Inverse Problems: From Regularization to Bayesian Inference

An Overview on Prior Modeling and Bayesian Computation Application to Computed Tomography

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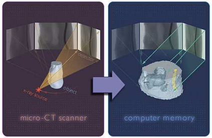
Content

- Image reconstruction in Computed Tomography :
  An ill posed invers problem
- Two main steps in Bayesian approach :
  Prior modeling and Bayesian computation
- Prior models for images :
  - Separable Gaussian, GG, ...
  - Gauss-Markov, General one layer Markovian models
  - Hierarchical Markovian models with hidden variables (contours and regions)
  - Gauss-Markov-Potts
- Bayesian computation
  - MCMC
  - Variational and Mean Field approximations (VBA, MFA)
- Application : Computed Tomography in NDT
- Conclusions and Work in Progress
- Questions and Discussion

Computed Tomography : Making an image of the interior of a body

- $f(x,y)$ a section of a real 3D body $f(x,y,z)$
- $g_\phi(r)$ a line of observed radiographe $g_\phi(r,z)$
- Forward model :
  Line integrals or Radon Transform



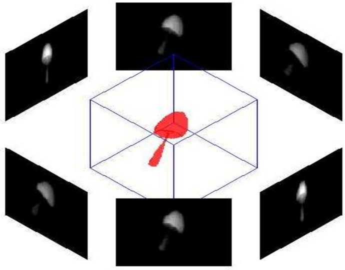
$$
\begin{aligned}
g_\phi(r) &= \int_{L_{r,\phi}} f(x,y)\, \mathrm{d}l + \epsilon_\phi(r) \\
&= \iint f(x,y)\, \delta(r - x\cos\phi - y\sin\phi)\, \mathrm{d}x\, \mathrm{d}y + \epsilon_\phi(r)
\end{aligned}
$$

- Inverse problem : Image reconstruction

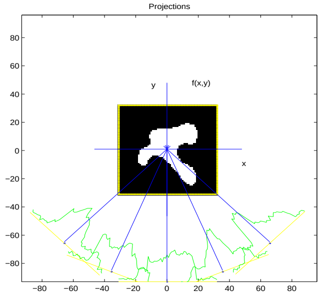
  Given the forward model $\mathcal{H}$ (Radon Transform) and
  a set of data $g_{\phi_i}(r), i = 1, \cdots, M$
  find $f(x,y)$

2D and 3D Computed Tomography

3D



2D

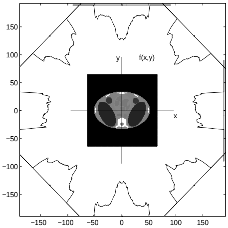


$$g_\phi(r_1,r_2) = \int_{\mathcal{L}_{r_1,r_2,\phi}} f(x,y,z) \, \mathrm{d}l \qquad g_\phi(r) = \int_{\mathcal{L}_{r,\phi}} f(x,y) \, \mathrm{d}l$$

Forward probelm : $f(x,y)$ or $f(x,y,z) \longrightarrow g_\phi(r)$ or $g_\phi(r_1,r_2)$

Inverse problem : $g_\phi(r)$ or $g_\phi(r_1,r_2) \longrightarrow f(x,y)$ or $f(x,y,z)$

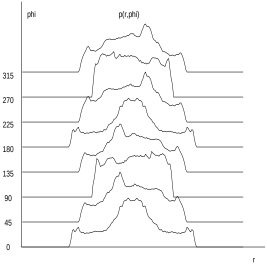
X ray Tomography



$$g(r,\phi) = -\ln\left(\frac{I}{I_0}\right) = \int_{L_{r,\phi}} f(x,y)\, \mathrm{d}l$$

$$g(r,\phi) = \iint_D f(x,y)\, \delta(r - x\cos\phi - y\sin\phi)\, \mathrm{d}x\, \mathrm{d}y$$

