Copula and Tomography

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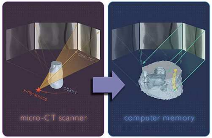
Content

- Tomography
- Radon Transform
- Classical methods of Tomography
- Tomography and Copula
- Some preliminary results
- Conclusions

Tomography : seeing interior of a body

- $f(x,y)$ a section of a real 3D body $f(x,y,z)$
- $g_\phi(r)$ a line of observed radiographe $g_\phi(r,z)$



- Forward model :
  Line integrals or Radon Transform

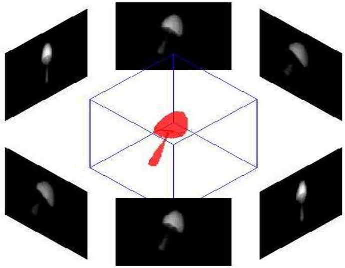
$$g_\phi(r) = \int_{L_{r,\phi}} f(x,y)\, \mathrm{d}l + \epsilon_\phi(r)$$
$$= \iint f(x,y)\, \delta(r - x\cos\phi - y\sin\phi)\, \mathrm{d}x\, \mathrm{d}y + \epsilon_\phi(r)$$

- Inverse problem : Image reconstruction

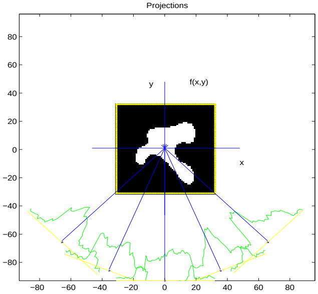
  Given the forward model $\mathcal{H}$ (Radon Transform) and
  a set of data $g_{\phi_i}(r), i = 1, \cdots, M$
  find $f(x,y)$

2D and 3D Computed Tomography

3D



2D

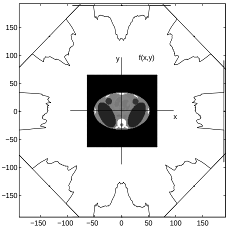


$$g_\phi(r_1,r_2) = \int_{\mathcal{L}_{r_1,r_2,\phi}} f(x,y,z) \, \mathrm{d}l \qquad g_\phi(r) = \int_{\mathcal{L}_{r,\phi}} f(x,y) \, \mathrm{d}l$$

Forward probelm : $f(x,y)$ or $f(x,y,z) \longrightarrow g_\phi(r)$ or $g_\phi(r_1,r_2)$

Inverse problem : $g_\phi(r)$ or $g_\phi(r_1,r_2) \longrightarrow f(x,y)$ or $f(x,y,z)$

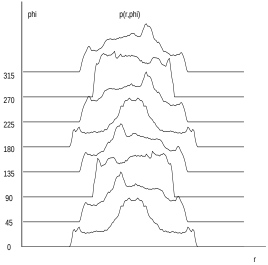
X ray Tomography and Radon Transform



$$g(r,\phi) = -\ln\left(\frac{I}{I_0}\right) = \int_{L_{r,\phi}} f(x,y)\ \mathrm{d}l$$

$$g(r,\phi) = \iint_D f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\ \mathrm{d}x\ \mathrm{d}y$$

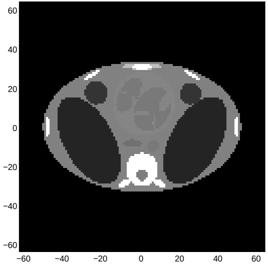
$$f(x,y) \longrightarrow \boxed{\text{RT}} \longrightarrow g(r,\phi)$$



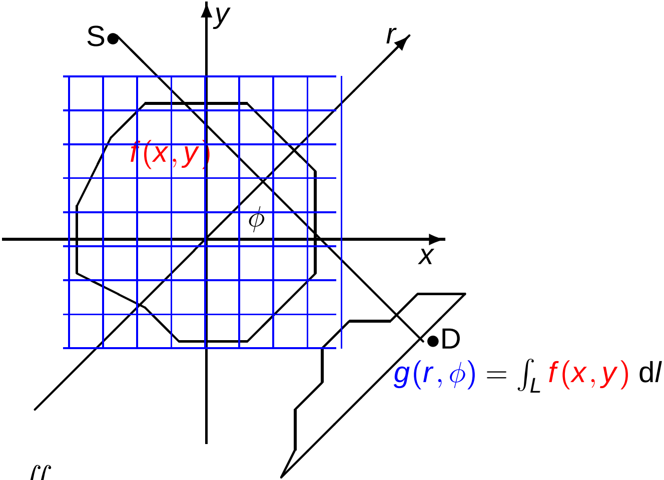
IRT

?

