Bayesian Inference Tools for Inverse Problem

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General inverse problem

$$g(t) = \mathcal{H} f(t) + \epsilon(t), \quad t \in [1, \cdots, T]$$

$$g(\boldsymbol{r}) = \mathcal{H} f(\boldsymbol{r}) + \epsilon(\boldsymbol{r}), \quad \boldsymbol{r} = (x, y) \in R^2$$

- f unknown quantity (input)
- $\mathcal{H}$ Forward operator:
  (Convolution, Radon, Fourier or any Linear operator)
- g observed quantity (output)
- ϵ represents the errors of modeling and measurement

Discretization:

$$\boldsymbol{g} = \boldsymbol{H} \boldsymbol{f} + \boldsymbol{\epsilon}$$

- Forward operation $\boldsymbol{H} \boldsymbol{f}$
- Adjoint operation $\boldsymbol{H}' \boldsymbol{g}$: $\quad < \boldsymbol{H}' \boldsymbol{g}, \boldsymbol{f} > = < \boldsymbol{H} \boldsymbol{f}, \boldsymbol{g} >$
- Inverse operation (if exists) $\boldsymbol{H}^{-1} \boldsymbol{g}$

General Bayesian Inference

- Bayesian inference:

$$p(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{\theta}) = \frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)}{p(\boldsymbol{g}|\boldsymbol{\theta})}$$

  with $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$

- Point estimators:
  Maximum A Posteriori (MAP) or Posterior Mean (PM) $\longrightarrow \widehat{\boldsymbol{f}}$

- Hyperparameter estimation: $\boldsymbol{\theta}$ unknown
- Simple prior models: $p(\boldsymbol{f}|\boldsymbol{\theta}_2)$

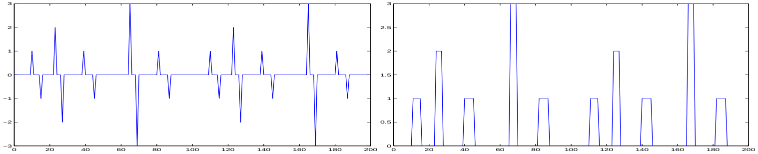
$$q(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)\, p(\boldsymbol{\theta})$$

- Prior models with hidden variables: $p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta}_2)\; p(\boldsymbol{z}|\boldsymbol{\theta}_3)$

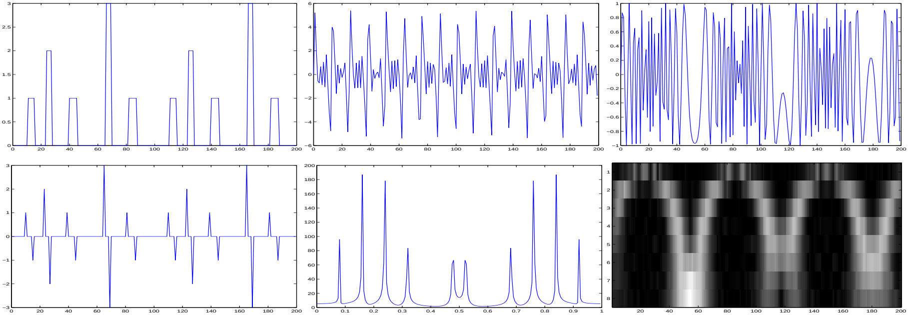
$$q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)\, p(\boldsymbol{z}|\boldsymbol{\theta}_3)\, p(\boldsymbol{\theta})$$

Sparsity enforcing prior models

- Sparse signals: Direct sparsity



- Sparse signals: Sparsity in a Transform domaine



Sparsity enforcing prior models

- Simple heavy tailed models:
  - Generalized Gaussian, Double Exponential
  - Student-t, Cauchy
  - Elastic net

  - Symmetric Weibull, Symmetric Rayleigh
  - Generalized hyperbolic

- Hierarchical mixture models:
  - Mixture of Gaussians
  - Bernoulli-Gaussian

  - Mixture of Gammas
  - Bernoulli-Gamma
  - Mixture of Dirichlet
  - Bernoulli-Multinomial

Simple heavy tailed models

- Generalized Gaussian, Double Exponential

$$p(\boldsymbol{f}|\gamma,\beta) = \prod_j \mathcal{GG}(f_j|\gamma,\beta) \propto \exp\left\{-\gamma \sum_j |f_j|^\beta\right\}$$

$\beta = 1$ Double exponential or Laplace.
$0 < \beta \leq 1$ are of great interest for sparsity enforcing.

- Student-t and Cauchy models

$$p(\boldsymbol{f}|\nu) = \prod_j \mathcal{S}t(f_j|\nu) \propto \exp\left\{-\frac{\nu+1}{2} \sum_j \log\left(1 + f_j^2/\nu\right)\right\}$$

Cauchy model is obtained when $\nu = 1$.