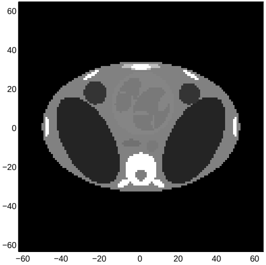
$f(x,y) \longrightarrow$ RT $\longrightarrow g(r,\phi)$



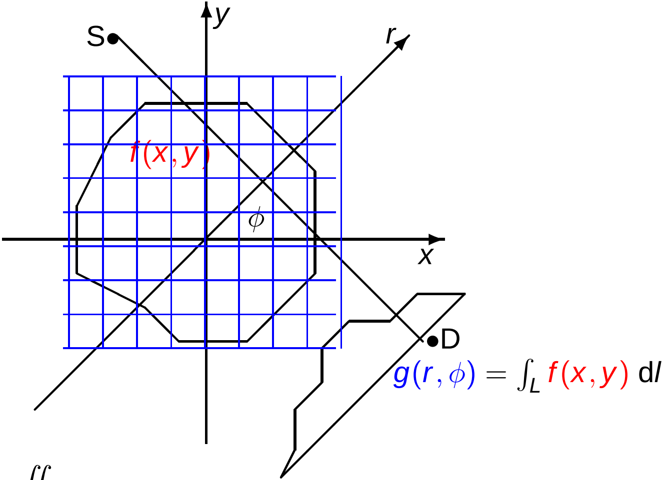
IRT

?

$\Longrightarrow$



Analytical Inversion methods



Radon :

$$g(r,\phi) = \iint_D f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right)\int_0^{\pi}\int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} g(r,\phi)}{(r - x\cos\phi - y\sin\phi)}\,\mathrm{d}r\,\mathrm{d}\phi$$

Filtered Backprojection method

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right) \int_0^{\pi} \int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} g(r,\phi)}{(r - x\cos\phi - y\sin\phi)} \, \mathrm{d}r \, \mathrm{d}\phi$$

Derivation $\mathcal{D}$: $\quad \overline{g}(r,\phi) = \frac{\partial g(r,\phi)}{\partial r}$

Hilbert Transform $\mathcal{H}$: $\quad g_1(r',\phi) = \frac{1}{\pi} \int_0^{\infty} \frac{\overline{g}(r,\phi)}{(r - r')} \, \mathrm{d}r$

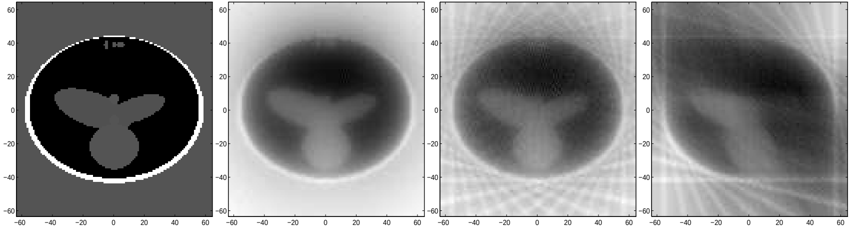
Backprojection $\mathcal{B}$: $\quad f(x,y) = \frac{1}{2\pi} \int_0^{\pi} g_1(r' = x\cos\phi + y\sin\phi, \phi) \, \mathrm{d}\phi$

$$f(x,y) = \mathcal{B}\,\mathcal{H}\,\mathcal{D}\,g(r,\phi) = \mathcal{B}\,\mathcal{F}_1^{-1}\,|\Omega|\,\mathcal{F}_1\,g(r,\phi)$$

- Backprojection of filtered projections :

$\xrightarrow{g(r,\phi)}$ [FT $\mathcal{F}_1$] $\longrightarrow$ [Filter $|\Omega|$] $\longrightarrow$ [IFT $\mathcal{F}_1^{-1}$] $\xrightarrow{g_1(r,\phi)}$ [Backprojection $\mathcal{B}$] $\xrightarrow{f(x,y)}$

