_1 q_2 : p) \} = \arg \max_{(q_1, q_2)} \{ \mathcal{F}(q_1 q_2) \} \]

KL\((q_1 q_2 : p)\) is convex in \(q_1\) for a given \(q_2\) and vise versa :

\[
\begin{aligned}
\hat{q}_1 & = \arg \min_{q_1} \{ \text{KL}(q_1 \hat{q}_2 : p) \} = \arg \max_{q_1} \{ \mathcal{F}(q_1 \hat{q}_2) \} \\
\hat{q}_2 & = \arg \min_{q_2} \{ \text{KL}(\hat{q}_1 q_2 : p) \} = \arg \max_{q_2} \{ \mathcal{F}(\hat{q}_1 q_2) \}
\end{aligned}
\]

\[
\begin{aligned}
\hat{q}_1(f) & \propto \exp \left\{ \langle \ln p(g, f, \theta; \mathcal{M}) \rangle \hat{q}_2(\theta) \right\} \\
\hat{q}_2(\theta) & \propto \exp \left\{ \langle \ln p(g, f, \theta; \mathcal{M}) \rangle \hat{q}_1(f) \right\}
\end{aligned}
\]
Separable Approximation:
Choice of appropriate families for $q_1$ and $q_2$

**Degenerate case 1:** Joint MAP

$$
\begin{align*}
\hat{q}_1(f|\bar{f}) &= \delta(f - \bar{f}) \quad \Rightarrow \quad \bar{f} = \arg \max_f \left\{ p(f, \bar{\theta}|g; \mathcal{M}) \right\} \\
\hat{q}_2(\theta|\bar{\theta}) &= \delta(\theta - \bar{\theta}) \quad \Rightarrow \quad \bar{\theta} = \arg \max_\theta \left\{ p(\bar{f}, \theta|g; \mathcal{M}) \right\}
\end{align*}
$$

**Degenerate case 2:** Expectation-Maximisation

$$
\begin{align*}
\hat{q}_1(f) &\propto p(f|\theta, g) \quad \Rightarrow \quad Q(\theta, \bar{\theta}) = \langle \ln p(f, \theta|g; \mathcal{M}) \rangle_{q_1(f|\bar{\theta})} \\
\hat{q}_2(\theta|\bar{\theta}) &= \delta(\theta - \bar{\theta}) \quad \Rightarrow \quad \bar{\theta} = \arg \max_\theta \left\{ Q(\theta, \bar{\theta}) \right\}
\end{align*}
$$

**Appropriate choice for inverse problems**

$$
\begin{align*}
\hat{q}_1(f) &\propto p(f|\bar{\theta}, g; \mathcal{M}) \\
\hat{q}_2(\theta) &\propto p(\theta|f, g; \mathcal{M})
\end{align*}
$$

Accounting for uncertainties of $\bar{\theta}$ in $\bar{f}$ and vice versa.
Which images I am looking for?
Which image I am looking for?

Gauss-Markov

Generalized GM

Piecewise Gaussian

Mixture of GM
Gauss-Markov-Potts prior models for images

\[ f(r) \quad z(r) \quad c(r) = 1 - \delta(z(r) - z(r')) \]

\[ p(f(r)|z(r) = k, m_k, v_k) = \mathcal{N}(m_k, v_k) \]

\[ p(f(r)) = \sum_k P(z(r) = k) \mathcal{N}(m_k, v_k) \text{ Mixture of Gaussians} \]

- Separable iid hidden variables: \[ p(z) = \prod_r p(z(r)) \]
- Markovian hidden variables: \[ p(z) \text{ Potts-Markov}: \]

\[ p(z(r)|z(r'), r' \in \mathcal{V}(r)) \propto \exp \left\{ \gamma \sum_{r' \in \mathcal{V}(r)} \delta(z(r) - z(r')) \right\} \]

\[ p(z) \propto \exp \left\{ \gamma \sum_{r \in \mathcal{R}} \sum_{r' \in \mathcal{V}(r)} \delta(z(r) - z(r')) \right\} \]
Four different cases

