Variational Bayesian Approximation for Linear Inverse Problems with a hierarchical prior models

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Seminar 1 given at
Tsinghua University, Beijing, China, December 9, 2013

General linear inverse problem

Signal processing : $g(t) = \mathcal{H} f(t) + \epsilon(t), \quad t \in [1, \cdots, T]$
Image processing : $g(\boldsymbol{r}) = \mathcal{H} f(\boldsymbol{r}) + \epsilon(\boldsymbol{r}), \ \boldsymbol{r} = (x, y) \in R^2$
More general : $g(\boldsymbol{s}) = [\mathcal{H} f(\boldsymbol{r})](\boldsymbol{s}) + \epsilon(\boldsymbol{s}), \ \boldsymbol{r} = (x, y), \boldsymbol{s} = (u, v)$

- f unknown quantity (input)
- $\mathcal{H}$ Forward operator:
  (Convolution, Radon, Fourier or any Linear operator)
- g observed quantity (output)
- ϵ represents the errors of modeling and measurement

Discretization:

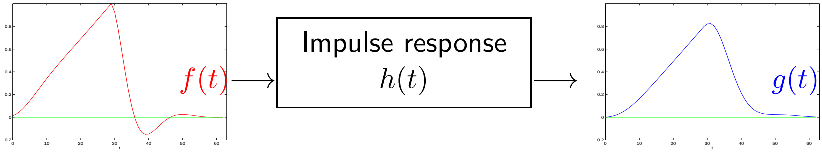
$$\boldsymbol{g} = \boldsymbol{H} \boldsymbol{f} + \boldsymbol{\epsilon}$$

- Forward operation $\boldsymbol{H} \boldsymbol{f}$
- Adjoint operation $\boldsymbol{H}' \boldsymbol{g}$: $\quad < \boldsymbol{H}' \boldsymbol{g}, \boldsymbol{f} > = < \boldsymbol{H} \boldsymbol{f}, \boldsymbol{g} >$
- Inverse operation (if exists) $\boldsymbol{H}^{-1} \boldsymbol{g}$, but this is never the case.

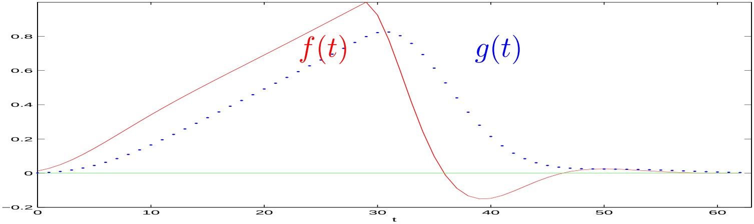
Example 1: Signal Deconvolution

Forward model: Convolution

$$g(t) = \int f(t')\, h(t-t')\, \mathrm{d}t' + \epsilon(t)$$



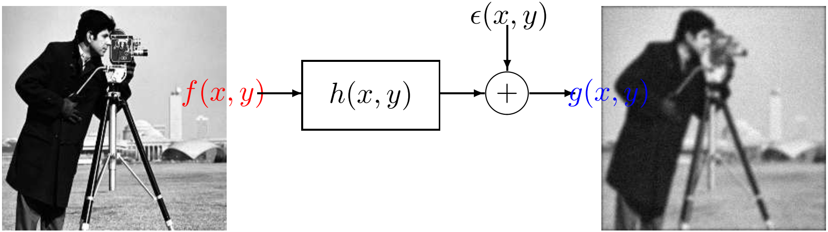
Inversion: Deconvolution



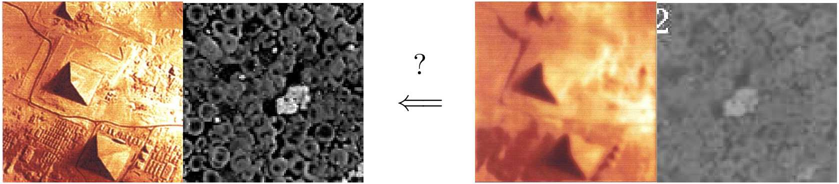
Example 2: Image restoration

Forward model: 2D Convolution

$$g(x, y) = \iint f(x', y')\, h(x - x', y - y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x, y)$$

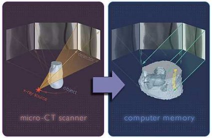


Inversion: Image Deconvolution or Restoration



Example 3: Image Reconstruction in Computed Tomography

- $f(x, y)$ a section of a real 3D body $f(x, y, z)$
- $g(r, \phi)$ a line of observed radiographe $g_\phi(r, z)$



- Forward model:
  Line integrals or Radon Transform

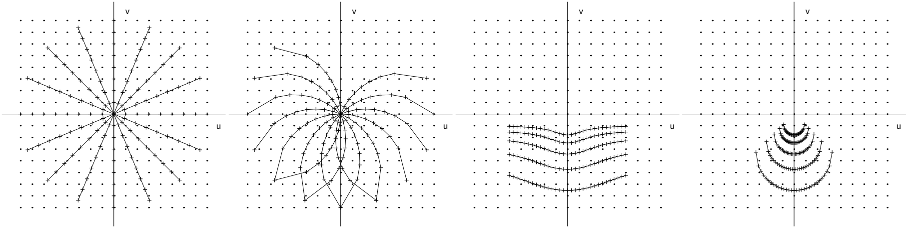
$$
\begin{aligned}
g(r, \phi) &= \int_{L_{r,\phi}} f(x, y) \, \mathrm{d}l + \epsilon_\phi(r) \\
&= \iint f(x, y) \, \delta(r - x \cos\phi - y \sin\phi) \, \mathrm{d}x \, \mathrm{d}y + \epsilon_\phi(r)
\end{aligned}
$$

- Inverse problem: Image reconstruction

  Given the forward model $\mathcal{H}$ (Radon Transform) and
  a set of data $g(r, \phi_i), i = 1, \cdots, M$
  find $f(x, y)$

Example 4: Fourier Synthesis in different imaging systems

$$G(u, v) = \iint f(x, y) \exp\left[-j\,(ux + vy)\right]\, \mathrm{d}x\, \mathrm{d}y$$



