Sparsity in signal and image processing: from modeling and representation to reconstruction and processing

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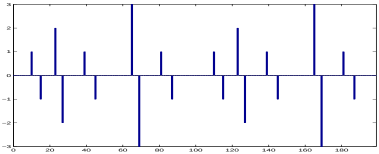
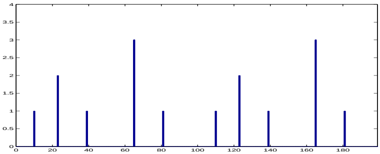
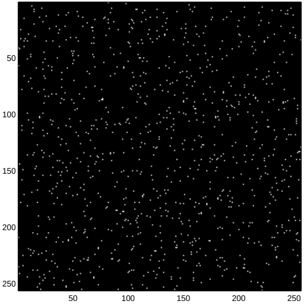
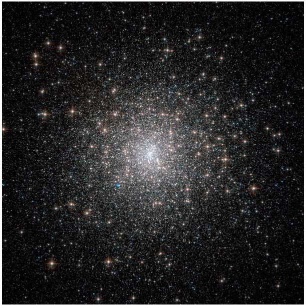




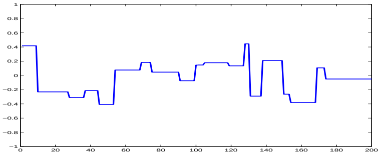
- Sparse images: Direct sparsity

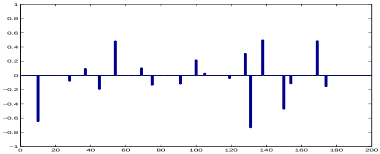




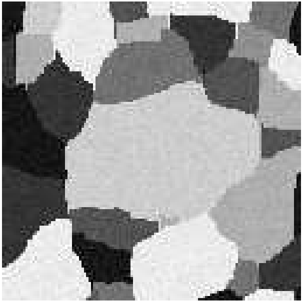
Sparse signals and images

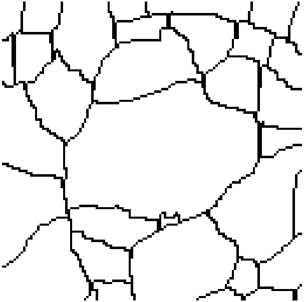
- Sparse signals in a Transform domain





- Sparse images in a Transform domain

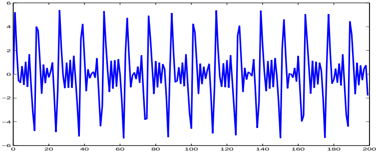




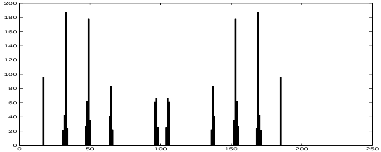
Sparse signals and images

- Sparse signals in Fourier domain

Time domain



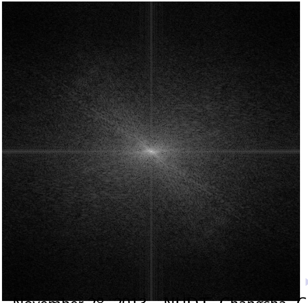
Fourier domain



- Sparse images in wavelet domain

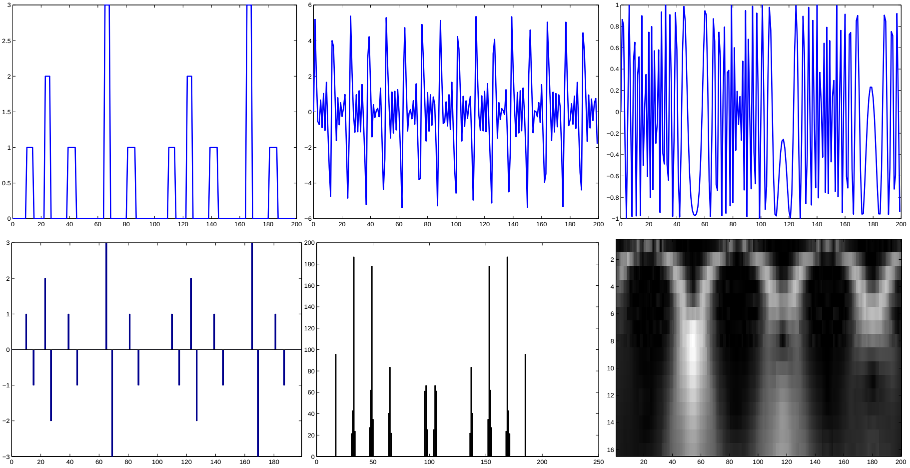
Space domain

Fourier domain

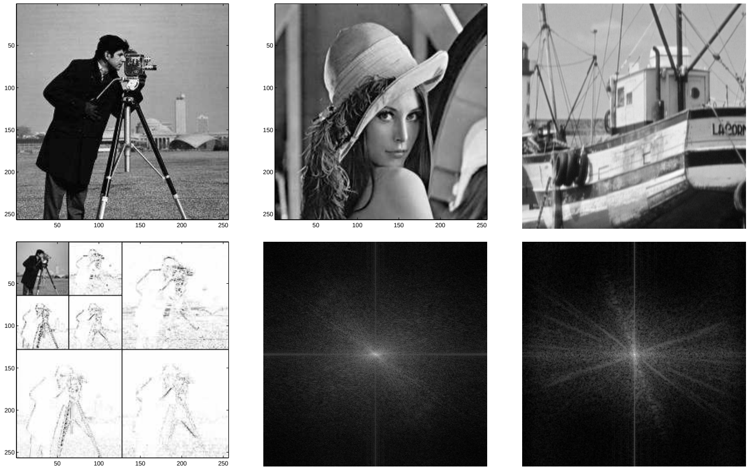


Sparse signals and images

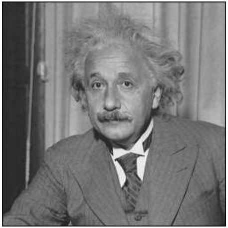
- Sparse signals: Sparsity in a Transform domaine



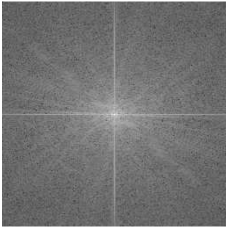
Sparse signals and images



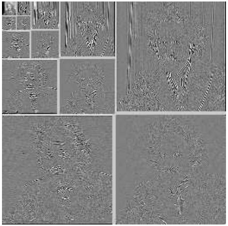
Sparse signals and images



Image



Fourier



Wavelets

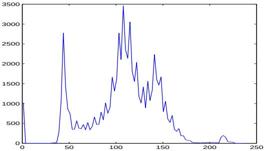
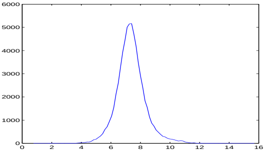
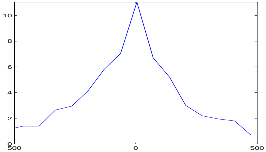


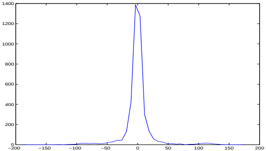
Image hist.



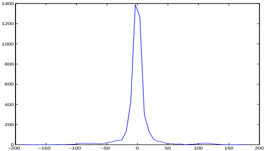
Fourier coeff. hist.



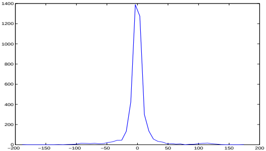
Wavelet coeff. hist.



bands 1-3



bands 4-6



bands 7-9

2. First ideas: some history

- 1948: Shannon:
  Sampling theorem and reconstruction of a band limited signal
- 1993-2007:
  - Mallat, Zhang, Candès, Romberg, Tao and Baraniuk:
    Non linear sampling, Compression and reconstruction,
  - Fuch: Sparse representation
  - Donoho, Elad, Tibshirani, Tropp, Duarte, Laska:
    Compressive Sampling, Compressive Sensing
- 2007-2012:
  Algorithms for sparse representation and compressive Sampling: Matching Pursuit (MP), Projection Pursuit Regression, Pure Greedy Algorithm, OMP, Basis Poursuit (BP), Dantzig Selector (DS), Least Absolute Shrinkage and Selection Operator (LASSO),...
- 2003-2012:
  Bayesian approach to sparse modeling
  Tipping, Bishop: Sparse Bayesian Learning,
  Relevance Vector Machine (RVM), ...