To each pixel of the image is associated 2 variables $f(r)$ and $z(r)$

- $f|z$ Gaussian iid, $z$ iid :
  Mixture of Gaussians

- $f|z$ Gauss-Markov, $z$ iid :
  Mixture of Gauss-Markov

- $f|z$ Gaussian iid, $z$ Potts-Markov :
  Mixture of Independent Gaussians
  (MIG with Hidden Potts)

- $f|z$ Markov, $z$ Potts-Markov :
  Mixture of Gauss-Markov
  (MGM with hidden Potts)
Case 1: \( f \mid z \) Gaussian iid, \( z \) iid

Independent Mixture of Independent Gaussians (IMIG):

\[
p(f(r) \mid z(r) = k) = \mathcal{N}(m_k, v_k), \quad \forall r \in \mathcal{R}
\]

\[
p(f(r)) = \sum_{k=1}^{K} \alpha_k \mathcal{N}(m_k, v_k), \text{ with } \sum_k \alpha_k = 1.
\]

\[
p(z) = \prod_r p(z(r) = k) = \prod_r \alpha_k = \prod_k \alpha_k^{n_k}
\]

Noting

\[
m_z(r) = m_k, v_z(r) = v_k, \alpha_z(r) = \alpha_k, \forall r \in \mathcal{R}_k
\]

we have:

\[
p(f \mid z) = \prod_{r \in \mathcal{R}} \mathcal{N}(m_z(r), v_z(r))
\]

\[
p(z) = \prod_r \alpha_z(r) = \prod_k \alpha_k^{\sum_{r \in \mathcal{R}} \delta(z(r) - k)} = \prod_k \alpha_k^{n_k}
\]
Case 2: \( f \mid z \sim \text{Gauss-Markov, } z \sim \text{iid} \)

Independent Mixture of Gauss-Markov (IMGM):

\[
p(f(r) \mid z(r), z(r'), f(r'), r', r' \in \mathcal{V}(r)) = \mathcal{N}(\mu_z(r), \nu_z(r)), \forall r \in \mathcal{R}
\]

\[
\begin{align*}
\mu_z(r) &= \frac{1}{|\mathcal{V}(r)|} \sum_{r' \in \mathcal{V}(r)} \mu^*_z(r') \\
\mu^*_z(r') &= \delta(z(r') - z(r)) f(r') + (1 - \delta(z(r') - z(r))) m_z(r') \\
&= (1 - c(r')) f(r') + c(r') m_z(r')
\end{align*}
\]

\[
p(f \mid z) \propto \prod_r \mathcal{N}(\mu_z(r), \nu_z(r)) \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \Sigma_k)
\]

\[
p(z) = \prod_r \nu_z(r) = \prod_k \alpha_k^{n_k}
\]

with \( 1_k = 1, \forall r \in \mathcal{R}_k \) and \( \Sigma_k \) a covariance matrix \((n_k \times n_k)\).
Case 3: \( f \mid z \text{ Gauss iid, } z \text{ Potts} \)

Gauss iid as in Case 1:

\[
p(f \mid z) = \prod_{r \in \mathcal{R}} \mathcal{N}(m_z(r), v_z(r)) = \prod_{k} \prod_{r \in \mathcal{R}_k} \mathcal{N}(m_k, v_k)
\]

Potts-Markov

\[
p(z(r) \mid z(r'), r' \in \mathcal{V}(r)) \propto \exp \left\{ \gamma \sum_{r' \in \mathcal{V}(r)} \delta(z(r) - z(r')) \right\}
\]

\[
p(z) \propto \exp \left\{ \gamma \sum_{r \in \mathcal{R}} \sum_{r' \in \mathcal{V}(r)} \delta(z(r) - z(r')) \right\}
\]
Case 4: $f|z$ Gauss-Markov, $z$ Potts