X ray Tomography Diffraction Eddy current SAR & Radar

Forward problem: Given $f(x, y)$ compute $G(u, v)$
Inverse problem : Given $G(u, v)$ on those algebraic lines, cercles or curves, estimate $f(x, y)$

General formulation of inverse problems

- General non linear inverse problems:

$$g(\boldsymbol{s}) = [\mathcal{H} f(\boldsymbol{r})](\boldsymbol{s}) + \epsilon(\boldsymbol{s}), \quad \boldsymbol{r} \in \mathcal{R}, \quad \boldsymbol{s} \in \mathcal{S}$$

- Linear models:

$$g(\boldsymbol{s}) = \int f(\boldsymbol{r})\, h(\boldsymbol{r}, \boldsymbol{s})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s})$$

  If $h(\boldsymbol{r}, \boldsymbol{s}) = h(\boldsymbol{r} - \boldsymbol{s}) \longrightarrow$ Convolution.

- Discrete data:

$$g(\boldsymbol{s}_i) = \int h(\boldsymbol{s}_i, \boldsymbol{r})\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s}_i), \quad i = 1, \cdots, m$$

- Inversion: Given the forward model $\mathcal{H}$ and the data $\boldsymbol{g} = \{g(\boldsymbol{s}_i), i = 1, \cdots, m)\}$ estimate $f(\boldsymbol{r})$
- Well-posed and Ill-posed problems (Hadamard): existance, uniqueness and stability
- Need for prior information

Bayesian inference for inverse problems

$$\mathcal{M}: \qquad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Observation model $\mathcal{M}$ + Hypothesis on the noise $\boldsymbol{\epsilon} \longrightarrow$
  $$p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M}) = p_{\boldsymbol{\epsilon}}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f})$$
- A priori information $\quad p(\boldsymbol{f}|\mathcal{M})$
- Bayes : $$p(\boldsymbol{f}|\boldsymbol{g};\mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M})\, p(\boldsymbol{f}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$$

Maximum A Posteriori (MAP):

$$\begin{aligned} \widehat{\boldsymbol{f}} &= \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{g}|\boldsymbol{f})\; p(\boldsymbol{f})\} \\ &= \arg\min_{\boldsymbol{f}} \{-\ln p(\boldsymbol{g}|\boldsymbol{f}) - \ln p(\boldsymbol{f})\} \end{aligned}$$

Posterior Mean (PM):

$$\widehat{\boldsymbol{f}} = \int \boldsymbol{f}\, p(\boldsymbol{f}|\boldsymbol{g})\, \mathrm{d}\boldsymbol{f}$$

Supervised and Unsupervised Bayesian Inference

- Bayesian inference:

$$p(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{\theta}) = \frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)}{p(\boldsymbol{g}|\boldsymbol{\theta})}$$

  with $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$

- Point estimators:
  Maximum A Posteriori (MAP) or Posterior Mean (PM) $\longrightarrow \widehat{\boldsymbol{f}}$

- Unsupervised Bayesian inference:
  Joint estimation of $\boldsymbol{f}$ and $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$

$$p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g}) = \frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)\, p(\boldsymbol{\theta})}{p(\boldsymbol{g})}$$

Full Bayesian approach

$$\mathcal{M}: \qquad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

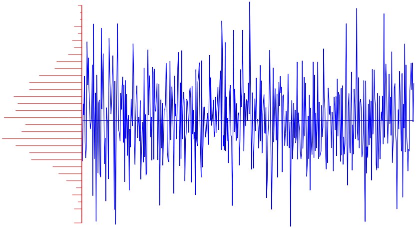
- Forward & errors model: $\longrightarrow p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1; \mathcal{M})$
- Prior models $\longrightarrow p(\boldsymbol{f}|\boldsymbol{\theta}_2; \mathcal{M})$
- Hyperparameters $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \longrightarrow p(\boldsymbol{\theta}|\mathcal{M})$
- Bayes: $\longrightarrow p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\, p(\boldsymbol{f}|\boldsymbol{\theta};\mathcal{M})\, p(\boldsymbol{\theta}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$
- Joint MAP: $\qquad (\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f},\boldsymbol{\theta})} \{p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\}$
- Marginalization: $\begin{cases} p(\boldsymbol{f}|\boldsymbol{g}; \mathcal{M}) & = \int p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{\theta} \\ p(\boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M}) & = \int p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f} \end{cases}$
- Posterior means: $\begin{cases} \widehat{\boldsymbol{f}} & = \int\int \boldsymbol{f}\, p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{\theta}\, \mathrm{d}\boldsymbol{f} \\ \widehat{\boldsymbol{\theta}} & = \int\int \boldsymbol{\theta}\, p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta} \end{cases}$
- Evidence of the model:

$$p(\boldsymbol{g}|\mathcal{M}) = \iint p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{f}|\boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{\theta}|\mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta}$$

Two main steps in the Bayesian approach

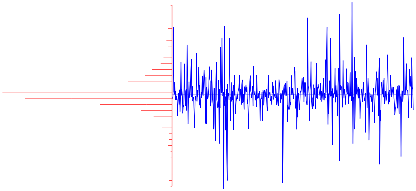
- Prior modeling: $p(\boldsymbol{f}|\boldsymbol{\theta}_1)$
  - Separable:
    Gaussian, Generalized Gaussian, Gamma,
    mixture of Gaussians, mixture of Gammas, ...
  - Markovian: Gauss-Markov, GGM, ...
  - Separable or Markovian with hidden variables
    (contours, region labels)
- Bayesian computational aspects
  - MAP needs optimization algorithms
  - Posterior mean needs integration methods
  - Marginal MAP and Hyperparameter estimation need
    integration and optimization
  - Approximations:
    - Gaussian approximation (Laplace)
    - Numerical exploration MCMC
    - Variational Bayes Approximation (VBA)