3. Modeling and representation

- Modeling via a basis (codebook, overcomplete dictionnary, Design Matrix)

$$g(t) = \sum_{j=1}^{N} f_j \, \phi_j(t), \; t = 1, \cdots, T \longrightarrow \boldsymbol{g} = \boldsymbol{\Phi}' \boldsymbol{f}$$

- When $T \geq N$

$$\widehat{f}_j = \arg\min_{f_j} \left\{ \sum_{t=1}^{T} \left| g(t) - \sum_{j=1}^{N} f_j \, \phi_j(t) \right|^2 \right\} \longrightarrow$$

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \left\{ \|\boldsymbol{g} - \boldsymbol{\Phi}' \boldsymbol{f}\|^2 \right\} = [\boldsymbol{\Phi}\boldsymbol{\Phi}']^{-1} \boldsymbol{\Phi} \boldsymbol{g}$$

- When orthogonal basis: $\boldsymbol{\Phi}\boldsymbol{\Phi}' = \boldsymbol{I} \longrightarrow \widehat{\boldsymbol{f}} = \boldsymbol{\Phi} \boldsymbol{g}$

$$\widehat{f}_j = \sum_{t=1}^{N} g(t) \, \phi_j(t) = < g(t), \phi_j(t) >$$

- Application in Compression, Transmission and Decompression

Modeling and representation

- When overcomplete basis $N > T$: Infinite number of solutions for $\boldsymbol{\Phi}'\boldsymbol{f} = \boldsymbol{g}$. We have to select one:

$$\widehat{\boldsymbol{f}} = \arg \min_{\boldsymbol{f}:\ \boldsymbol{\Phi}'\boldsymbol{f}=\boldsymbol{g}} \left\{\|\boldsymbol{f}\|_2^2\right\}$$

  or writing differently:

$$\text{minimize} \quad \|\boldsymbol{f}\|_2^2 \quad \text{subject to} \quad \boldsymbol{\Phi}'\boldsymbol{f} = \boldsymbol{g}$$

  resulting to:

$$\widehat{\boldsymbol{f}} = \boldsymbol{\Phi}[\boldsymbol{\Phi}'\boldsymbol{\Phi}]^{-1}\boldsymbol{g}$$

- Again if $\boldsymbol{\Phi}'\boldsymbol{\Phi} = \boldsymbol{I} \longrightarrow \widehat{\boldsymbol{f}} = \boldsymbol{\Phi}\boldsymbol{g}$.
- No real interest if we have to keep all the $N > T$ coefficients.
- Sparsity:

$$\text{minimize} \quad \|\boldsymbol{f}\|_0 \quad \text{subject to} \quad \boldsymbol{\Phi}'\boldsymbol{f} = \boldsymbol{g}$$

  or

$$\text{minimize} \quad \|\boldsymbol{f}\|_1 \quad \text{subject to} \quad \boldsymbol{\Phi}'\boldsymbol{f} = \boldsymbol{g}$$

Sparse decomposition

- Strict sparsity and exact reconstruction

$$\text{minimize} \quad \|\boldsymbol{f}\|_0 \quad \text{subject to} \quad \boldsymbol{\Phi}'\boldsymbol{f} = \boldsymbol{g}$$

$\|\boldsymbol{f}\|_0$ is the number of non-zero elements of $\boldsymbol{f}$

  - Matching Pursuit (MP) [Mallat & Zhang, 1993]
  - Orthogonal Matching Pursuit (OMP) [Lin, Huang et al., 1993]
  - Projection Pursuit Regression
  - Greedy Algorithms

- Sparsity enforcing and exact reconstruction

$$\text{minimize} \quad \|\boldsymbol{f}\|_1 \quad \text{subject to} \quad \boldsymbol{\Phi}'\boldsymbol{f} = \boldsymbol{g}$$

  - Basis Pursuit (BP)
  - Block Coordinate Relaxation

Sparse decomposition

- Strict sparsity and approximate reconstruction

$$\text{minimize} \quad \|\boldsymbol{f}\|_0 \quad \text{subject to} \quad \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 < c$$

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \left\{\|\boldsymbol{f}\|_0 + \mu\|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2\right\} = \arg\min_{\boldsymbol{f}} \left\{\|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{f}\|_0\right\}$$

- Sparsity enforcing and approximate reconstruction

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \left\{\|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{f}\|_1\right\}$$

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{f}\|_1 = \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \lambda\sum_j |f_j|$$

  - Main Algorithm: LASSO [Tibshirani 2003]

$$\text{minimize} \quad \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 \quad \text{subject to} \quad \|\boldsymbol{f}\|_1 < \tau$$

Sparse Decomposition Algorithms

- LASSO:

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \lambda \sum_j |f_j|$$

- Other Criteria
  - L_p

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \lambda_1 \sum_j |f_j|^p, \quad 1 < p \leq 2$$

  - Elastic net

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \sum_j \left(\lambda_1 |f_j| + \lambda_2 |f_j|^2\right)$$

  - Group LASSO

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \lambda_1 \sum_j |f_j| + \lambda_2 \sum_j |f_j - f_{j-1}|^2$$

  - Weighted L1:

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \lambda \sum_j |w_j f_j|$$

Multi Dimensional signals: PCA, SPCA, BSS, ...

$$g_i(t) = \sum_{j=1}^{N} \boldsymbol{\Phi}_{ij} f_j(t), \; i = 1, \cdots, M, \; t = 1, \cdots, T$$

$$\boldsymbol{g}(t) = \boldsymbol{\Phi}\, \boldsymbol{f}(t), \; t = 1, \cdots, T$$

$$\boldsymbol{G} = \boldsymbol{\Phi}\, \boldsymbol{F}, \quad \text{with } \boldsymbol{G}\ [M \times T], \quad \boldsymbol{\Phi}\ [M \times N], \quad \boldsymbol{F}\ [N \times T]$$

- $f_j(t)$ factors, sources, codes
- $\boldsymbol{\Phi}$ Loading matrix (Factor Analysis),
  Mixing matrix (Blind Sources Separation),
  Design matrix (Sparse coding, Compressed Sensing)
- Objective: Find $\boldsymbol{\Phi}$ and $f_j(t)$

$$J(\boldsymbol{f}(t), \boldsymbol{\Phi}) = \sum_t \|\boldsymbol{g}(t) - \boldsymbol{\Phi}\boldsymbol{f}(t)\|^2 + \lambda_1 \sum_i \sum_j |\boldsymbol{\Phi}_{ij}| + \lambda_2 \sum_t \sum_j |f_j(t)|$$

$$J(\boldsymbol{F}, \boldsymbol{\Phi}) = \|\boldsymbol{G} - \boldsymbol{\Phi}\boldsymbol{F}\|^2 + \lambda_1 \|\boldsymbol{\Phi}\|_1 + \lambda_2 \|\boldsymbol{F}\|_1$$

Matrix Decomposition or Approximation

- Matrix approximation:
  Find an approximate matrix $\widehat{\boldsymbol{G}} = \boldsymbol{\Phi F}$ for $\boldsymbol{G}$ with some degrees of sparsity in the elements of $\boldsymbol{\Phi}$ and $\boldsymbol{F}$.

$$J(\boldsymbol{F}, \boldsymbol{\Phi}) = \|\boldsymbol{G} - \boldsymbol{\Phi F}\|^2 + \lambda_1 \|\boldsymbol{\Phi}\|_1 + \lambda_2 \|\boldsymbol{F}\|_1$$