- Elastic net prior model

$$p(\boldsymbol{f}|\nu) = \prod_j \mathcal{EN}(f_j|\nu) \propto \exp\left\{-\sum_j (\gamma_1|f_j| + \gamma_2 f_j^2)\right\}$$

Mixture models

- Mixture of two Gaussians (MoG2) model

$$p(\boldsymbol{f}|\lambda, v_1, v_0) = \prod_j \left(\lambda \mathcal{N}(f_j|0, v_1) + (1-\lambda)\mathcal{N}(f_j|0, v_0)\right)$$

- Bernoulli-Gaussian (BG) model

$$p(\boldsymbol{f}|\lambda, v) = \prod_j p(f_j) = \prod_j \left(\lambda \mathcal{N}(f_j|0, v) + (1-\lambda)\delta(f_j)\right)$$

- Mixture of Gammas

$$p(\boldsymbol{f}|\lambda, v_1, v_0) = \prod_j \left(\lambda \mathcal{G}(f_j|\alpha_1, \beta_1) + (1-\lambda)\mathcal{G}(f_j|\alpha_2, \beta_2)\right)$$

- Bernoulli-Gamma model

$$p(\boldsymbol{f}|\lambda, \alpha, \beta) = \prod_j \left[\lambda \mathcal{G}(f_j|\alpha, \beta) + (1-\lambda)\delta(f_j)\right]$$

MAP and Joint MAP

- Inverse problems: $\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$
- Posterior law:

$$p(\boldsymbol{f}|\boldsymbol{\theta}, \boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)$$

- Examples:

  Gaussian noise, Gaussian prior and MAP:

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \text{ with } J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|_2^2 + \lambda\|\boldsymbol{f}\|_2^2$$

  Gaussian noise, Double Exponential prior and MAP:

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \text{ with } J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|_2^2 + \lambda\|\boldsymbol{f}\|_1$$

- Full Bayesian: Joint Posterior:

$$p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)\, p(\boldsymbol{\theta})$$

- Joint MAP:

$$(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f}, \boldsymbol{\theta})} \{p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g})\}$$

Marginal MAP, EM and VBA

- Marginal MAP: $\widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \{p(\boldsymbol{\theta}|\boldsymbol{g})\}$ where

$$p(\boldsymbol{\theta}|\boldsymbol{g}) = \int p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) \, \mathrm{d}\boldsymbol{f} = \int p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1) \, p(\boldsymbol{f}|\boldsymbol{\theta}_2) \, \mathrm{d}\boldsymbol{f}$$

  and then
  $\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \left\{p(\boldsymbol{f}|\widehat{\boldsymbol{\theta}}, \boldsymbol{g})\right\}$ or $\widehat{\boldsymbol{f}} = \int \boldsymbol{f} \, p(\boldsymbol{f}|\widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \, \mathrm{d}\boldsymbol{f}$
- Analytic expression for $p(\boldsymbol{\theta}|\boldsymbol{g})$ not always possible
- Expectation-Maximization (EM) and GEM Algorithms

- Variational Bayesian Approximation:
  Approximate $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g})$ by $q(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f}|\boldsymbol{g}) \, q_2(\boldsymbol{\theta}|\boldsymbol{g})$
  and then use these approximate laws for other computations.
- JMAP, EM and GEM correspond to different approximations.

Hierarchical models and hidden variables

- All the mixture models and some of simple models can be modeled via hidden variables $\boldsymbol{z}$.
- Example 1: Student-t model

$$\left\{ \begin{array}{l} p(\boldsymbol{f}|\boldsymbol{z}) = \prod_j p(f_j|z_j) = \prod_j \mathcal{N}(f_j|0, 1/z_j) \propto \exp\left\{-\frac{1}{2}\sum_j z_j f_j^2\right\} \\ p(z_j|a,b) = \mathcal{G}(z_j|a,b) \propto z_j{}^{(a-1)} \exp\{-b z_j\} \text{ with } a = b = \nu/2 \end{array} \right.$$

- Example 2: MoG model:

$$\left\{ \begin{array}{l} p(\boldsymbol{f}|\boldsymbol{z}) = \prod_j p(f_j|z_j) = \prod_j \mathcal{N}\left(f_j|0, v_{z_j}\right) \propto \exp\left\{-\frac{1}{2}\sum_j \frac{f_j^2}{v_{z_j}}\right\} \\ P(z_j = 1) = \lambda, \qquad P(z_j = 0) = 1 - \lambda \end{array} \right.$$

- With these models we have:

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta}_2)\, p(\boldsymbol{z}|\boldsymbol{\theta}_3)\, p(\boldsymbol{\theta})$$

Bayesian Computation and Algorithms

- Often, the expression of $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g})$ is complex.
- Its optimization (for Joint MAP) or
  its marginalization or integration (for Marginal MAP or PM)
  is not easy
- Two main techniques:
  - MCMC:
    Needs the expressions of the conditionals
    $p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta}, \boldsymbol{g})$, $p(\boldsymbol{z}|\boldsymbol{f}, \boldsymbol{\theta}, \boldsymbol{g})$, and $p(\boldsymbol{\theta}|\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{g})$
  - Variational Bayesian Approximation (VBA):
    Approximate $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g})$ by a separable one

    $$q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta})$$

    and do any computations with these separable ones.