Prior models: Separable $p(\boldsymbol{f}) = \prod_j p(f_j)$



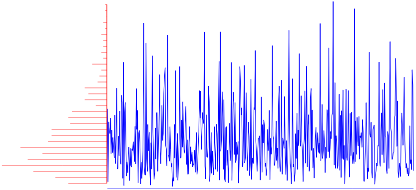
Gaussian

$p(f_j|\alpha) \propto \exp\left[-\alpha|f_j|^2\right]$



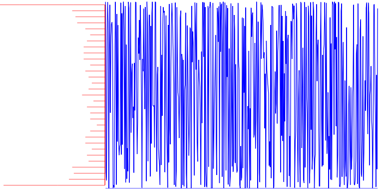
Generalized Gaussian

$p(f_j|\alpha,\beta) \propto \exp\left[-\alpha|f_j|^\beta\right], \quad 1 \leq \beta \leq 2$



Gamma

$p(f_j|\alpha,\beta) \propto f_j^\alpha \exp\left[-\beta f_j\right]$



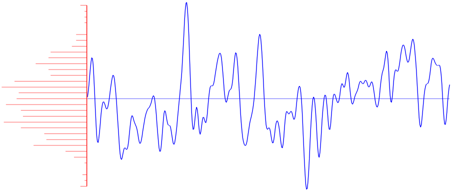
Beta

$p(f_j|\alpha,\beta) \propto f_j^\alpha (1-f_j)^\beta$

Prior models: Markovian models

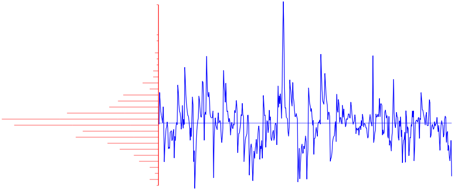
Gauss-Markov (G-M)

$$p(\boldsymbol{f}) \propto \exp\left[-\gamma \sum_j |f_j - f_{j-1}|^2\right]$$



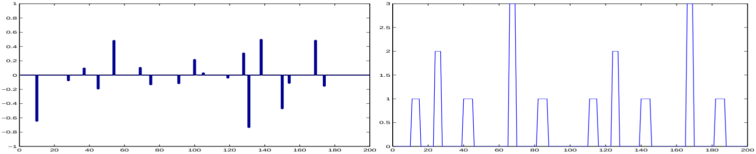
Generalized G-M

$$p(\boldsymbol{f}) \propto \exp\left[-\gamma \sum_j |f_j - f_{j-1}|^\beta\right]$$

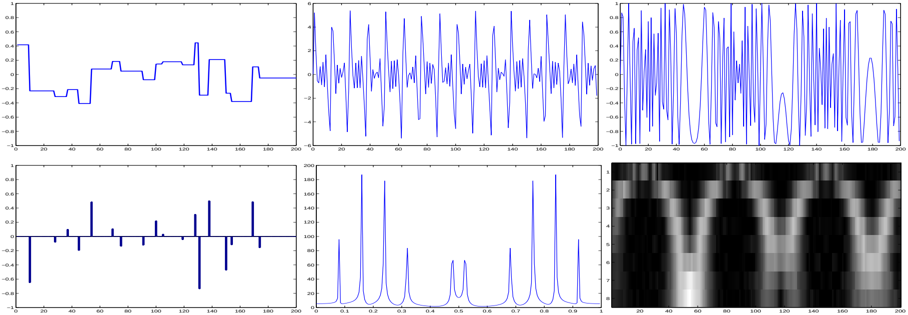


Sparsity enforcing prior models

- Sparse signals: Direct sparsity



- Sparse signals: Sparsity in a Transform domaine



Prior models: Sparsity enforcing models

- Simple heavy tailed models:
  - Generalized Gaussian, Double Exponential
  - Student-t, Cauchy

  - Elastic net
  - Symmetric Weibull, Symmetric Rayleigh
  - Generalized hyperbolic

- Hierarchical mixture models:
  - Mixture of Gaussians
  - Bernoulli-Gaussian

  - Mixture of Gammas
  - Bernoulli-Gamma
  - Mixture of Dirichlet
  - Bernoulli-Multinomial

Simple heavy tailed models

- Generalized Gaussian, Double Exponential

$$p(\boldsymbol{f}|\gamma,\beta) = \prod_j \mathcal{GG}(f_j|\gamma,\beta) \propto \exp\left[-\gamma \sum_j |f_j|^\beta\right]$$

$\beta = 1$ Double exponential or Laplace.
$0 < \beta \leq 1$ are of great interest for sparsity enforcing.

- Student-t and Cauchy models

$$p(\boldsymbol{f}|\nu) = \prod_j \mathcal{S}t(f_j|\nu) \propto \exp\left[-\frac{\nu+1}{2} \sum_j \log\left(1 + f_j^2/\nu\right)\right]$$

Cauchy model is obtained when $\nu = 1$.

Mixture models

- Mixture of two Gaussians (MoG2) model

$$p(\boldsymbol{f}|\lambda, v_1, v_0) = \prod_j \left(\lambda \mathcal{N}(f_j|0, v_1) + (1-\lambda)\mathcal{N}(f_j|0, v_0)\right)$$

- Bernoulli-Gaussian (BG) model

$$p(\boldsymbol{f}|\lambda, v) = \prod_j p(f_j) = \prod_j \left(\lambda \mathcal{N}(f_j|0, v) + (1-\lambda)\delta(f_j)\right)$$

- Mixture of Gammas

$$p(\boldsymbol{f}|\lambda, v_1, v_0) = \prod_j \left(\lambda \mathcal{G}(f_j|\alpha_1, \beta_1) + (1-\lambda)\mathcal{G}(f_j|\alpha_2, \beta_2)\right)$$

- Bernoulli-Gamma model

$$p(\boldsymbol{f}|\lambda, \alpha, \beta) = \prod_j \left[\lambda \mathcal{G}(f_j|\alpha, \beta) + (1-\lambda)\delta(f_j)\right]$$

MAP, Joint MAP

- Inverse problems: $\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$
- Posterior law: $p(\boldsymbol{f}|\boldsymbol{\theta}, \boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)$
- MAP:

$$\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{\theta}, \boldsymbol{g})\} = \arg\min_{\boldsymbol{f}} \{J(f) = -\ln p(\boldsymbol{f}|\boldsymbol{\theta}, \boldsymbol{g})\}$$