- Low rank Matrice decomposition:

$$\widehat{\boldsymbol{G}} = \sum_{k=1}^{K} d_k \boldsymbol{u}_k \boldsymbol{v}'_k = \boldsymbol{U D V}$$

  with some degrees of sparsity in the elements of $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$J(\boldsymbol{U}, \boldsymbol{V}) = \|\boldsymbol{G} - \boldsymbol{U D V}\|^2 + \lambda_1 \|\boldsymbol{U}\|_1 + \lambda_2 \|\boldsymbol{V}\|_1$$

4. Sparse decomposition: Bayesian MAP interpretation

- Sparsity enforcing and approximate reconstruction

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \left\{ \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \lambda \|\boldsymbol{f}\|_1 \right\}$$

- Equivalent to MAP estimate in a Bayesian approach
  - Bayesian approach:

$$p(\boldsymbol{f}|\boldsymbol{g}) = \frac{p(\boldsymbol{g}|\boldsymbol{f})\, p(\boldsymbol{f})}{p(\boldsymbol{g})} \propto p(\boldsymbol{g}|\boldsymbol{f})\, p(\boldsymbol{f})$$

$$\begin{cases} p(\boldsymbol{g}|\boldsymbol{f}) = \mathcal{N}(\boldsymbol{\Phi}'\boldsymbol{f}, \sigma_\epsilon^2) \propto \exp\left\{ \frac{-1}{2\sigma_\epsilon^2} \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 \right\} \\ p(\boldsymbol{f}) = \mathcal{DE}(-\gamma) \propto \exp\left\{ \gamma \|\boldsymbol{f}\|_1 \right\} \end{cases}$$

  - Maximum A Posteriori (MAP):

$$\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \left\{ p(\boldsymbol{f}|\boldsymbol{g}) \right\} = \arg\min_{\boldsymbol{f}} \left\{ J(\boldsymbol{f}) \right\}$$

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \lambda \|\boldsymbol{f}\|_1 \quad \text{with} \quad \lambda = 2\gamma\sigma_\epsilon^2$$

Sparse decomposition: Regularization or MAP

- Regularization: $J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{f}\|_1$
- With fixed λ: Find a good optimization algorithm
- How to choose λ ?
- L-Curve, Cross Validation, adhoc $\lambda = 1$, ...
- ---
- MAP: $J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{\Phi}'\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{f}\|_1$ with $\lambda = 2\gamma\sigma_\epsilon^2$
- How to estimate γ and σ_ϵ^2?
- Bayesian: Joint MAP, Expectation-Maximization, MCMC, Variational Bayesian Approximation,...
- Advantages of the Bayesian approach:
  - More probabilistic modeling for sparsity enforcing
  - Hyperparameter estimation
  - Uncertainty handling

5. Sparsity enforcing prior models

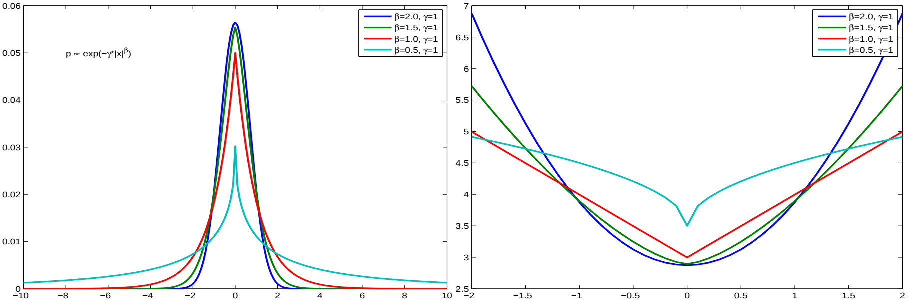
- Simple heavy tailed models:
  - Generalized Gaussian, Double Exponential
  - Student-t, Cauchy
  - Elastic net
  - Generalized hyperbolic
  - Symmetric Weibull, Symmetric Rayleigh
- Hierarchical mixture models:
  - Mixture of Gaussians
  - Bernoulli-Gaussian
  - Mixture of Gammas
  - Bernoulli-Gamma
  - Mixture of Dirichlet
  - Bernoulli-Multinomial

Simple heavy tailed models

- Generalized Gaussian, Double Exponential

$$p(\boldsymbol{f}|\gamma,\beta)=\prod_j \mathcal{GG}(f_j|\gamma,\beta) \propto \exp\left\{-\gamma \sum_j |f_j|^\beta\right\}$$

$\beta=1$ Double exponential or Laplace.
$0<\beta<2$ are of great interest for sparsity enforcing.



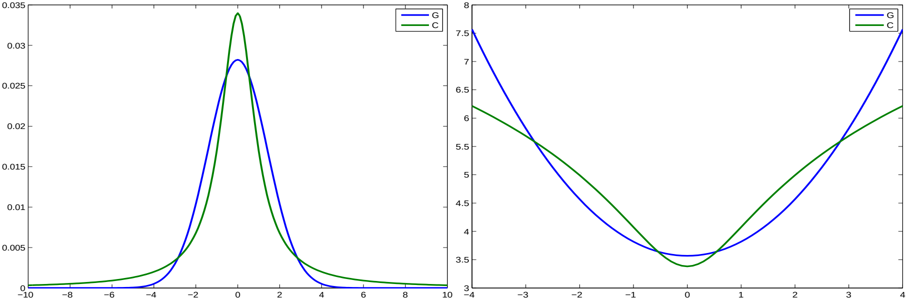
Generalized Gaussian family

Simple heavy tailed models

- Student-t and Cauchy models

$$p(\boldsymbol{f}|\nu) = \prod_j \mathcal{S}t(f_j|\nu) \propto \exp\left\{ -\frac{\nu+1}{2} \sum_j \log\left(1 + f_j^2/\nu\right) \right\}$$

Cauchy model is obtained when $\nu = 1$.

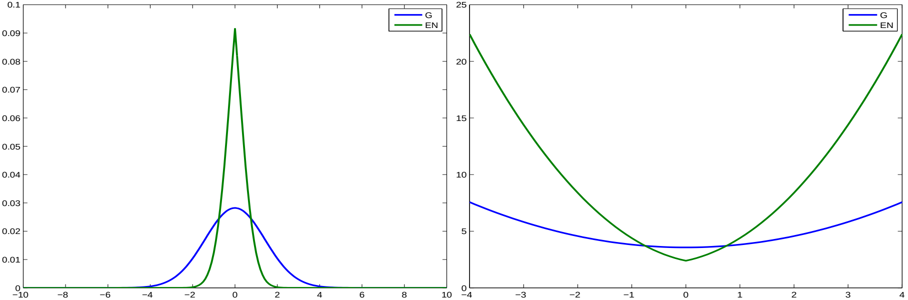


Student-t and Cauchy families

Simple heavy tailed models

- Elastic net prior model

$$p(\boldsymbol{f}|\nu) = \prod_j \mathcal{EN}(f_j|\nu) \propto \exp\left\{ -\sum_j (\gamma_1 |f_j| + \gamma_2 f_j^2) \right\}$$

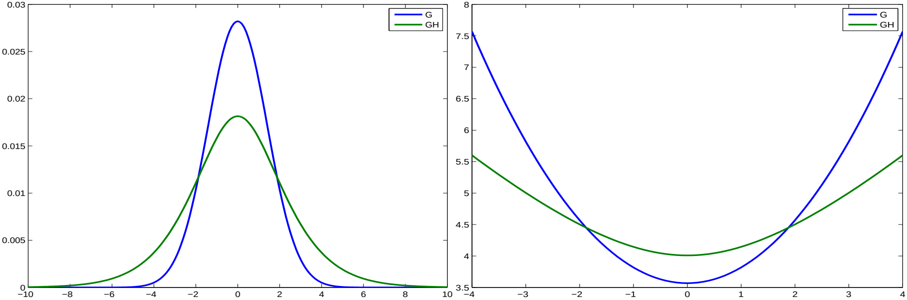


Elastic Net family

Simple heavy tailed models

- Generalized hyperbolic (GH) models

$$p(\boldsymbol{f}|\delta,\nu,\beta) = \prod_j (\delta^2 + f_j^2)^{(\nu-1/2)/2} \exp\{\beta x)\} \, K_{\nu-1/2}(\alpha\sqrt{\delta^2 + f_j^2})$$