$\Longrightarrow$



Analytical Inversion methods



Radon :

$$g(r,\phi) = \iint_D f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right)\int_0^{\pi}\int_{-\infty}^{+\infty}\frac{\frac{\partial}{\partial r}g(r,\phi)}{(r - x\cos\phi - y\sin\phi)}\,\mathrm{d}r\,\mathrm{d}\phi$$

Filtered Backprojection method

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right) \int_0^{\pi} \int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} g(r,\phi)}{(r - x\cos\phi - y\sin\phi)} \, \mathrm{d}r \, \mathrm{d}\phi$$

Derivation $\mathcal{D}$: $\quad \bar{g}(r,\phi) = \dfrac{\partial g(r,\phi)}{\partial r}$

Hilbert Transform $\mathcal{H}$: $\quad g_1(r',\phi) = \dfrac{1}{\pi} \displaystyle\int_0^{\infty} \frac{\bar{g}(r,\phi)}{(r - r')} \, \mathrm{d}r$

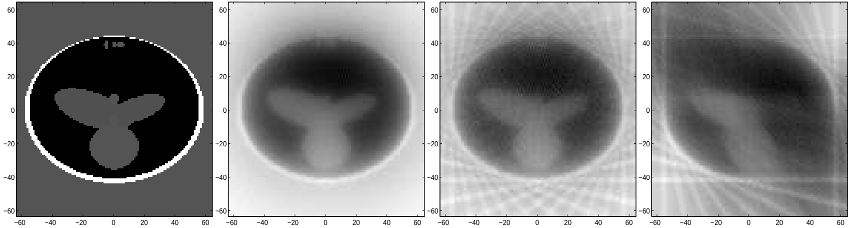
Backprojection $\mathcal{B}$: $\quad f(x,y) = \dfrac{1}{2\pi} \displaystyle\int_0^{\pi} g_1(r' = x\cos\phi + y\sin\phi, \phi) \, \mathrm{d}\phi$

$$f(x,y) = \mathcal{B}\,\mathcal{H}\,\mathcal{D}\, g(r,\phi) = \mathcal{B}\,\mathcal{F}_1^{-1}\,|\Omega|\,\mathcal{F}_1\, g(r,\phi)$$

- Backprojection of filtered projections :

$g(r,\phi) \longrightarrow$ [FT $\mathcal{F}_1$] $\longrightarrow$ [Filter $|\Omega|$] $\longrightarrow$ [IFT $\mathcal{F}_1^{-1}$] $\overset{g_1(r,\phi)}{\longrightarrow}$ [Backprojection $\mathcal{B}$] $\overset{f(x,y)}{\longrightarrow}$

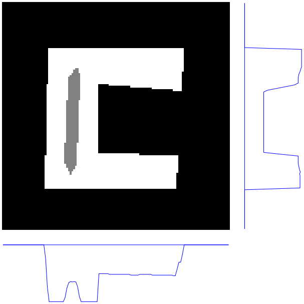
Limitations : Limited angle or noisy data



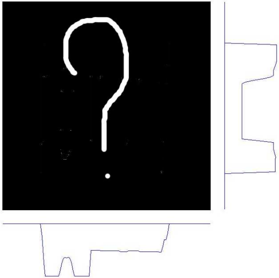
Original | 64 proj. | 16 proj. | 8 proj. $[0, \pi/2]$

- Limited angle or noisy data
- Accounting for detector size
- Other measurement geometries : fan beam, ...

Tomography with only Two projections



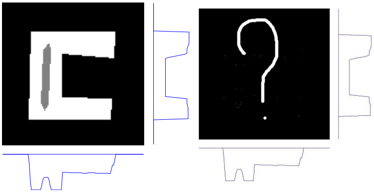
Forward problem :
Given $f(x,y)$ find $f_1(x)$ and $f_2(y)$



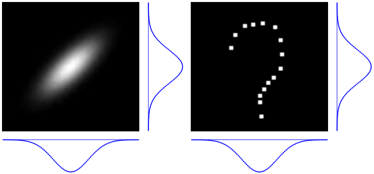
Inverse problem :
Given $f_1(x)$ and $f_2(y)$ find $f(x,y)$

Link between Tomography and Copula

Tomography :
Given the two horizontal and vertical projections $f_1(x)$ and $f_2(y)$, find $f(x,y)$



Copula :
Given the two marginal pdfs $f_1(x)$ and $f_2(y)$, find the joint pdf $f(x,y)$



Ill-posed Inverse problems : Infinite number of solutions
There is a need to reduce the space of possible solutions

Link between Tomography and Copula

Copula and Tomography

Given $f_1(x)$ and $f_2(y)$ find $f(x,y)$ such that

$$f_1(x) = \int f(x,y) \, dy \quad \text{and} \quad f_2(y) = \int f(x,y) \, dx$$

Tomography :

If we find a solution $f_0(x,y)$, we can add any function $\omega(x,y)$ such that

$$\int \omega(x,y) \, dx = \int \omega(x,y) \, dy = 0$$

and we get again another solution.