Gauss-Markov as in Case 2:

$$p(f(r)|z(r), z(r'), f(r'), r' \in V(r)) = \mathcal{N}(\mu_z(r), \nu_z(r)), \forall r \in \mathcal{R}$$

$$\mu_z(r) = \frac{1}{|V(r)|} \sum_{r' \in V(r)} \mu^*_z(r')$$

$$\mu^*_z(r') = \delta(z(r') - z(r)) f(r') + (1 - \delta(z(r') - z(r))) m_z(r')$$

$$p(f|z) \propto \prod_r \mathcal{N}(\mu_z(r), \nu_z(r)) \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \Sigma_k)$$

Potts-Markov as in Case 3:

$$p(z) \propto \exp \left\{ \gamma \sum_{r \in \mathcal{R}} \sum_{r' \in V(r)} \delta(z(r) - z(r')) \right\}$$
Summary of the two proposed models

$$f \mid z \sim \text{Gaussian iid}$$
$$z \sim \text{Potts-Markov}$$
(MIG with Hidden Potts)

$$f \mid z \sim \text{Markov}$$
$$z \sim \text{Potts-Markov}$$
(MGM with hidden Potts)
Bayesian Computation

\[ p(f, z, \theta | g) \propto p(g | f, z, \nu) p(f | z, m, v) p(z | \gamma, \alpha) p(\theta) \]

\[ \theta = \{ \nu, (\alpha_k, m_k, \nu_k), k = 1, \ldots, K \} \quad p(\theta) \quad \text{Conjugate priors} \]

- Direct computation and use of \( p(f, z, \theta | g; \mathcal{M}) \) is too complex

- Possible approximations:
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)

- Main idea in Variational Bayesian methods:
  Approximate
  \[ p(f, z, \theta | g; \mathcal{M}) \quad \text{by} \quad q(f, z, \theta) = q_1(f) q_2(z) q_3(\theta) \]

  - Choice of approximation criterion: \( KL(q : p) \)
  - Choice of appropriate families of probability laws for \( q_1(f), q_2(z) \) and \( q_3(\theta) \)
**MCMC based algorithm**

\[
p(f, z, \theta | g) \propto p(g | f, z, \theta) p(f | z, \theta) p(z) p(\theta)
\]

General scheme:

\[
\hat{f} \sim p(f | \hat{z}, \hat{\theta}, g) \rightarrow \hat{z} \sim p(z | \hat{f}, \hat{\theta}, g) \rightarrow \hat{\theta} \sim (\theta | \hat{f}, \hat{z}, g)
\]

- Estimate \( f \) using \( p(f | \hat{z}, \hat{\theta}, g) \propto p(g | f, \theta) p(f | \hat{z}, \hat{\theta}) \)
  Needs optimisation of a quadratic criterion.

- Estimate \( z \) using \( p(z | \hat{f}, \hat{\theta}, g) \propto p(g | \hat{f}, \hat{z}, \hat{\theta}) p(z) \)
  Needs sampling of a Potts Markov field.

- Estimate \( \theta \) using
  \[
p(\theta | \hat{f}, \hat{z}, g) \propto p(g | \hat{f}, \sigma^2 \epsilon I) p(\hat{f} | \hat{z}, (m_k, v_k)) p(\theta)
  \]
  Conjugate priors \( \rightarrow \) analytical expressions.
Variational Bayes for Separable Approximation

- **Objective**: Approximate \( p(f, z, \theta | g; M) \) by \( q(f, z, \theta) = q_1(f) q_2(z) q_3(\theta) \)

- **Criterion**:
  \[
  \text{KL}(q : p) = \sum_z \int \int q(f, z, \theta) \ln \frac{p(f, z, \theta | g; M)}{q(f, z, \theta)} \, df \, d\theta
  \]