Generalized hyperbolic family

Simple heavy tailed models

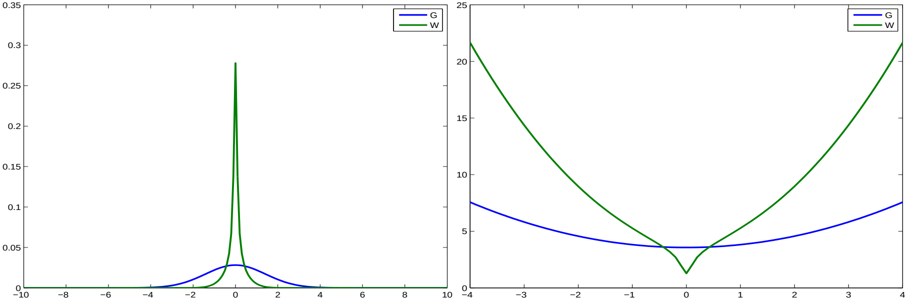
- Symmetric Weibull

$$p(\boldsymbol{f}|\gamma,\beta) = \prod_j \mathcal{W}(f_j|\gamma,\beta) \propto \exp\left\{-\gamma \sum_j |f_j|^\beta + (\beta-1)\log|f_j|\right\}$$

$\beta = 2$ is the Symmetric Rayleigh distribution.

$\beta = 1$ is the Double exponential and

$0 < \beta < 2$ are of great interest for sparsity enforcing.



Symmetric Weibull family

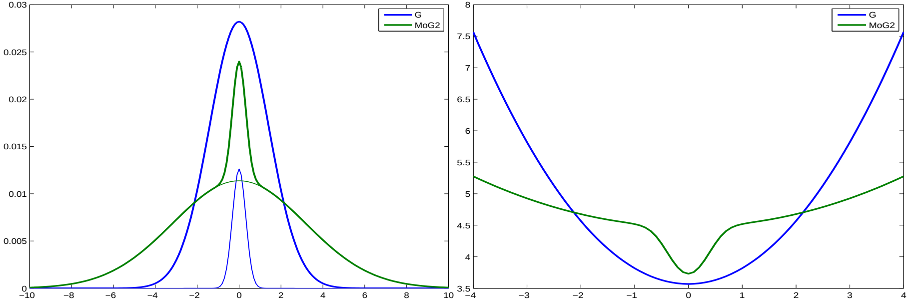
Mixture models

- Mixture of two Gaussians (MoG2) model

$$p(\boldsymbol{f}|\alpha, v_1, v_0) = \prod_j \left[\alpha \mathcal{N}(f_j|0, v_1) + (1-\alpha)\mathcal{N}(f_j|0, v_0)\right]$$

- Bernoulli-Gaussian (BG) model

$$p(\boldsymbol{f}|\alpha, v) = \prod_j p(f_j) = \prod_j \left[\alpha \mathcal{N}(f_j|0, v) + (1-\alpha)\delta(f_j)\right]$$



Mixture of 2 Gaussians families

- Mixture of Gammas

$$p(\boldsymbol{f}|\lambda, v_1, v_0) = \prod_j [\lambda \mathcal{G}(f_j|\alpha_1, \beta_1) + (1-\lambda)\mathcal{G}(f_j|\alpha_2, \beta_2)]$$

- Bernoulli-Gamma model

$$p(\boldsymbol{f}|\lambda, \alpha, \beta) = \prod_j [\lambda \mathcal{G}(f_j|\alpha, \beta) + (1-\lambda)\delta(f_j)]$$

- Mixture of Dirichlets model

$$p(\boldsymbol{f}|\lambda, \boldsymbol{H}_1, \boldsymbol{\alpha}_1, \boldsymbol{H}_2, \boldsymbol{\alpha}_2) = \prod_j [\lambda \mathcal{D}(f_j|\boldsymbol{H}_1, \boldsymbol{\alpha}_1) + (1-\lambda)\mathcal{D}(f_j|\boldsymbol{H}_2, \boldsymbol{\alpha}_2)]$$

$$\mathcal{D}(f_j|\boldsymbol{H}, \boldsymbol{\alpha}) = \prod_{k=1}^{K} \frac{\Gamma(\alpha)}{\Gamma(\alpha_0)\Gamma(\alpha_K)} a_k^{\alpha_k - 1}, \quad \alpha_k \geq 0, \quad a_k \geq 0$$

where $\boldsymbol{H} = \{a_1, \cdots, a_K\}$ and $\boldsymbol{\alpha} = \{\alpha_1, \cdots, \alpha_K\}$
with $\sum_k \alpha_k = \alpha$ and $\sum_k a_k = 1$.

- Bernoulli-Multinomial (BMultinomial) model

$$p(\boldsymbol{f}|\lambda, \boldsymbol{H}, \boldsymbol{\alpha}) = \prod_j [\lambda \delta(f_j) + (1-\lambda)\mathcal{M}ult(f_j|\boldsymbol{H}, \boldsymbol{\alpha})]$$

Hierarchical models and hidden variables

- All the mixture models and some of simple models can be modeled via hidden variables $\boldsymbol{z}$.

$$p(f) = \sum_{k=1}^{K} \alpha_k p_k(f) \longrightarrow \left\{ \begin{array}{l} p(f|z=k) = p_k(f), \\ P(z=k) = \alpha_k, \quad \sum_k \alpha_k = 1 \end{array} \right.$$

- Example 1: MoG model: $p_k(f) = \mathcal{N}(f|m_k, v_k)$
  2 Gaussians: $p_0 = \mathcal{N}(0, v_0), p_1 = \mathcal{N}(0, v_1), \alpha_0 = \lambda, \alpha_1 = 1 - \lambda$

$$p(f_j|\lambda, v_1, v_0) = \lambda \mathcal{N}(f_j|0, v_1) + (1-\lambda)\mathcal{N}(f_j|0, v_0)$$

$$\left\{ \begin{array}{l} p(f_j|z_j=0, v_0) = \mathcal{N}(f_j|0, v_0), \\ p(f_j|z_j=1, v_1) = \mathcal{N}(f_j|0, v_1), \end{array} \right. \text{ and } \left\{ \begin{array}{l} P(z_j=0) = \lambda, \\ P(z_j=1) = 1-\lambda \end{array} \right.$$

$$\left\{ \begin{array}{l} p(\boldsymbol{f}|\boldsymbol{z}) = \prod_j p(f_j|z_j) = \prod_j \mathcal{N}\left(f_j|0, v_{z_j}\right) \propto \exp\left\{-\frac{1}{2}\sum_j \frac{f_j^2}{v_{z_j}}\right\} \\ p(\boldsymbol{z}) = \lambda^{n_1}(1-\lambda)^{n_0}, \quad n_1 = \sum_j \delta(z_j - 1), \quad n_0 = \sum_j \delta(z_j) \end{array} \right.$$

Hierarchical models and hidden variables

- Example 2: Student-t model

$$\mathcal{S}t(f|\nu) \propto \exp\left\{-\frac{\nu+1}{2}\log\left(1+f^2/\nu\right)\right\}$$