Copula :

Any function $f(x,y)$ given by

$$f(x,y) = f_1(x) \, f_2(y) \, c\left(F_1(x), F_2(y)\right)$$

where $c(u,v)$ is any copula, is a solution to the problem.

$$\int c(x,y) \, dx = \int c(x,y) \, dy = 1$$

Link between Tomography and Copula

Tomography :

Back Projection (BP) :

$$f(x,y) = f(x) + f(y)$$

Copula :

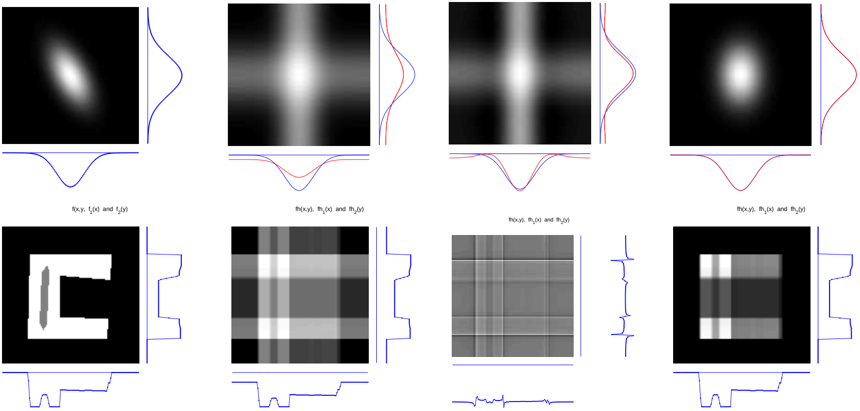
Multiplicative Back Projection (MBP) :

$$f(x,y) = f(x)\,f(y)$$

Copula Back Projection (MBP) :

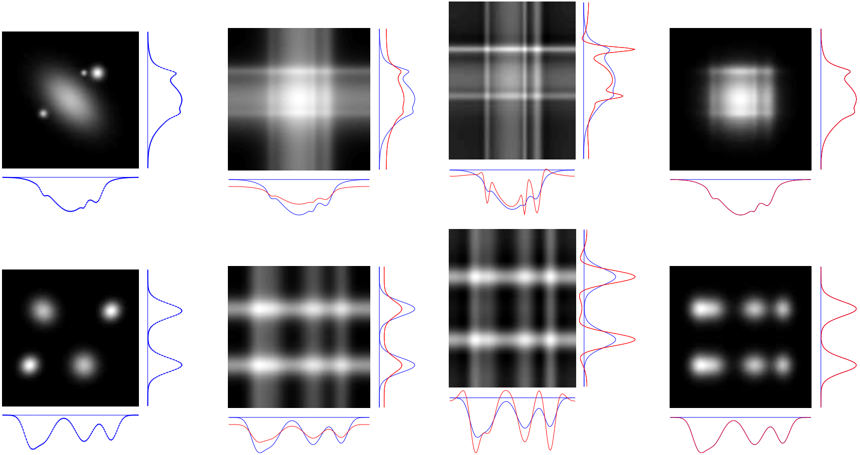
$$f(x,y) = f(x)\,f(y)\,c\,(F_1(x),F_2(y))$$

Preliminary results



Originals $f(x,y)$ BP $\widehat{f}(x,y)$ FBP $\widehat{f}(x,y)$ MBP $\widehat{f}(x,y)$

Preliminary results



Originals $f(x,y)$ BP $\widehat{f}(x,y)$ FBP $\widehat{f}(x,y)$ MBP $\widehat{f}(x,y)$

Conclusions

- There is a link between Copula and Tomography.
- Both problems are ill-posed.
- We are sure that the methods in one domain can be useful in other one.
- With only two projections we can only reconstruct simple images
- There is a need for more data or for more constraintes (maximum entropy, minimum energy, ...)
- How to use more than 2 projections with Copula ?