- Examples:

  Gaussian noise, Gaussian prior:

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|_2^2 + \lambda\|\boldsymbol{f}\|_2^2$$

  Gaussian noise, Double Exponential prior:

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|_2^2 + \lambda\|\boldsymbol{f}\|_1$$

- Full Bayesian: Joint Posterior:

$$p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)\, p(\boldsymbol{\theta})$$

- Joint MAP:

$$(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f}, \boldsymbol{\theta})} \{p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g})\}$$

Marginal MAP and PM estimates

- Joint posterior: $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g})$
- Marginal MAP: $\widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \{p(\boldsymbol{\theta}|\boldsymbol{g})\}$ where

$$p(\boldsymbol{\theta}|\boldsymbol{g}) = \int p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) \,\mathrm{d}\boldsymbol{f} = \int p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2) \,\mathrm{d}\boldsymbol{f}$$

  and then:
  - MAP: $\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \left\{p(\boldsymbol{f}|\widehat{\boldsymbol{\theta}}, \boldsymbol{g})\right\}$
  - Posterior Mean: $\widehat{\boldsymbol{f}} = \int \boldsymbol{f}\, p(\boldsymbol{f}|\widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \,\mathrm{d}\boldsymbol{f}$
- EM and BEM Algorithms

- Variational Bayesian Approximation:
  Main idea: Approximate $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g})$ by a simpler, for example a separable one:

$$q(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{\theta})$$

  and then continue computations with $q_1(\boldsymbol{f})$ and $q_2(\boldsymbol{\theta})$.

Variational Bayesian Approximation

- Full Bayesian: $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)\, p(\boldsymbol{\theta})$
- Approximate $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g})$ by $q(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f}|\boldsymbol{g})\, q_2(\boldsymbol{\theta}|\boldsymbol{g})$ and then continue computations.
- Criterion $\mathrm{KL}(q(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) : p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}))$
- $\mathrm{KL}(q : p) = \int\int q \ln q/p = \int\int q_1 q_2 \ln \frac{q_1 q_2}{p}$
- Iterative algorithm $q_1 \longrightarrow q_2 \longrightarrow q_1 \longrightarrow q_2, \cdots$

$$\left\{ \begin{array}{ll} \widehat{q}_1(\boldsymbol{f}) & \propto \exp\left[\langle \ln p(\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M})\rangle_{\widehat{q}_2(\boldsymbol{\theta})}\right] \\ \widehat{q}_2(\boldsymbol{\theta}) & \propto \exp\left[\langle \ln p(\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M})\rangle_{\widehat{q}_1(\boldsymbol{f})}\right] \end{array} \right.$$

$p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) \longrightarrow$ **Variational Bayesian Approximation** $\longrightarrow q_1(\boldsymbol{f}) \longrightarrow \widehat{\boldsymbol{f}}$

$\longrightarrow q_2(\boldsymbol{\theta}) \longrightarrow \widehat{\boldsymbol{\theta}}$

VBA: Choice of family of laws q_1 and q_2

- Case 1 : $\longrightarrow$ Joint MAP

$$\left\{\begin{array}{ll}\widehat{q}_1(\boldsymbol{f}|\widetilde{\boldsymbol{f}}) & =\delta(\boldsymbol{f}-\widetilde{\boldsymbol{f}}) \\ \widehat{q}_2(\boldsymbol{\theta}|\widetilde{\boldsymbol{\theta}}) & =\delta(\boldsymbol{\theta}-\widetilde{\boldsymbol{\theta}})\end{array}\right. \longrightarrow \left\{\begin{array}{l}\widetilde{\boldsymbol{f}}= \arg\max_{\boldsymbol{f}}\left\{p(\boldsymbol{f},\widetilde{\boldsymbol{\theta}}|\boldsymbol{g};\mathcal{M})\right\} \\ \widetilde{\boldsymbol{\theta}}= \arg\max_{\boldsymbol{\theta}}\left\{p(\widetilde{\boldsymbol{f}},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})\right\}\end{array}\right.$$

- Case 2 : $\longrightarrow$ EM

$$\left\{\begin{array}{ll}\widehat{q}_1(\boldsymbol{f}) & \propto p(\boldsymbol{f}|\boldsymbol{\theta},\boldsymbol{g}) \\ \widehat{q}_2(\boldsymbol{\theta}|\widetilde{\boldsymbol{\theta}}) & =\delta(\boldsymbol{\theta}-\widetilde{\boldsymbol{\theta}})\end{array}\right. \longrightarrow \left\{\begin{array}{ll}Q(\boldsymbol{\theta},\widetilde{\boldsymbol{\theta}}) & = \langle \ln p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})\rangle_{q_1(\boldsymbol{f}|\widetilde{\boldsymbol{\theta}})} \\ \widetilde{\boldsymbol{\theta}} & = \arg\max_{\boldsymbol{\theta}}\left\{Q(\boldsymbol{\theta},\widetilde{\boldsymbol{\theta}})\right\}\end{array}\right.$$

- Appropriate choice for inverse problems

$$\left\{\begin{array}{lll}\widehat{q}_1(\boldsymbol{f}) & \text{in the same family than} & p(\boldsymbol{f}|\boldsymbol{\theta},\boldsymbol{g};\mathcal{M}) \\ \widehat{q}_2(\boldsymbol{\theta}) & \text{in the same family than} & p(\boldsymbol{\theta}|\boldsymbol{f},\boldsymbol{g};\mathcal{M})\end{array}\right.$$

Variational Bayesian Approximation and model selection

- Joint posterior law:

$$p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g},\mathcal{M}) = \frac{p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g},\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})} = \frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1,\mathcal{M})\,p(\boldsymbol{f}|\boldsymbol{\theta}_2,\mathcal{M})\,p(\boldsymbol{\theta}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$$

where

$$p(\boldsymbol{g}|\mathcal{M}) = \int\int p(\boldsymbol{f},\boldsymbol{\theta},\boldsymbol{g}|\mathcal{M})\,\mathsf{d}\boldsymbol{f}\;\mathsf{d}\boldsymbol{\theta}$$