- Infinite mixture

$$\mathcal{S}t(f|\nu) \propto= \int_0^\infty \mathcal{N}(f|,0,1/z)\,\mathcal{G}(z|\alpha,\beta)\,\mathrm{d}z, \quad \text{with } \alpha=\beta=\nu/2$$

$$\left\{\begin{array}{ll} p(\boldsymbol{f}|\boldsymbol{z}) & = \prod_j p(f_j|z_j) = \prod_j \mathcal{N}(f_j|0,1/z_j) \propto \exp\left\{-\frac{1}{2}\sum_j z_j f_j^2\right\} \\ p(\boldsymbol{z}|\alpha,\beta) & = \prod_j \mathcal{G}(z_j|\alpha,\beta) \propto \prod_j {z_j}^{(\alpha-1)} \exp\{-\beta z_j\} \\ & \propto \exp\left\{\sum_j (\alpha-1)\ln z_j - \beta z_j\right\} \\ p(\boldsymbol{f},\boldsymbol{z}|\alpha,\beta) & \propto \exp\left\{-\frac{1}{2}\sum_j z_j f_j^2 + (\alpha-1)\ln z_j - \beta z_j\right\} \end{array}\right.$$

Hierarchical models and hidden variables

- Example 3: Laplace (Double Exponential) model

$$\mathcal{DE}(f|a) = \frac{a}{2}\exp\{-a|f|\} = \int_0^\infty \mathcal{N}(f|,0,z)\,\mathcal{E}(z|a^2/2)\;\mathrm{d}z, \quad a>0$$

$$\begin{cases} p(\boldsymbol{f}|\boldsymbol{z}) & = \prod_j p(f_j|z_j) = \prod_j \mathcal{N}(f_j|0,z_j) \propto \exp\left\{-\frac{1}{2}\sum_j f_j^2/z_j\right\} \\ p(\boldsymbol{z}|\frac{a^2}{2}) & = \prod_j \mathcal{E}(z_j|\frac{a^2}{2}) \propto \exp\left\{\sum_j \frac{a^2}{2} z_j\right\} \\ p(\boldsymbol{f},\boldsymbol{z}|\frac{a^2}{2}) & \propto \exp\left\{-\frac{1}{2}\sum_j f_j^2/z_j + \frac{a^2}{2} z_j\right\} \end{cases}$$

- With these models we have:
  - Simple priors

$$p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\; p(\boldsymbol{f}|\boldsymbol{\theta}_2)\; p(\boldsymbol{\theta})$$

  - Hierarchical priors

$$p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\; p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta}_2)\; p(\boldsymbol{z}|\boldsymbol{\theta}_3)\; p(\boldsymbol{\theta})$$

Bayesian Computation and Algorithms

- When the expression of $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g})$ or of $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g})$ is obtained, we have following options:
- Joint MAP: (needs optimization algorithms)

$$(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}) = \arg \max_{(\boldsymbol{f}, \boldsymbol{\theta})} \{p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g})\}$$

- MCMC: Needs the expressions of the conditionals $p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta}, \boldsymbol{g})$, $p(\boldsymbol{z}|\boldsymbol{f}, \boldsymbol{\theta}, \boldsymbol{g})$, and $p(\boldsymbol{\theta}|\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{g})$
- Variational Bayesian Approximation (VBA): Approximate $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g})$ by a separable one

$$q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta})$$

and do any computations with these separable ones.

Joint MAP

$$p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)\, p(\boldsymbol{\theta})$$

- Objective:

$$(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f},\boldsymbol{\theta})} \{p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g})\}$$

- Alternate optimization:

$$\left\{\begin{array}{l} \widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \left\{p(\boldsymbol{f},\widehat{\boldsymbol{\theta}}|\boldsymbol{g})\right\} \\ \widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \left\{p(\widehat{\boldsymbol{f}},\boldsymbol{\theta}|\boldsymbol{g})\right\} \end{array}\right.$$

$$\begin{array}{ccccc} \boldsymbol{\theta}^{(0)} \longrightarrow & \widehat{\boldsymbol{\theta}} \longrightarrow & \boxed{\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \left\{p(\boldsymbol{f}|\widehat{\boldsymbol{\theta}},\boldsymbol{g})\right\}} & \longrightarrow & \widehat{\boldsymbol{f}} \\ & \uparrow & & & \downarrow \\ & \widehat{\boldsymbol{\theta}} \longleftarrow & \boxed{\widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \left\{p(\boldsymbol{\theta}|\widehat{\boldsymbol{f}},\boldsymbol{g})\right\}} & \longleftarrow & \widehat{\boldsymbol{f}} \end{array}$$

- Uncertainties are not propagated.

MCMC based algorithm

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{z})\, p(\boldsymbol{\theta})$$

General scheme (Gibbs Sampling):

- Generate samples from the conditionals:

$$\widehat{\boldsymbol{f}} \sim p(\boldsymbol{f}|\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{z}} \sim p(\boldsymbol{z}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \boldsymbol{g})$$

- Waite for convergency
- Compute empirical statistics (means, modes, variances) from the samples

$$\mathsf{E}\{\boldsymbol{f}\} \approx \frac{1}{N} \sum_{n} \boldsymbol{f}^{(n)}$$

Variational Bayesian Approximation

- Approximate $p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g})$ by $q(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f}|\boldsymbol{g})\, q_2(\boldsymbol{\theta}|\boldsymbol{g})$ and then continue computations.
- Criterion $\mathsf{KL}(q(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g}) : p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g}))$

$$\mathsf{KL}(q:p) = \iint q \ln \frac{q}{p} = \iint q_1 q_2 \ln \frac{q_1 q_2}{p}$$

- Iterative algorithm $q_1 \longrightarrow q_2 \longrightarrow q_1 \longrightarrow q_2, \cdots$

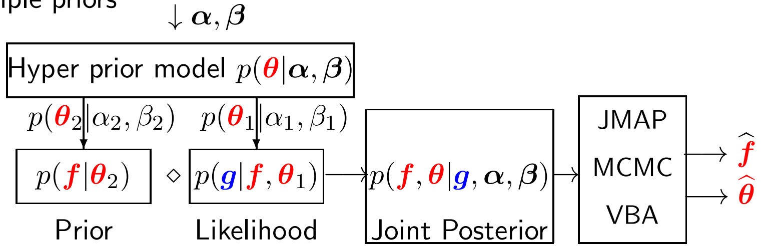
$$\left\{ \begin{array}{ll} q_1(\boldsymbol{f}) & \propto \exp\left\{ \langle \ln p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\rangle_{q_2(\boldsymbol{\theta})} \right\} \\ q_2(\boldsymbol{\theta}) & \propto \exp\left\{ \langle \ln p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\rangle_{q_1(\boldsymbol{f})} \right\} \end{array} \right.$$

$$\begin{array}{ccccc} \widehat{q}_2^{(0)} \longrightarrow \widehat{q}_2 \longrightarrow & \boxed{q_1(\boldsymbol{f}) \propto \exp\left\{ \langle \ln p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\rangle_{q_2} \right\}} & \longrightarrow \widehat{q}_1(\boldsymbol{f}) \longrightarrow \widehat{\boldsymbol{f}} \\ \uparrow & & \downarrow \\ \widehat{\boldsymbol{\theta}} \leftarrow \widehat{q}_2(\boldsymbol{\theta}) \longleftarrow & \boxed{q_2(\boldsymbol{\theta}) \propto \exp\left\{ \langle \ln p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\rangle_{q_1} \right\}} & \longleftarrow \widehat{q}_1 \end{array}$$

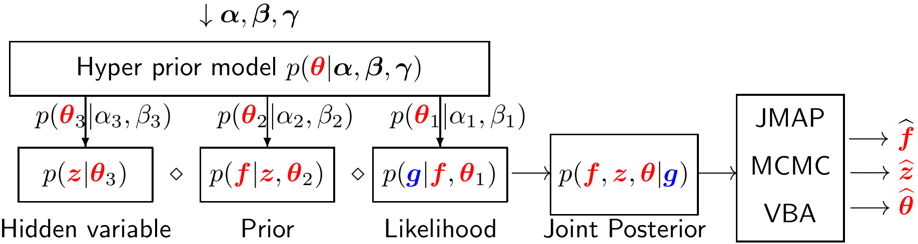
- Uncertainties are propagated (Message Passing methods)