- **Algorithm**: Iterate:
  \[
  (\hat{q}_1, \hat{q}_2, \hat{q}_3) = \arg \min_{(q_1, q_2, q_3)} \{ \text{KL}(q_1 q_2 q_3 : p) \}
  \]
  \[
  \begin{align*}
  \hat{q}_1(f) & \propto \exp \left\{ \langle \ln p(g, f, \theta; M) \rangle \hat{q}_2(z) \hat{q}_3(\theta) \right\} \\
  \hat{q}_2(z) & \propto \exp \left\{ \langle \ln p(g, f, \theta; M) \rangle \hat{q}_1(f) \hat{q}_3(\theta) \right\} \\
  \hat{q}_3(\theta) & \propto \exp \left\{ \langle \ln p(g, f, \theta; M) \rangle \hat{q}_1(f) \hat{q}_2(z) \right\}
  \end{align*}
  \]
Bayesian computation with Gauss-Markov-Potts prior models

\[ p(f, z, \theta | g) \propto p(g | f, \theta) \ p(f | z, \theta) \ p(z) \ p(\theta) \]

Approximations :

- \( f | z \) Gaussian iid, \( z \) iid :

  \[ q(f, z, \theta | g) = q_1(f | z) \ q_2(z) \ q_3(\theta) \]
  \[ = \sum_r q_1(f(r) | z(r)) \ \sum_r q_2(z(r) | z(r')) \ \sum_l q_3(\theta_l) \]

- \( f | z \) Gaussian iid, \( z \) Markov :

  \[ q(f, z, \theta | g) = q_1(f | z) \ q_{2w}(z_w) \ q_{2b}(z_b) \ q_3(\theta) \]
  \[ = \sum_r q_1(f(r) | z(r)) \]
  \[ \sum_{r \in \mathcal{R}_W} q_{2w}(z(r) | z(r')) \ \sum_{r \in \mathcal{R}_B} q_{2b}(z(r) | z(r')) \ \sum_l q_3(\theta_l) \]
Bayesian computation with Gauss-Markov-Potts prior models

In all cases:

\[
q(f|z) = \prod_r \mathcal{N}(\tilde{\mu}_z(r), \tilde{\nu}_z(r))
\]

\[
p(z) = \prod_r \hat{\alpha}_k = \prod_r p(z(r)|\tilde{Z}(r'), r' \in \mathcal{V}(r))
\]

\[
p(z(r)|\tilde{Z}(r')) \propto \tilde{c} \tilde{d}_1 \tilde{d}_2(r) \tilde{e}(r)
\]

\[
q(\theta_e|\tilde{\alpha}_e, \tilde{\beta}_e) = \mathcal{G}(\tilde{\alpha}_e, \tilde{\beta}_e)
\]

\[
q(m_k|\tilde{m}_k, \tilde{\nu}_k) = \mathcal{N}(\tilde{m}_k, \tilde{\nu}_k), \forall k
\]

\[
q(v_k^{-1}|\tilde{a}_k, \tilde{b}_k) = \mathcal{G}(\tilde{a}_k, \tilde{b}_k), \forall k
\]

\[
q(\alpha) \propto \mathcal{D}(\tilde{\alpha}_1, \cdots, \tilde{\alpha}_K)
\]

The expressions of right hand sides depends on the case and are given in the paper.
Bayesian computation with Gauss-Markov-Potts prior models

\[ p(f, z, \theta | g) = \frac{p(g | f, \theta) p(f | z, \theta) p(z)}{p(g | \theta)} \]

Approximations:

- \( f | z \) iid, \( z \) iid:
  \[ q(f, z, \theta | g) = q_1(f | z) q_2(z) q_3(\theta). \]

- \( f | z \) iid, \( z \) Markov:
  \[ q(f, z, \theta | g) = q_1(f | z) q_{2w}(z_w) q_{2b}(z_b) q_3(\theta). \]

- \( f | z \) Markov, \( z \) iid:
  \[ q(f, z, \theta | g) = q_{1w}(f_w | z) q_{1b}(f_b | z) q_2(z) q_3(\theta). \]

- \( f | z \) Markov, \( z \) iid:
  \[ q(f, z, \theta | g) = q_{1w}(f_w | z) q_{1b}(f_b | z) q_{2w}(z_w) q_{2b}(z_b) q_3(\theta) \]
Application of CT in NDT