- Approximate $p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g},\mathcal{M})$ by a simpler $q(\boldsymbol{f},\boldsymbol{\theta})$ by minimizing

$$\mathsf{KL}(q:p) = \int q \ln \frac{q}{p} = \left\langle \ln \frac{q}{p} \right\rangle_q$$

- Free energy:

$$\mathcal{F}(q) = \left\langle \ln \frac{p(\boldsymbol{f},\boldsymbol{\theta},\boldsymbol{g}|\mathcal{M})}{q(\boldsymbol{f},\boldsymbol{\theta})} \right\rangle_q$$

- Link between the evidence of the model $\ln p(\boldsymbol{g}|\mathcal{M})$, $\mathsf{KL}(q:p)$ and $\mathcal{F}(q)$:

$$\mathsf{KL}(q:p) = \ln p(\boldsymbol{g}|\mathcal{M}) - \mathcal{F}(q)$$

Variational Bayesian Approximation

- Alternate optimization of $\mathcal{F}(q)$ or $\text{KL}(q:p)$ with respect to q_1 and q_2 results in

$$\begin{cases} q_1(\boldsymbol{f}) & \propto \exp\left[-\langle \ln p(\boldsymbol{f},\boldsymbol{\theta},\boldsymbol{g})\rangle_{q_2(\boldsymbol{\theta})}\right] \\ q_2(\boldsymbol{\theta}) & \propto \exp\left[-\langle \ln p(\boldsymbol{f},\boldsymbol{\theta},\boldsymbol{g})\rangle_{q_1(\boldsymbol{f})}\right] \end{cases}$$

- $q_1 = \delta(\boldsymbol{f}-\widetilde{\boldsymbol{f}}), \quad q_2 = \delta(\boldsymbol{\theta}-\widetilde{\boldsymbol{\theta}}) \rightarrow$ JMAP:

$$\begin{cases} \widetilde{\boldsymbol{\theta}} & = \arg\max_{\boldsymbol{\theta}} \{p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g})\} \\ \widetilde{\boldsymbol{f}} & = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g})\} \end{cases}$$

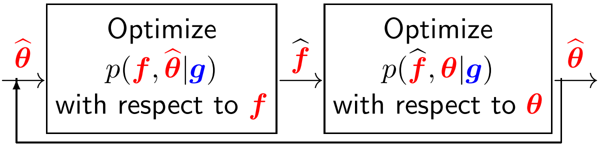
- q_1 free but $q_2 = \delta(\boldsymbol{\theta}-\widetilde{\boldsymbol{\theta}}) \leftarrow$ Expectation-Maximization (EM):

$$\begin{cases} \text{E stap:} & Q(\boldsymbol{\theta}) = \langle \ln p(\boldsymbol{f},\boldsymbol{\theta},\boldsymbol{g})\rangle_{p(\boldsymbol{f}|\boldsymbol{g},\widetilde{\boldsymbol{\theta}})} \\ \text{M stap:} & \widetilde{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \{Q(\boldsymbol{\theta})\} \end{cases}$$

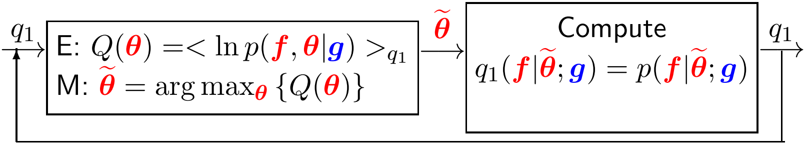
- If p is in the exponential family, then q will be too.

Comparison between JMAP, BEM and VBA

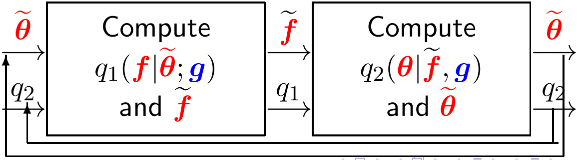
- JMAP:



- BEM:



- VBA:



Hierarchical models and hidden variables

- All the mixture models and some of simple models can be modeled via hidden variables $\boldsymbol{z}$.
- Example 1: Student-t model

$$\begin{cases} p(\boldsymbol{f}|\boldsymbol{z}) = \prod_j p(f_j|z_j) = \prod_j \mathcal{N}(f_j|0, 1/z_j) \propto \exp\left[-\frac{1}{2}\sum_j z_j f_j^2\right] \\ p(z_j|a,b) = \mathcal{G}(z_j|a,b) \propto z_j^{(a-1)} \exp\left[-b z_j\right] \text{ with } a = b = \nu/2 \end{cases}$$

- Example 2: MoG model:

$$\begin{cases} p(\boldsymbol{f}|\boldsymbol{z}) = \prod_j p(f_j|z_j) = \prod_j \mathcal{N}\left(f_j|0, v_{z_j}\right) \propto \exp\left[-\frac{1}{2}\sum_j \frac{f_j^2}{v_{z_j}}\right] \\ P(z_j = 1) = \lambda, \qquad P(z_j = 0) = 1 - \lambda \end{cases}$$

- With these models we have:

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta}_2)\, p(\boldsymbol{z}|\boldsymbol{\theta}_3)\, p(\boldsymbol{\theta})$$

Bayesian Variational Approximation

- Alternate optimization of $\mathcal{F}(q)$ with respect to q_1, q_2 and q_3 results in

$$
\begin{array}{rcl}
q(\boldsymbol{f}) & \propto & \exp\left[-\left\langle \ln p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta},\boldsymbol{g})\right\rangle_{q(\boldsymbol{z})q(\boldsymbol{\theta})}\right] \\
q(\boldsymbol{z}) & \propto & \exp\left[-\left\langle \ln p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta},\boldsymbol{g})\right\rangle_{q(\boldsymbol{f})q(\boldsymbol{\theta})}\right] \\
q(\boldsymbol{\theta}) & \propto & \exp\left[-\left\langle \ln p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta},\boldsymbol{g})\right\rangle_{q(\boldsymbol{f})q(\boldsymbol{z})}\right]
\end{array}
$$