Summary of Bayesian approach

- Simple priors



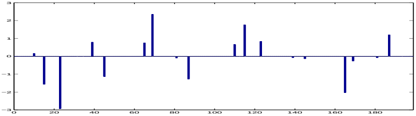
- Hierarchical priors



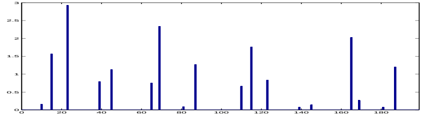
Advantages of the Bayesian Approach

- More possibilities to model sparsity
- More tools to handle hyperparameters
- More tools to account for uncertainties
- More possibilities to understand and to control many ad hoc deterministic algorithms
- Hierarchical models give still more modeling possibilities
  - Bernouilli-Gaussian: strict sparsity
  - Bernouilli-Gamma: strict sparsity + positivity
  - Bernouilli-Multinomial: strict sparsity + discrete values (finite states)
  - Independent Mixture models: sparsity enforcing
  - Mixture of multivariate models: group sparsity enforcing
  - Gauss-Markov-Potts models: sparsity in transform domains

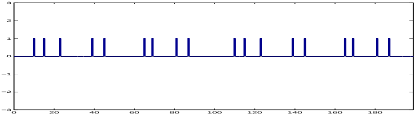
Examples of Hierarchical models



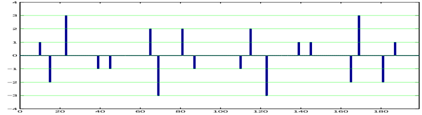
Bernouilli-Gaussian



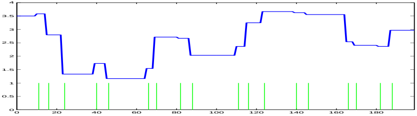
Bernouilli-Gamma



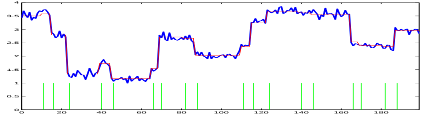
Bernouilli-Binomial



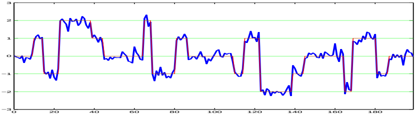
Bernouilli-Multinomial



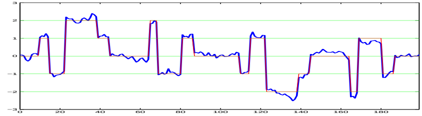
Piecewise constant



Piecewise Gaussian

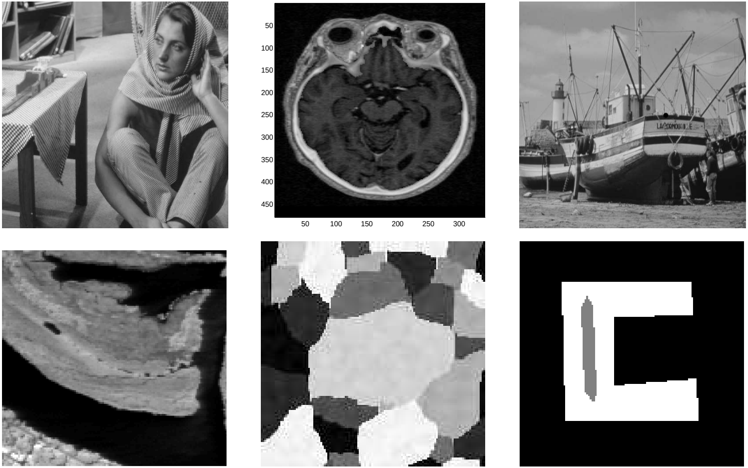


Gauss-Markov-Potts 1

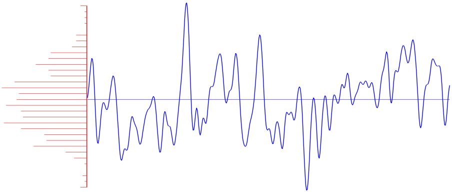


Gauss-Markov-Potts 2

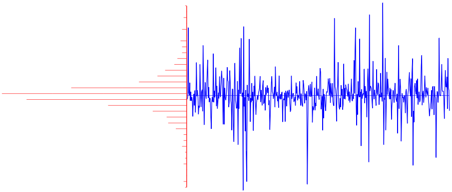
Which class of images I am looking for?



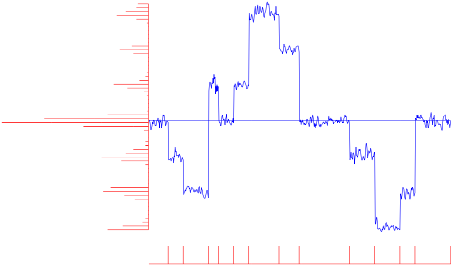
Which class of signals I am looking for?



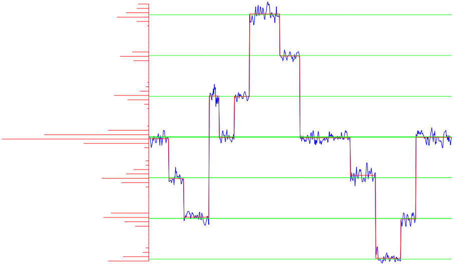
Gauss-Markov



Generalized GM

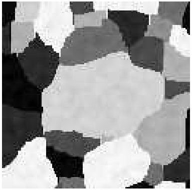


Piecewize Gaussian

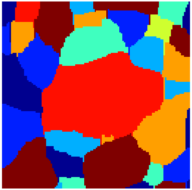


Mixture of GM: Gauss-Markov-Potts

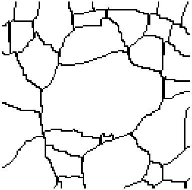
Gauss-Markov-Potts prior models for images



$f(\boldsymbol{r})$



$z(\boldsymbol{r})$



$q(\boldsymbol{r}) = 1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k, m_k, v_k) = \mathcal{N}(m_k, v_k)$$

$$p(f(\boldsymbol{r})) = \sum_k P(z(\boldsymbol{r}) = k)\,\mathcal{N}(m_k, v_k) \quad \text{Mixture of Gaussians}$$

- Separable iid hidden variables: $p(\boldsymbol{z}) = \prod_{\boldsymbol{r}} p(z(\boldsymbol{r}))$
- Markovian hidden variables: $p(\boldsymbol{z})$ Potts-Markov:

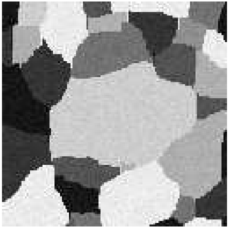
$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r} \in \mathcal{R}} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

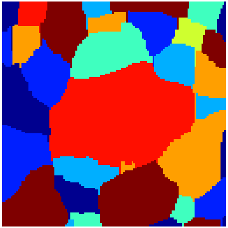
Four different cases

To each pixel of the image is associated 2 variables $f(\boldsymbol{r})$ and $z(\boldsymbol{r})$

- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ iid :
  Mixture of Gaussians
- $\boldsymbol{f}|\boldsymbol{z}$ Gauss-Markov, $\boldsymbol{z}$ iid :
  Mixture of Gauss-Markov
- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ Potts-Markov :
  Mixture of Independent Gaussians
  (MIG with Hidden Potts)
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\boldsymbol{z}$ Potts-Markov :
  Mixture of Gauss-Markov
  (MGM with hidden Potts)