Reconstruction from only 2 projections
Application in CT

\[
g | f \\
g = H f + \epsilon \\
g | f \sim \mathcal{N}(H f, \sigma^2 I)
\]

Gaussian

\[
f | z \\
\text{iid Gaussian}
\]

or

Gauss-Markov

\[
z
\]

iid

or

Potts

\[
c
\]

\(c(r) \in \{0, 1\}\)

or

\(1 - \delta(z(r) - z(r'))\)

binary
Proposed algorithm

\[ p(f, z, \theta | g) \propto p(g | f, z, \theta) p(f | z, \theta) p(\theta) \]

General scheme :

\[ \hat{f} \sim p(f | \hat{z}, \hat{\theta}, g) \longrightarrow \hat{z} \sim p(z | \hat{f}, \hat{\theta}, g) \longrightarrow \hat{\theta} \sim (\theta | \hat{f}, \hat{z}, g) \]

- Estimate \( f \) using \( p(f | \hat{z}, \hat{\theta}, g) \propto p(g | f, \theta) p(f | \hat{z}, \hat{\theta}) \)
  Needs optimisation of a quadratic criterion.

- Estimate \( z \) using \( p(z | \hat{f}, \hat{\theta}, g) \propto p(g | \hat{f}, \hat{z}, \hat{\theta}) p(z) \)
  Needs sampling of a Potts Markov field.

- Estimate \( \theta \) using
  \[ p(\theta | \hat{f}, \hat{z}, g) \propto p(g | \hat{f}, \sigma^2_\epsilon I) p(\hat{f} | \hat{z}, (m_k, v_k)) p(\theta) \]
  Conjugate priors \( \longrightarrow \) analytical expressions.
Results

Original
Backprojection
Filtered BP
LS
Gauss-Markov+pos
GM+Line process
GM+Label process

z

\frac{20}{35}
Application in Microwave imaging

\[
g(\omega) = \int f(r) \exp \{-j(\omega \cdot r)\} \, dr + \epsilon(\omega)
\]

\[
g(u, v) = \int f(x, y) \exp \{-j(ux + vy)\} \, dx \, dy + \epsilon(u, v)
\]

\[
g = H f + \epsilon
\]

f(x, y)  
g(u, v)  
\(\hat{f}\) IFT  
\(\hat{f}\) Proposed method
Conclusions

- Bayesian Inference for inverse problems
- Approximations (Laplace, MCMC, Variational)
- Gauss-Markov-Potts are useful prior models for images incorporating regions and contours
- Separable approximations for Joint posterior with Gauss-Markov-Potts priors
- Application in different CT (X ray, US, Microwaves, PET, SPECT)

Perspectives :
- Efficient implementation in 2D and 3D cases
- Evaluation of performances and comparison with MCMC methods
- Application to other linear and non linear inverse problems:
  (PET, SPECT or ultrasound and microwave imaging)
Some references


Questions and Discussions

- Thanks for your attentions
- ...
- ...
- Questions ?
- Discussions ?
- ...
- ...
CT from two projections = Joint distribution froms its marginals

\[ g_1(x) = \int f(x, y) \, dy \]
\[ g_2(y) = \int f(x, y) \, dx \]

Given the marginals \( g_1(x) \) and \( g_2(y) \) find the joint distribution \( f(x, y) \)

Infinite number of solutions

\[ f(x, y) = g_1(x) g_2(y) \Omega(G_1(x), G_2(y)) \]

\( \Omega(u, v) \) is a Copula:
\[ \Omega(u, 0) = 0, \; \Omega(u, 1) = 1, \]
\[ \Omega(0, u) = 0, \; \Omega(1, u) = 1 \]

\((x, y) \in [0, 1]^2, \; G_1(x) \) and \( G_2(y) \) are CDFs of \( g_1(x) \) and \( g_2(y) \)
Example : \( \Omega(u, v) = uv \)

Any link between geometrical structure of \( f \) and Copula functions?