- If p is in the exponential family, then q will be too.
- Other separable decompositions:

$$q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}) = q_1(\boldsymbol{f}|\boldsymbol{z})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta})$$

$$q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}) = \prod_j q_{1j}(f_j|z_j)\, q_{2j}(z_j) \prod_k q_{3k}(\boldsymbol{\theta}_k)$$

BVA with Student-t priors

Scale Mixture Model of Student-t:

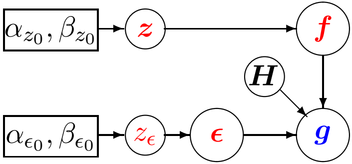
$$\mathcal{S}t(f_j|\nu) = \int_0^\infty \mathcal{N}(f_j|0, \frac{1}{z_j})\, \mathcal{G}(z_j|\nu/2, \nu/2)\, \mathrm{d}z_j$$

Hidden variables z_j:

$$\begin{array}{ll} p(\boldsymbol{f}|\boldsymbol{z}) & = \prod_j p(f_j|z_j) = \prod_j \mathcal{N}(f_j|0, \frac{1}{z_j}) \propto \exp\left[-\frac{1}{2}\sum_j z_j f_j^2\right] \\ p(z_j|\alpha,\beta) & = \mathcal{G}(z_j|\alpha,\beta) \propto z_j^{(\alpha-1)} \exp\left[-\beta z_j\right] \text{ with } \alpha = \beta = \nu/2 \end{array}$$

Cauchy model is obtained when $\nu = 1$:

- Graphical model:



BVA with Student-t priors Algorithm

Likelihood:

$$p(\boldsymbol{g}|\boldsymbol{f}, z_\epsilon) = \mathcal{N}(\boldsymbol{g}|\boldsymbol{H}\boldsymbol{f}, \tfrac{1}{z_\epsilon}\boldsymbol{I})$$
$$p(z_\epsilon|\alpha_{z0}, \beta_{z0}) = \mathcal{G}(z_\epsilon|\alpha_{z0}, \beta_{z0})$$

Prior laws:

$$p(\boldsymbol{f}|\boldsymbol{z}) = \prod_j \mathcal{N}\left(f_j|0, \tfrac{1}{z_j}\right)$$
$$p(\boldsymbol{z}|\alpha_0, \beta_0) = \prod_j \mathcal{G}(z_j|\alpha_0, \beta_0)$$

Joint posterior law:

$$p(\boldsymbol{f}, \boldsymbol{z}, z_\epsilon|\boldsymbol{g}|) \propto p(\boldsymbol{g}|\boldsymbol{f}, z_\epsilon)p(\boldsymbol{f}|\boldsymbol{z})p(\boldsymbol{z})$$

Approximation by:

$$q(\boldsymbol{f}, z_\epsilon) = q_1(\boldsymbol{f}|z_\epsilon)\prod_j q_2(z_j)q_3(z_\epsilon)$$

$$q_1(\boldsymbol{f}|\widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{\Sigma}}) = \mathcal{N}(\boldsymbol{f}|\widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{\Sigma}})$$
$$\widetilde{\boldsymbol{\mu}} = \langle\lambda\rangle_q \widetilde{\boldsymbol{\Sigma}}\boldsymbol{H}'\boldsymbol{g}$$
$$\widetilde{\boldsymbol{\Sigma}} = (\langle\lambda\rangle_q \boldsymbol{H}'\boldsymbol{H} + \widetilde{\boldsymbol{Z}})^{-1} \text{with} \quad \widetilde{\boldsymbol{Z}} = \operatorname{diag}[\widetilde{\boldsymbol{z}}]$$
$$q_{2j}(z_j) = \mathcal{G}(z_j|\widetilde{\alpha}_j, \widetilde{\beta}_j)$$
$$\widetilde{\alpha}_j = \alpha_{00} + \tfrac{1}{2}$$
$$\widetilde{\beta}_j = \beta_{00} + < f_j^2 > /2$$

$$q_3(z_\epsilon) = \mathcal{G}(z_\epsilon|\widetilde{\alpha}_{z_\epsilon}, \widetilde{\beta}_{z_\epsilon}),$$
$$\widetilde{\alpha}_{z_\epsilon} = \alpha_{z_0} + (n+1)/2$$
$$\widetilde{\beta}_{z_\epsilon} = \beta_{z_0} + \tfrac{1}{2}[\|\boldsymbol{g}\|^2$$
$$-2\langle\boldsymbol{f}\rangle_q' \boldsymbol{H}'\boldsymbol{g} + \boldsymbol{H}'\langle\boldsymbol{f}\boldsymbol{f}'\rangle_q \boldsymbol{H}]$$

$$< \boldsymbol{f} > = \widetilde{\boldsymbol{\mu}}, \quad < \boldsymbol{f}\boldsymbol{f}' > = \widetilde{\boldsymbol{\Sigma}} + \widetilde{\boldsymbol{\mu}}\widetilde{\boldsymbol{\mu}}', \quad < f_j^2 > = [\widetilde{\boldsymbol{\Sigma}}]_{jj} + \widetilde{\mu}_j^2, \quad \widetilde{\lambda} = \frac{\widetilde{\alpha}_z}{\widetilde{\beta}_z}, \quad \tilde{z}_j = \frac{\widetilde{\alpha}_j}{\widetilde{\beta}_j}$$

$\widetilde{\lambda} \rightarrow$, $\widetilde{\boldsymbol{z}} \rightarrow$

$$q_1(\boldsymbol{f}|\widetilde{\boldsymbol{z}}, \widetilde{\lambda}) = \mathcal{N}(\widetilde{\boldsymbol{f}}, \widetilde{\boldsymbol{\Sigma}})$$
$$\widetilde{\boldsymbol{f}} = \widetilde{\lambda}\widetilde{\boldsymbol{\Sigma}}\boldsymbol{H}'\boldsymbol{g}$$
$$\widetilde{\boldsymbol{\Sigma}} = (\widetilde{\lambda}\boldsymbol{H}'\boldsymbol{H} + \widetilde{\boldsymbol{z}}^{-1})^{-1}$$