Bayesian Variational Approximation

- Objective: Approximate $p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g})$ by a separable one $q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta})$
- Criterion:
$$\mathrm{KL}(q:p) = \int q \ln \frac{q}{p} = \left\langle \ln \frac{q}{p} \right\rangle_q$$
- Free energy: $\mathrm{KL}(q:p) = \ln p(\boldsymbol{g}|\mathcal{M}) - \mathcal{F}(q)$ where:
$$p(\boldsymbol{g}|\mathcal{M}) = \int\int\int p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta},\boldsymbol{g}|\mathcal{M})\, \mathrm{d}\boldsymbol{f}\ \mathrm{d}\boldsymbol{z}\ \mathrm{d}\boldsymbol{\theta}$$
with $p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta},\boldsymbol{g}|\mathcal{M}) = p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta})\, p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta})\, p(\boldsymbol{z}|\boldsymbol{\theta})\, p(\boldsymbol{\theta})$ and $\mathcal{F}(q)$ is the free energy associated to q defined as
$$\mathcal{F}(q) = \left\langle \ln \frac{p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta},\boldsymbol{g}|\mathcal{M})}{q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta})} \right\rangle_q$$
- For a given model $\mathcal{M}$, minimizing $\mathrm{KL}(q:p)$ is equivalent to maximizing $\mathcal{F}(q)$ and when optimized, $\mathcal{F}(q^*)$ gives a lower bound for $\ln p(\boldsymbol{g}|\mathcal{M})$.

BVA with Student-t priors

Scale Mixture Model of Student-t:

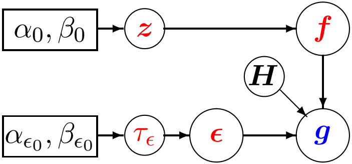
$$\mathcal{S}t(f_j|\nu) = \int_0^\infty \mathcal{N}(f_j|0, 1/z_j)\,\mathcal{G}(z_j|\nu/2, \nu/2)\;\mathrm{d}z_j$$

Hidden variables z_j:

$$\begin{aligned} p(\boldsymbol{f}|\boldsymbol{z}) &= \prod_j p(f_j|z_j) = \prod_j \mathcal{N}(f_j|0, 1/z_j) \propto \exp\left\{-\tfrac{1}{2}\sum_j z_j f_j^2\right\} \\ p(z_j|\alpha_0,\beta_0) &= \mathcal{G}(z_j|\alpha_0,\beta_0) \propto z_j{}^{(\alpha_0-1)} \exp\{-\beta_0 z_j\} \text{ with } \alpha_0=\beta_0=\nu/2 \end{aligned}$$

Cauchy model is obtained when $\nu = 1$:

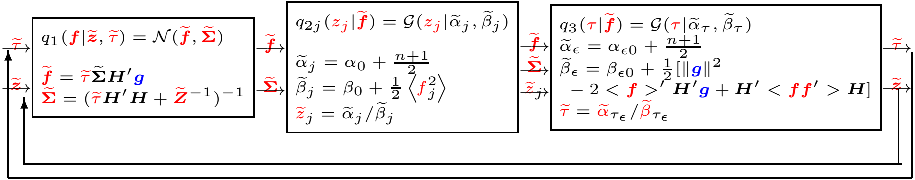
- Graphical model: $\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$, $\quad p(\boldsymbol{\epsilon}) = \mathcal{N}(\boldsymbol{\epsilon}|0, \frac{1}{\tau_\epsilon}\boldsymbol{I})$