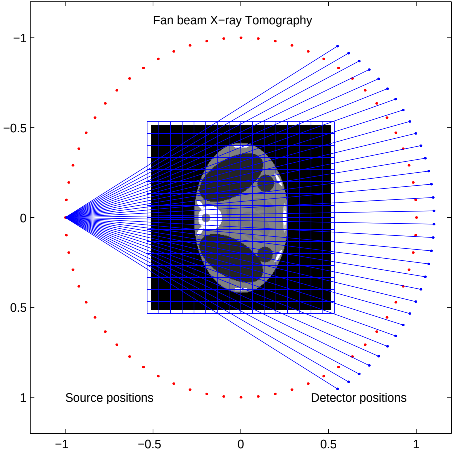
Limitations : Limited angle or noisy data



Original | 64 proj. | 16 proj. | 8 proj. $[0, \pi/2]$

- Limited angle or noisy data
- Accounting for detector size
- Other measurement geometries : fan beam, ...

CT as a linear inverse problem



$$g(s_i) = \int_{L_i} f(\boldsymbol{r}) \; \mathrm{d}l_i + \epsilon(s_i) \longrightarrow \text{Discretization} \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- $\boldsymbol{g}$, $\boldsymbol{f}$ and $\boldsymbol{H}$ are huge dimensional

Inversion : Deterministic methods

Data matching

- Observation model
  $g_i = h_i(\boldsymbol{f}) + \epsilon_i, \quad i = 1, \ldots, M \longrightarrow \boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) + \boldsymbol{\epsilon}$
- Misatch between data and output of the model $\Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f}))$

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{\Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f}))\}$$

- Examples :

– LS $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \|\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})\|^2 = \sum_i |g_i - h_i(\boldsymbol{f})|^2$

– L_p $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \|\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})\|^p = \sum_i |g_i - h_i(\boldsymbol{f})|^p, \quad 1 < p < 2$

– KL $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \sum_i g_i \ln \frac{g_i}{h_i(\boldsymbol{f})}$

- In general, does not give satisfactory results for inverse problems.

Inversion : Regularization theory

Inverse problems = Ill posed problems

$\longrightarrow$ Need for prior information

Functional space (Tikhonov) :

$$g = \mathcal{H}(f) + \epsilon \longrightarrow J(f) = ||g - \mathcal{H}(f)||_2^2 + \lambda||\mathcal{D}f||_2^2$$

Finite dimensional space (Philips & Towmey) : $\boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) + \boldsymbol{\epsilon}$

- Minimum norme LS (MNLS) : $J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda||\boldsymbol{f}||^2$
- Classical regularization : $J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda||\boldsymbol{D}\boldsymbol{f}||^2$
- More general regularization :

$$J(\boldsymbol{f}) = \mathcal{Q}(\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})) + \lambda\Omega(\boldsymbol{D}\boldsymbol{f})$$

or

$$J(\boldsymbol{f}) = \Delta_1(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) + \lambda\Delta_2(\boldsymbol{f}, \boldsymbol{f}_\infty)$$

Limitations :

- Errors are considered implicitly white and Gaussian
- Limited prior information on the solution
- Lack of tools for the determination of the hyperparameters

Bayesian estimation approach

$$\mathcal{M}: \qquad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Observation model $\mathcal{M}$ + Hypothesis on the noise $\boldsymbol{\epsilon} \longrightarrow$
  $p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M}) = p_{\boldsymbol{\epsilon}}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f})$
- A priori information $\quad p(\boldsymbol{f}|\mathcal{M})$
- Bayes : $\quad p(\boldsymbol{f}|\boldsymbol{g};\mathcal{M}) = \dfrac{p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M})\, p(\boldsymbol{f}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$

Link with regularization :