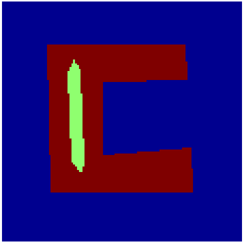
$f(\boldsymbol{r})$

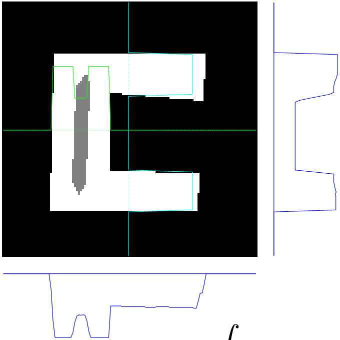


$z(\boldsymbol{r})$

Application of CT in NDT

Reconstruction from only 2 projections



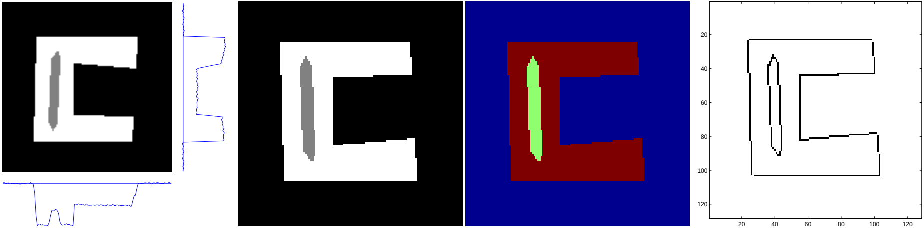


$$g_1(x) = \int f(x,y) \, \mathrm{d}y, \qquad g_2(y) = \int f(x,y) \, \mathrm{d}x$$

- Given the marginals $g_1(x)$ and $g_2(y)$ find the joint distribution $f(x,y)$.
- Infinite number of solutions : $f(x,y) = g_1(x) \, g_2(y) \, \Omega(x,y)$
  $\Omega(x,y)$ is a Copula:

$$\int \Omega(x,y) \, \mathrm{d}x = 1 \quad \text{and} \quad \int \Omega(x,y) \, \mathrm{d}y = 1$$

Application in CT



$\boldsymbol{g}\vert\boldsymbol{f}$	$\boldsymbol{f}\vert\boldsymbol{z}$	$\boldsymbol{z}$	$\boldsymbol{c}$
$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$	iid Gaussian	iid	$q(\boldsymbol{r}) \in \{0, 1\}$
$\boldsymbol{g}\vert\boldsymbol{f} \sim \mathcal{N}(\boldsymbol{H}\boldsymbol{f}, \sigma_\epsilon^2 \boldsymbol{I})$	or	or	$1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$
Gaussian	Gauss-Markov	Potts	binary

Proposed algorithm

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta} | \boldsymbol{g}) \propto p(\boldsymbol{g} | \boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{f} | \boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{\theta})$$

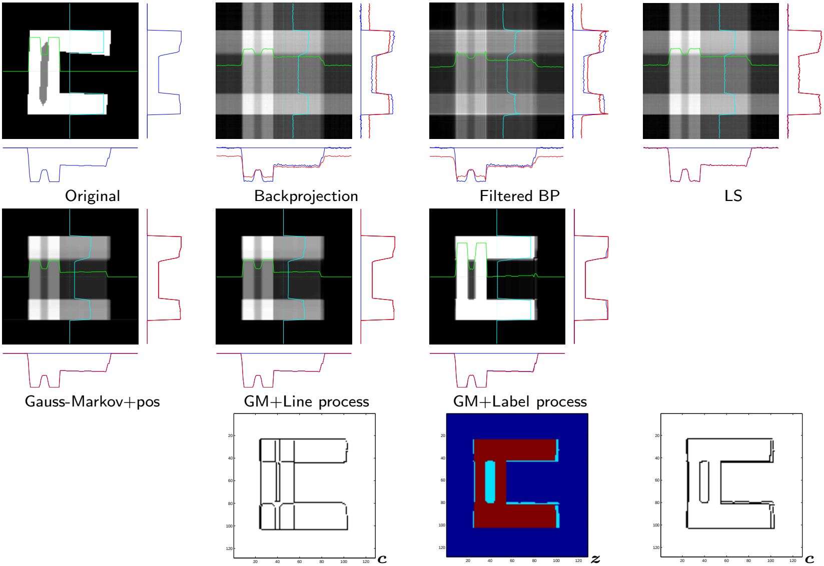
General scheme:

$$\widehat{\boldsymbol{f}} \sim p(\boldsymbol{f} | \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{z}} \sim p(\boldsymbol{z} | \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta} | \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \boldsymbol{g})$$

Iterative algorithme:

- Estimate $\boldsymbol{f}$ using $p(\boldsymbol{f} | \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \propto p(\boldsymbol{g} | \boldsymbol{f}, \boldsymbol{\theta})\, p(\boldsymbol{f} | \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}})$
  Needs optimisation of a quadratic criterion.
- Estimate $\boldsymbol{z}$ using $p(\boldsymbol{z} | \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \propto p(\boldsymbol{g} | \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}})\, p(\boldsymbol{z})$
  Needs sampling of a Potts Markov field.
- Estimate $\boldsymbol{\theta}$ using
  $p(\boldsymbol{\theta} | \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \boldsymbol{g}) \propto p(\boldsymbol{g} | \widehat{\boldsymbol{f}}, \sigma_\epsilon^2 \boldsymbol{I})\, p(\widehat{\boldsymbol{f}} | \widehat{\boldsymbol{z}}, (m_k, v_k))\, p(\boldsymbol{\theta})$
  Conjugate priors $\longrightarrow$ analytical expressions.

Results



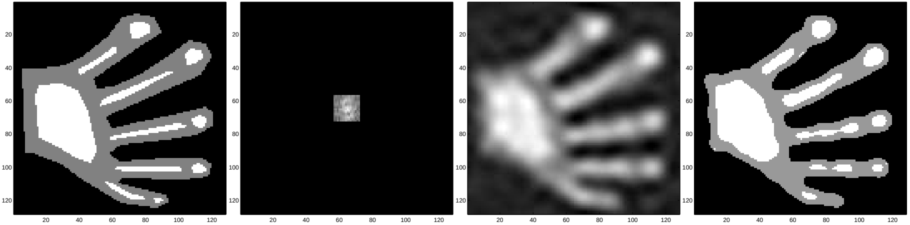
Original
Backprojection
Filtered BP
LS
Gauss-Markov+pos
GM+Line process
GM+Label process
c
z
c

Application in Microwave imaging

$$g(\boldsymbol{\omega}) = \int f(\boldsymbol{r}) \exp\{-j(\boldsymbol{\omega}.\boldsymbol{r})\} \; \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{\omega})$$

$$g(u,v) = \iint f(x,y) \exp\{-j(ux+vy)\} \; \mathrm{d}x \; \mathrm{d}y + \epsilon(u,v)$$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$



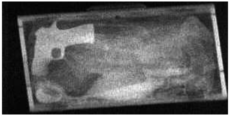
$f(x,y)$ $\quad$ $g(u,v)$ $\quad$ $\widehat{\boldsymbol{f}}$ IFT $\quad$ $\widehat{\boldsymbol{f}}$ Proposed method

Images fusion and joint segmentation

(with O. Féron)

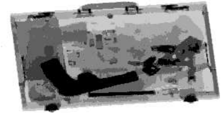
$$\begin{cases} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_{ik}, \sigma_{i\,k}^2) \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}) \end{cases}$$

$\boldsymbol{g}_1$

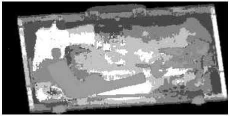


$\boldsymbol{g}_2$

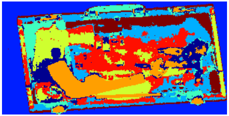
$\longrightarrow$



$\widehat{\boldsymbol{f}}_1$



$\widehat{\boldsymbol{f}}_2$

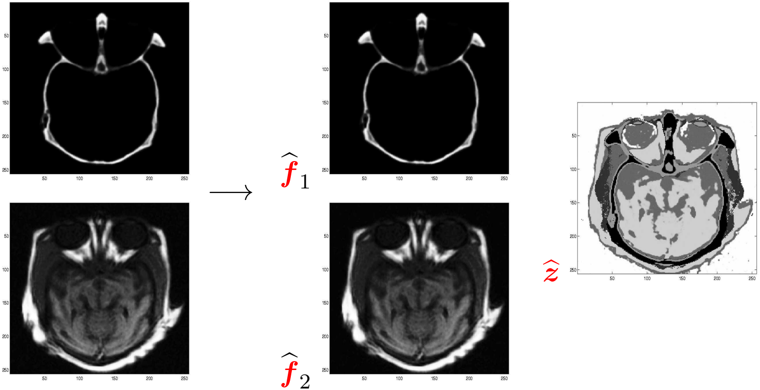


$\widehat{\boldsymbol{z}}$

Data fusion in medical imaging

(with O. Féron)

$$
\begin{cases}
g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\
p(f_i(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma_{i\,k}^2) \\
p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z})
\end{cases}
$$