$\widetilde{\boldsymbol{f}} \rightarrow$, $\widetilde{\boldsymbol{\Sigma}} \rightarrow$

$$q_{2j}(z_j|\widetilde{\boldsymbol{f}}) = \mathcal{G}(z_j|\widetilde{\alpha}_j, \widetilde{\beta}_j)$$
$$\widetilde{\alpha}_j = \alpha_{00} + \frac{n+1}{2}$$
$$\widetilde{\beta}_j = \beta_{00} + \frac{1}{2}\left\langle f_j^2\right\rangle_q$$
$$\tilde{z}_j = \widetilde{\alpha}_j/\widetilde{\beta}_j$$

$\widetilde{\boldsymbol{f}} \rightarrow$, $\widetilde{\boldsymbol{\Sigma}} \rightarrow$, $\tilde{z}_j \rightarrow$

$$q_3(z|\widetilde{\boldsymbol{f}}) = \mathcal{G}(z|\widetilde{\alpha}_z, \widetilde{\beta}_z)$$
$$\widetilde{\alpha}_z = \alpha_{z_0} + \frac{n+1}{2}$$
$$\widetilde{\beta}_z = \beta_{z_0} + \frac{1}{2}[\|\boldsymbol{g}\|^2$$
$$-2 < \boldsymbol{f} >' \boldsymbol{H}'\boldsymbol{g} + \boldsymbol{H}' < \boldsymbol{f}\boldsymbol{f}' > \boldsymbol{H}]$$
$$\widetilde{\lambda} = \widetilde{\alpha}_z/\widetilde{\beta}_z$$

$\widetilde{\lambda} \rightarrow$, $\widetilde{\boldsymbol{z}} \rightarrow$

Implementation issues

- To compute $\boldsymbol{f}$, at each iteration, a gradient based optimization algorithm is used. This step is a common step with all the methods. The criterion to optimize is often quadratic:

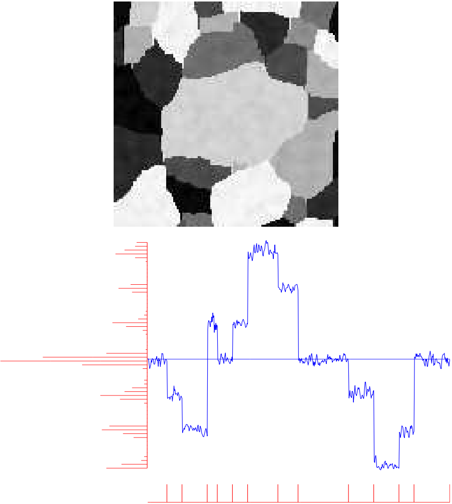
$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2$$

  and its gradient is

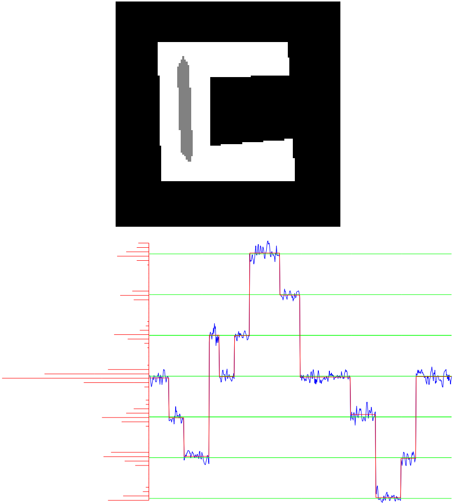
$$\nabla J(\boldsymbol{f}) = 2\boldsymbol{H}'(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}) + 2\lambda\boldsymbol{D}'\boldsymbol{D}\boldsymbol{f}$$

- In inverse problems, often we do not have access directly to the matrix $\boldsymbol{H}$. But, we can compute:
  - Forward operator : $\boldsymbol{H}\boldsymbol{f} \longrightarrow \boldsymbol{g}$ `g=forward(f,...)`
  - Adjoint operator : $\boldsymbol{H}'\boldsymbol{g} \longrightarrow \boldsymbol{f}$ `f=adjoint(g,...)`
- For any particular application, we have to write specific programs (`forward & adjoint`).
- Often the main computational cost is in these two programs. being Bayesian does not cost much more
- The main computational cost for BEM and VBA is the computation and inversion of the covariances.
- Complete factorization may reduce this to the computation of

Two other Hierarchical models

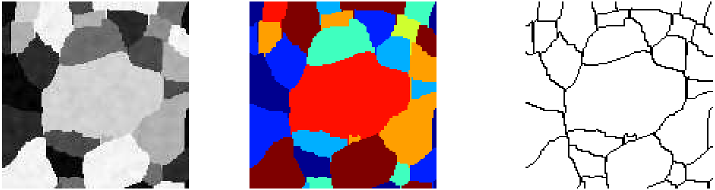


Contours: $p(f_j|(1-q_j)f_{j-1})$



Regions: $p(f_j|z_j = k) = \mathcal{N}(m_k, v_k)$

Gauss-Markov-Potts prior models for images



$f(\boldsymbol{r})$ $\qquad z(\boldsymbol{r})$ $\qquad c(\boldsymbol{r}) = 1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k, m_k, v_k) = \mathcal{N}(m_k, v_k)$$

$$p(f(\boldsymbol{r})) = \sum_k P(z(\boldsymbol{r}) = k)\, \mathcal{N}(m_k, v_k) \quad \text{Mixture of Gaussians}$$

- Separable iid hidden variables: $\quad p(\boldsymbol{z}) = \prod_{\boldsymbol{r}} p(z(\boldsymbol{r}))$
- Markovian hidden variables: $\quad p(\boldsymbol{z})$ Potts-Markov:

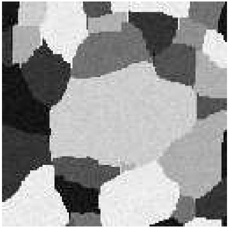
$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left[\gamma \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right]$$

$$p(\boldsymbol{z}) \propto \exp\left[\gamma \sum_{\boldsymbol{r} \in \mathcal{R}} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right]$$