Maximum A Posteriori (MAP) :

$$\begin{aligned} \widehat{\boldsymbol{f}} &= \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{g}|\boldsymbol{f})\, p(\boldsymbol{f})\} \\ &= \arg\min_{\boldsymbol{f}} \{-\ln p(\boldsymbol{g}|\boldsymbol{f}) - \ln p(\boldsymbol{f})\} \end{aligned}$$

with $\quad Q(\boldsymbol{g}, \boldsymbol{H}\boldsymbol{f}) = -\ln p(\boldsymbol{g}|\boldsymbol{f}) \quad$ and $\quad \lambda\Omega(\boldsymbol{f}) = -\ln p(\boldsymbol{f})$

But, Bayesian inference is not only limited to MAP

Case of linear models and Gaussian priors

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Hypothesis on the noise : $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma_\epsilon^2 \boldsymbol{I}) \longrightarrow$
  $p(\boldsymbol{g}|\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2\right\}$
- Hypothesis on $\boldsymbol{f}$: $\boldsymbol{f} \sim \mathcal{N}(0, \sigma_f^2 (\boldsymbol{D}^t\boldsymbol{D})^{-1}) \longrightarrow$
  $p(\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_f^2}\|\boldsymbol{D}\boldsymbol{f}\|^2\right\}$
- A posteriori :
  $$p(\boldsymbol{f}|\boldsymbol{g}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 \frac{1}{2\sigma_f^2}\|\boldsymbol{D}\boldsymbol{f}\|^2\right\}$$
- MAP : $\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$
  with $J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2, \qquad \lambda = \frac{\sigma_\epsilon^2}{\sigma_f^2}$
- Advantage : characterization of the solution

$$\boldsymbol{f}|\boldsymbol{g} \sim \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{P}}) \text{ with } \widehat{\boldsymbol{f}} = \widehat{\boldsymbol{P}}\boldsymbol{H}^t\boldsymbol{g}, \quad \widehat{\boldsymbol{P}} = \left(\boldsymbol{H}^t\boldsymbol{H} + \lambda\boldsymbol{D}^t\boldsymbol{D}\right)^{-1}$$

MAP estimation with other priors :

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{avec} \quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\Omega(\boldsymbol{f})$$

Separable priors :

- Gaussian : $p(f_j) \propto \exp\left\{-\alpha|f_j|^2\right\} \longrightarrow \Omega(\boldsymbol{f}) = \alpha \sum_j |f_j|^2$
- Gamma : $p(f_j) \propto f_j^{\alpha} \exp\left\{-\beta f_j\right\} \longrightarrow \Omega(\boldsymbol{f}) = \alpha \sum_j \ln f_j + \beta f_j$
- Beta :
  $p(f_j) \propto f_j^{\alpha}(1-f_j)^{\beta} \longrightarrow \Omega(\boldsymbol{f}) = \alpha \sum_j \ln f_j + \beta \sum_j \ln(1-f_j)$
- Generalized Gaussian :
  $p(f_j) \propto \exp\left\{-\alpha|f_j|^p\right\}, \quad 1 < p < 2 \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j |f_j|^p,$

Markovian models :

$$p(f_j|\boldsymbol{f}) \propto \exp\left\{-\alpha \sum_{i \in N_j} \phi(f_j, f_i)\right\} \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j \sum_{i \in N_j} \phi(f_j, f_i),$$

MAP estimation with markovien priors

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{with} \quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\Omega(\boldsymbol{f})$$

$$\Omega(\boldsymbol{f}) = \sum_j \phi(f_j - f_{j-1})$$

with $\phi(t)$:

Convex functions :

$$|t|^\alpha, \ \sqrt{1+t^2} - 1, \ \log(\cosh(t)), \ \begin{cases} t^2 & |t| \leq T \\ 2T|t| - T^2 & |t| > T \end{cases}$$

or Non convex functions :

$$\log(1+t^2), \quad \frac{t^2}{1+t^2}, \quad \arctan(t^2), \quad \begin{cases} t^2 & |t| \leq T \\ T^2 & |t| > T \end{cases}$$