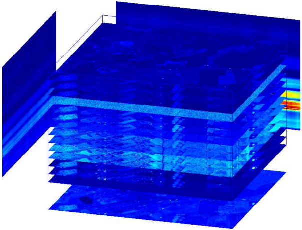
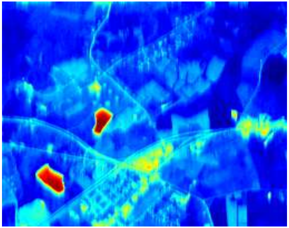
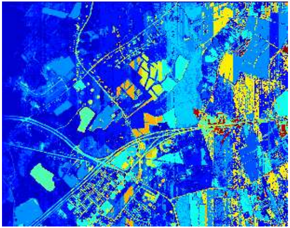


$\boldsymbol{g}_1$ $\boldsymbol{g}_2$ $\longrightarrow$ $\widehat{\boldsymbol{f}}_1$ $\widehat{\boldsymbol{f}}_2$ $\widehat{\boldsymbol{z}}$

Joint segmentation of hyper-spectral images

(with N. Bali & A. Mohammadpour)

$$
\begin{cases}
g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\
p(f_i(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_{ik}, \sigma_{i\,k}^2), \quad k = 1, \cdots, K \\
p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}) \\
m_{ik} \text{ follow a Markovian model along the index } i
\end{cases}
$$

Segmentation of a video sequence of images

(with P. Brault)

$$
\begin{cases}
g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\
p(f_i(\boldsymbol{r})|z_i(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma_{i\,k}^2), \quad k = 1, \cdots, K \\
p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}_i) \\
z_i(\boldsymbol{r}) \ \text{follow a Markovian model along the index} \ i
\end{cases}
$$

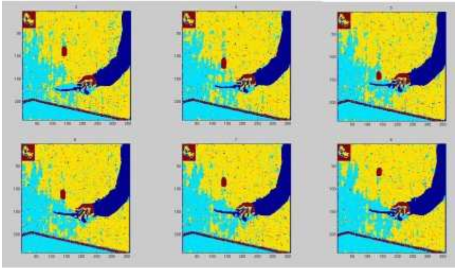
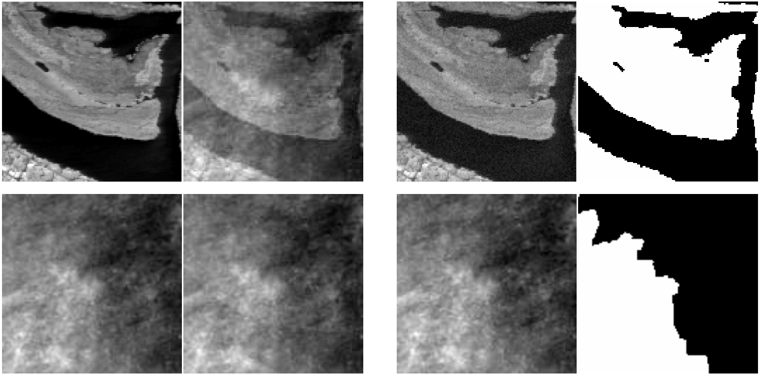


Image separation in Sattelite imaging

(with H. Snoussi & M. Ichir)

$$\begin{cases} g_i(\boldsymbol{r}) = \sum_{j=1}^{N} A_{ij} f_j(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_j(\boldsymbol{r}) | z_j(\boldsymbol{r}) = k) = \mathcal{N}(m_{j\,k}, \sigma^2_{j\,k}) \\ p(A_{ij}) = \mathcal{N}(A_{0ij}, \sigma^2_{0ij}) \end{cases}$$



$\boldsymbol{f}$ $\boldsymbol{g}$ $\widehat{\boldsymbol{f}}$ $\widehat{\boldsymbol{z}}$

Conclusions

- Sparsity: a great property to use in signal and image processing
- Origine: Sampling theory and reconstruction, modeling and representation Compressed Sensing, Approximation theory
- Deterministic Algorithms: Optimization of a two termes criterion, penalty term, regularization term
- Probabilistic: Bayesian approach
- Sprasity enforcing priors: Simple heavy tailed and Hierarchical with hidden variables.
- Gauss-Markov-Potts models for images incorporating hidden regions and contours
- Main Bayesian computation tools: JMAP, MCMC and VBA
- Application in different imaging system (X ray CT, Microwaves, PET, ultrasound and microwave imaging)

Current Projects and Perspectives :

- Efficient implementation in 2D and 3D cases
- Evaluation of performances and comparison between MCMC and VBA methods

Some references

- A. Mohammad-Djafari (Ed.) Problèmes inverses en imagerie et en vision (Vol. 1 et 2), Hermes-Lavoisier, Traité Signal et Image, IC2, 2009,
- A. Mohammad-Djafari (Ed.) Inverse Problems in Vision and 3D Tomography, ISTE, Wiley and sons, ISBN: 9781848211728, December 2009, Hardback, 480 pp.
- H. Ayasso and Ali Mohammad-Djafari Joint NDT Image Restoration and Segmentation using Gauss-Markov-Potts Prior Models and Variational Bayesian Computation, To appear in IEEE Trans. on Image Processing, TIP-04815-2009.R2, 2010.
- H. Ayasso, B. Duchene and A. Mohammad-Djafari, Bayesian Inversion for Optical Diffraction Tomography Journal of Modern Optics, 2008.
- A. Mohammad-Djafari, Gauss-Markov-Potts Priors for Images in Computer Tomography Resulting to Joint Optimal Reconstruction and segmentation, International Journal of Tomography & Statistics 11: W09. 76-92, 2008.
- A Mohammad-Djafari, Super-Resolution : A short review, a new method based on hidden Markov modeling of HR image and future challenges, The Computer Journal doi:10,1093/comjnl/bxn005:, 2008.
- O. Féron, B. Duchène and A. Mohammad-Djafari, Microwave imaging of inhomogeneous objects made of a finite number of dielectric and conductive materials from experimental data, Inverse Problems, 21(6):95-115, Dec 2005.
- M. Ichir and A. Mohammad-Djafari, Hidden markov models for blind source separation, IEEE Trans. on Signal Processing, 15(7):1887-1899, Jul 2006.
- F. Humblot and A. Mohammad-Djafari, Super-Resolution using Hidden Markov Model and Bayesian Detection Estimation Framework, EURASIP Journal on Applied Signal Processing, Special number on Super-Resolution Imaging: Analysis, Algorithms, and Applications:ID 36971, 16 pages, 2006.
- O. Féron and A. Mohammad-Djafari, Image fusion and joint segmentation using an MCMC algorithm, Journal of Electronic Imaging, 14(2):paper no. 023014, Apr 2005.
- H. Snoussi and A. Mohammad-Djafari, Fast joint separation and segmentation of mixed images, Journal of Electronic Imaging, 13(2):349-361, April 2004.
- A. Mohammad-Djafari, J.F. Giovannelli, G. Demoment and J. Idier, Regularization, maximum entropy and probabilistic methods in mass spectrometry data processing problems, Int. Journal of Mass Spectrometry, 215(1-3):175-193, April 2002.

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- D. Pougaza (Tomography and Copula)
- —————–
- Sh. Zhu (SAR Imaging)
- D. Fall (Emission Positon Tomography, Non Parametric Bayesian)

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- E. Barat (CEA-LIST) (Positon Emission Tomography, Non Parametric Bayesian)
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Questions and Discussions