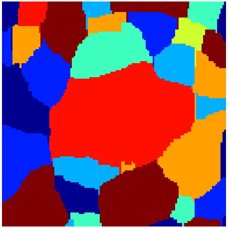
Four different cases

To each pixel of the image is associated 2 variables $f(\boldsymbol{r})$ and $z(\boldsymbol{r})$

- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ iid :
  Mixture of Gaussians
- $\boldsymbol{f}|\boldsymbol{z}$ Gauss-Markov, $\boldsymbol{z}$ iid :
  Mixture of Gauss-Markov
- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ Potts-Markov :
  Mixture of Independent Gaussians
  (MIG with Hidden Potts)
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\boldsymbol{z}$ Potts-Markov :
  Mixture of Gauss-Markov
  (MGM with hidden Potts)



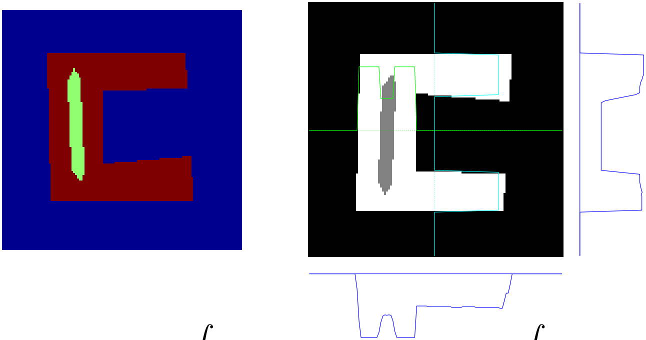
$f(\boldsymbol{r})$



$z(\boldsymbol{r})$

Application of CT in NDT

Reconstruction from only 2 projections

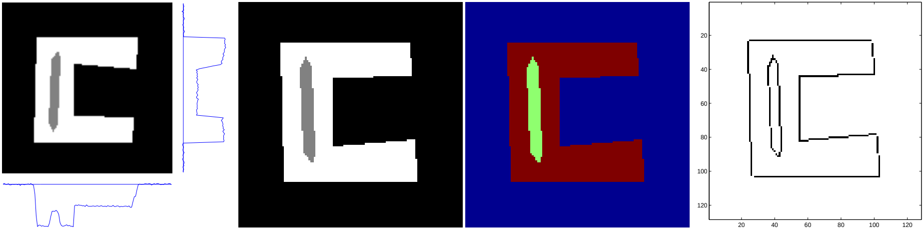


$$g_1(x) = \int f(x,y) \, \mathrm{d}y, \qquad g_2(y) = \int f(x,y) \, \mathrm{d}x$$

- Given the marginals $g_1(x)$ and $g_2(y)$ find the joint distribution $f(x,y)$.
- Infinite number of solutions : $f(x,y) = g_1(x)\, g_2(y)\, \Omega(x,y)$
  $\Omega(x,y)$ is a Copula:

$$\int \Omega(x,y) \, \mathrm{d}x = 1 \quad \text{and} \quad \int \Omega(x,y) \, \mathrm{d}y = 1$$

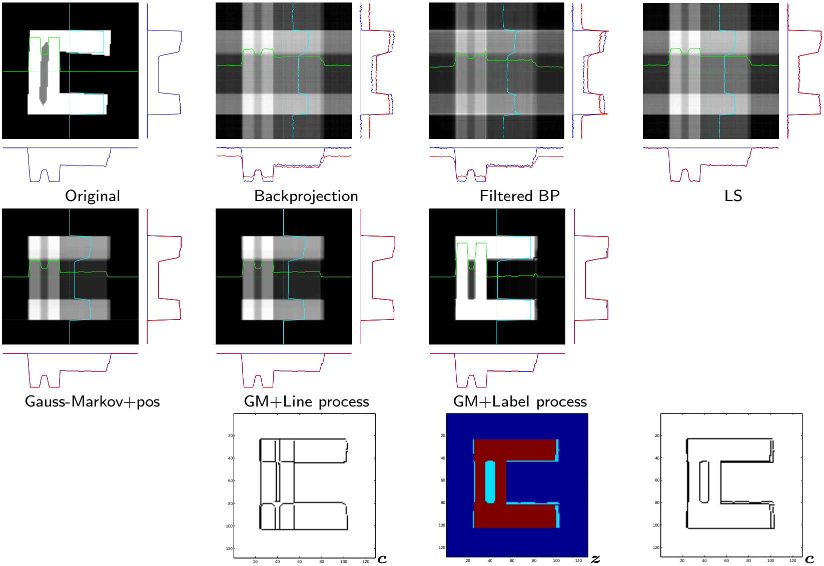
Application in CT



$\boldsymbol{g}\vert\boldsymbol{f}$	$\boldsymbol{f}\vert\boldsymbol{z}$	$\boldsymbol{z}$	$\boldsymbol{c}$
$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$	iid Gaussian	iid	$c(\boldsymbol{r}) \in \{0,1\}$
$\boldsymbol{g}\vert\boldsymbol{f} \sim \mathcal{N}(\boldsymbol{H}\boldsymbol{f}, \sigma_\epsilon^2 \boldsymbol{I})$	or	or	$1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$
Gaussian	Gauss-Markov	Potts	binary

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta}_2)\, p(\boldsymbol{z}|\boldsymbol{\theta}_3)\, p(\boldsymbol{\theta})$$

Results



Original
Backprojection
Filtered BP
LS
Gauss-Markov+pos
GM+Line process
GM+Label process
c
z
c

Conclusions

- Inverse problems arise in many signal and image processing applications.
- Two main steps in the Bayesian approach: Prior modeling and Bayesian computation
- Different prior models: Separable, Markovian and hierarchical models
- Different Bayesian computational tools: Gaussian approximation, MCMC and VBA
- We use these models for inverse problems in different signal and image processing applications such as:
  - Spectral and periodical components estimation in biological time series
  - X ray Computed Tomography,
  - Signal deconvolution in Proteomic and molecular imaging
  - Diffraction Optical Tomography
  - Microwave Imaging, Acoustic imaging and sources localization
  - Synthetic Aperture Radar (SAR) Imaging

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