MAP estimation with different prior models

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon},\ \boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma_\epsilon^2 \boldsymbol{I})$$
$$\boldsymbol{f} = \boldsymbol{C}\boldsymbol{f} + \boldsymbol{z},\ \boldsymbol{z} \sim \mathcal{N}(0, \sigma_f^2 \boldsymbol{I})$$
$$\boldsymbol{f} = (\boldsymbol{I} - \boldsymbol{C})^{-1}\boldsymbol{z} = \boldsymbol{D}\boldsymbol{z}$$

$$p(\boldsymbol{g}|\boldsymbol{f}) = \mathcal{N}(\boldsymbol{H}\boldsymbol{f}, \sigma_\epsilon^2 \boldsymbol{I})$$
$$p(\boldsymbol{f}) = \mathcal{N}(0, \sigma_f^2 (\boldsymbol{D}^t\boldsymbol{D})^{-1})$$

$$p(\boldsymbol{f}|\boldsymbol{g}) = \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{P}}_f)$$
$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$$
$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2$$

$$\widehat{\boldsymbol{f}} = \widehat{\boldsymbol{P}}\boldsymbol{H}^t\boldsymbol{g}$$
$$\widehat{\boldsymbol{P}}_f = \left(\boldsymbol{H}^t\boldsymbol{H} + \lambda\boldsymbol{D}^t\boldsymbol{D}\right)^{-1}$$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon},\ \boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma_\epsilon^2 \boldsymbol{I})$$
$$\boldsymbol{f} = \boldsymbol{W}\boldsymbol{z},\ \boldsymbol{z} \sim \mathcal{N}(0, \sigma_z^2 \boldsymbol{I})$$
$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{W}\boldsymbol{z} + \boldsymbol{\epsilon}$$

$$p(\boldsymbol{g}|\boldsymbol{z}) = \mathcal{N}(\boldsymbol{H}\boldsymbol{W}\boldsymbol{z}, \sigma_\epsilon^2 \boldsymbol{I})$$
$$p(\boldsymbol{z}) = \mathcal{N}(0, \sigma_z^2 \boldsymbol{I})$$

$$p(\boldsymbol{z}|\boldsymbol{g}) = \mathcal{N}(\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{P}}_z)$$
$$\widehat{\boldsymbol{z}} = \arg\min_{\boldsymbol{z}} \{J(\boldsymbol{z})\}$$
$$J(\boldsymbol{z}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{W}\boldsymbol{z}\|^2 + \lambda\|\boldsymbol{z}\|^2$$

$$\widehat{\boldsymbol{z}} = \widehat{\boldsymbol{P}}_z\boldsymbol{H}^t\boldsymbol{W}^t\boldsymbol{g}$$
$$\widehat{\boldsymbol{P}}_z = \left(\boldsymbol{H}^t\boldsymbol{W}^t\boldsymbol{H} + \lambda\boldsymbol{I}\right)^{-1}$$

$$\widehat{\boldsymbol{f}} = \boldsymbol{W}\widehat{\boldsymbol{z}}$$
$$\widehat{\boldsymbol{P}}_f = \boldsymbol{W}\widehat{\boldsymbol{P}}_z\boldsymbol{W}^t$$

MAP estimation with different prior models

$$\left\{\begin{array}{l} \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \\ \boldsymbol{f} \sim \mathcal{N}\left(0, \sigma_f^2 (\boldsymbol{D}^t\boldsymbol{D})^{-1}\right) \end{array}\right. = \left\{\begin{array}{l} \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \\ \boldsymbol{f} = \boldsymbol{C}\boldsymbol{f} + \boldsymbol{z} \quad \text{with} \quad \boldsymbol{z} \sim \mathcal{N}(0, \sigma_f^2 \boldsymbol{I}) \\ \boldsymbol{D}\boldsymbol{f} = \boldsymbol{z} \text{ with } \boldsymbol{D} = (\boldsymbol{I} - \boldsymbol{C}) \end{array}\right.$$