BVA with Student-t priors Algorithm

$$\begin{cases} p(\boldsymbol{g}|\boldsymbol{f},\tau_\epsilon) = \mathcal{N}(\boldsymbol{g}|\boldsymbol{H}\boldsymbol{f},(1/\tau_\epsilon)\boldsymbol{I}) \\ p(\tau_\epsilon|\alpha_{\epsilon0},\beta_{\epsilon0}) = \mathcal{G}(\tau_\epsilon|\alpha_{\epsilon0},\beta_{\epsilon0}) \\ p(\boldsymbol{f}|\boldsymbol{z}) = \prod_j \mathcal{N}(f_j|0,1/z_j) \\ p(\boldsymbol{z}|\alpha_0,\beta_0) = \prod_j \mathcal{G}(z_j|\alpha_0,\beta_0) \end{cases} \quad \begin{cases} q_1(\boldsymbol{f}|\widetilde{\boldsymbol{\mu}},\widetilde{\boldsymbol{\Sigma}}) = \mathcal{N}(\boldsymbol{f}|\widetilde{\boldsymbol{\mu}},\widetilde{\boldsymbol{\Sigma}}), \quad \widetilde{\boldsymbol{\mu}} = \widetilde{\tau}\,\widetilde{\boldsymbol{\Sigma}}\boldsymbol{H}'\boldsymbol{g} \\ \widetilde{\boldsymbol{\Sigma}} = \left(\widetilde{\tau}\,\boldsymbol{H}'\boldsymbol{H} + \widetilde{\boldsymbol{Z}}\right)^{-1} \text{ with } \widetilde{\boldsymbol{Z}} = \text{diag}\left[\widetilde{\boldsymbol{z}}^{-1}\right] \\ q_{2j}(z_j) = \mathcal{G}(z_j|\widetilde{\alpha}_j,\widetilde{\beta}_j); \quad \widetilde{\alpha}_j = \alpha_0 + \frac{1}{2} \\ \widetilde{\beta}_j = \beta_0 + < f_j^2 > /2 \\ q_3(\tau_\epsilon) = \mathcal{G}(\tau_\epsilon|\widetilde{\alpha}_\epsilon,\widetilde{\beta}_\epsilon) \\ \widetilde{\alpha}_\epsilon = \alpha_{\epsilon0} + (n+1)/2 \\ \widetilde{\beta}_\epsilon = \beta_{\epsilon0} + \frac{1}{2}[\|\boldsymbol{g}\|^2 - 2\langle\boldsymbol{f}\rangle'\boldsymbol{H}'\boldsymbol{g} + \boldsymbol{H}'\langle\boldsymbol{f}\boldsymbol{f}'\rangle\boldsymbol{H}] \end{cases}$$

$$< \boldsymbol{f} > = \widetilde{\boldsymbol{\mu}}, \; < \boldsymbol{f}\boldsymbol{f}' > = \widetilde{\boldsymbol{\Sigma}} + \widetilde{\boldsymbol{\mu}}\widetilde{\boldsymbol{\mu}}', \; < f_j^2 > = [\widetilde{\boldsymbol{\Sigma}}]_{jj} + \widetilde{\mu}_j^2, \; \widetilde{\tau} = \frac{\widetilde{\alpha}_{\tau_\epsilon}}{\widetilde{\beta}_{\tau_\epsilon}}, \; \widetilde{z}_j = \frac{\widetilde{\alpha}_j}{\widetilde{\beta}_j}$$



Implementation issues

- In inverse problems, often we do not have access directly to the matrix $\boldsymbol{H}$. But, we can compute:
  - Forward operator : $\boldsymbol{H}\boldsymbol{f} \longrightarrow \boldsymbol{g}$ `g=direct(f,...)`
  - Adjoint operator : $\boldsymbol{H}'\boldsymbol{g} \longrightarrow \boldsymbol{f}$ `f=transp(g,...)`
- For any particular application, we can always write two programs (`direct & transp`) corresponding to the application of these two operators.
- To compute $\widetilde{\boldsymbol{f}}$, we use a gradient based optimization algorithm which will use these operators.
- We may also need to compute the diagonal elements of $[\boldsymbol{H}'\boldsymbol{H}]$.. We also developped algorithms which computes these diagonal elements with the same programs (`direct & transp`)

Conclusions and Perspectives

- We proposed a list of different probabilistic prior models which can be used for sparsity enforcing.
- We classified these models in two categories: simple heavy tails and hierarchical mixture models
- We showed how to use these models for inverse problems where the desired solutions are sparse
- Different algorithms have been developed and their relative performances are compared.
- We use these models for inverse problems in different signal and image processing applications such as:
  - Period estimation in biological time series
  - X ray Computed Tomography,
  - Signal deconvolution in Proteomic and molecular imaging
  - Diffraction Optical Tomography
  - Microwave Imaging, Acoustic imaging and sources localization
  - Synthetic Aperture Radar (SAR) Imaging

Related presentations in MaxEnt2012 workshop

- Mircea Dumitru: (Talk on Wednesday)
  Estimating the period of a signal through inverse problem modeling and Bayesian inference with sparsity enforcing prior

- Leila Gharsali: (Poster on Tuseday)
  Variational Bayesian Approximation with scale mixture priors: A comparison between three algorithms

- Rémi Pronon: (Talk on Tusesday)
  MCMC-based Bayesian estimation algorithm dedicated to NEMS Mass Spectrometry

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