$$\boldsymbol{f}|\boldsymbol{g} \sim \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{P}}_f) \quad \text{with} \quad \widehat{\boldsymbol{f}} = \widehat{\boldsymbol{P}}_f \boldsymbol{H}^t \boldsymbol{g}, \quad \widehat{\boldsymbol{P}}_f = \left(\boldsymbol{H}^t\boldsymbol{H} + \lambda \boldsymbol{D}^t\boldsymbol{D}\right)^{-1}$$

$$J(\boldsymbol{f}) = -\ln p(\boldsymbol{f}|\boldsymbol{g}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda \|\boldsymbol{D}\boldsymbol{f}\|^2$$

$$\left\{\begin{array}{l} \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \\ \boldsymbol{f} \sim \mathcal{N}\left(0, \sigma_f^2 (\boldsymbol{W}\boldsymbol{W}^t)\right) \end{array}\right. = \left\{\begin{array}{l} \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \\ \boldsymbol{f} = \boldsymbol{W}\boldsymbol{z} \quad \text{with} \quad \boldsymbol{z} \sim \mathcal{N}(0, \sigma_f^2 \boldsymbol{I}) \end{array}\right.$$

$$\boldsymbol{z}|\boldsymbol{g} \sim \mathcal{N}(\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{P}}_z) \text{ with } \widehat{\boldsymbol{z}} = \widehat{\boldsymbol{P}}_z \boldsymbol{W}^t \boldsymbol{H}^t \boldsymbol{g}, \ \widehat{\boldsymbol{P}}_z = \left(\boldsymbol{W}^t\boldsymbol{H}^t\boldsymbol{H}\boldsymbol{W} + \lambda \boldsymbol{I}\right)^{-1}$$

$$J(\boldsymbol{z}) = -\ln p(\boldsymbol{z}|\boldsymbol{g}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{W}\boldsymbol{z}\|^2 + \lambda \|\boldsymbol{z}\|^2 \longrightarrow \widehat{\boldsymbol{f}} = \boldsymbol{W}\widehat{\boldsymbol{z}}$$

$\boldsymbol{z}$ decomposqition coefficients

MAP estimation and Compressed Sensing

$$\begin{cases} \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \\ \boldsymbol{f} = \boldsymbol{W}\boldsymbol{z} \end{cases}$$

- $\boldsymbol{W}$ a code book matrix, $\boldsymbol{z}$ coefficients
- Gaussian :

$$p(\boldsymbol{z}) = \mathcal{N}(0, \sigma_z^2 \boldsymbol{I}) \propto \exp\left\{-\frac{1}{2\sigma_z^2}\sum_j |z_j|^2\right\}$$
$$J(\boldsymbol{z}) = -\ln p(\boldsymbol{z}|\boldsymbol{g}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{W}\boldsymbol{z}\|^2 + \lambda \sum_j |z_j|^2$$

- Generalized Gaussian (sparsity, $\beta = 1$) :

$$p(\boldsymbol{z}) \propto \exp\left\{-\lambda \sum_j |z_j|^\beta\right\}$$
$$J(\boldsymbol{z}) = -\ln p(\boldsymbol{z}|\boldsymbol{g}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{W}\boldsymbol{z}\|^2 + \lambda \sum_j |z_j|^\beta$$

- $\boldsymbol{z} = \arg\min_{\boldsymbol{z}} \{J(\boldsymbol{z})\} \longrightarrow \widehat{\boldsymbol{f}} = \boldsymbol{W}\widehat{\boldsymbol{z}}$

Main advantages of the Bayesian approach

- MAP = Regularization
- Posterior mean ? Marginal MAP ?
- More information in the posterior law than only its mode or its mean
- Meaning and tools for estimating hyper parameters
- Meaning and tools for model selection
- More specific and specialized priors, particularly through the hidden variables
- More computational tools :
  - Expectation-Maximization for computing the maximum likelihood parameters
  - MCMC for posterior exploration
  - Variational Bayes for analytical computation of the posterior marginals
  - ...

Full Bayesian approach

$$\mathcal{M}: \qquad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

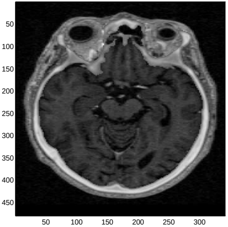
- Forward & errors model : $\longrightarrow p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1; \mathcal{M})$
- Prior models $\longrightarrow p(\boldsymbol{f}|\boldsymbol{\theta}_2; \mathcal{M})$
- Hyperparameters $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \longrightarrow p(\boldsymbol{\theta}|\mathcal{M})$
- Bayes : $\longrightarrow p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\, p(\boldsymbol{f}|\boldsymbol{\theta};\mathcal{M})\, p(\boldsymbol{\theta}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$
- Joint MAP : $\qquad (\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f},\boldsymbol{\theta})} \{p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\}$
- Marginalization : $\left\{ \begin{array}{ll} p(\boldsymbol{f}|\boldsymbol{g}; \mathcal{M}) & = \int p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f} \\ p(\boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M}) & = \int p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{\theta} \end{array} \right.$
- Posterior means : $\left\{ \begin{array}{ll} \widehat{\boldsymbol{f}} & = \int \boldsymbol{f}\, p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta} \\ \widehat{\boldsymbol{\theta}} & = \int \boldsymbol{\theta}\, p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta} \end{array} \right.$
- Evidence of the model :

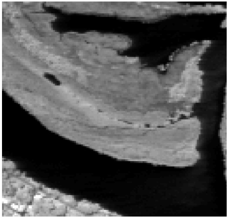
$$p(\boldsymbol{g}|\mathcal{M}) = \iint p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{f}|\boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{\theta}|\mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta}$$

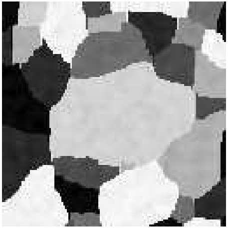
Two main steps in the Bayesian approach

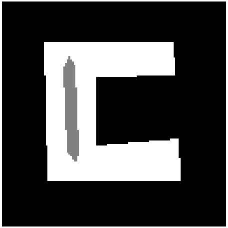
- Prior modeling
  - Separable :
    Gaussian, Generalized Gaussian, Gamma,
    mixture of Gaussians, mixture of Gammas, ...
  - Markovian : Gauss-Markov, GGM, ...
  - Separable or Markovian with hidden variables
    (contours, region labels)
- Choice of the estimator and computational aspects
  - MAP, Posterior mean, Marginal MAP
  - MAP needs optimization algorithms
  - Posterior mean needs integration methods
  - Marginal MAP needs integration and optimization
  - Approximations :
    - Gaussian approximation (Laplace)
    - Numerical exploration MCMC
    - Variational Bayes (Separable approximation)

Which images I am looking for ?

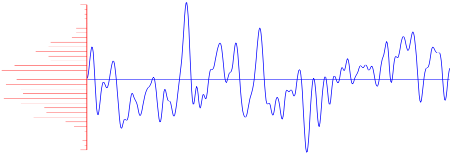






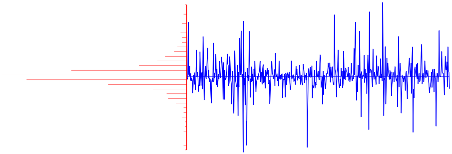


Which image I am looking for ?



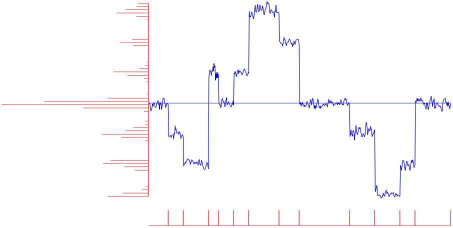
Gaussian

$p(f_j|f_{j-1}) \propto \exp\left\{-\alpha|f_j - f_{j-1}|^2\right\}$



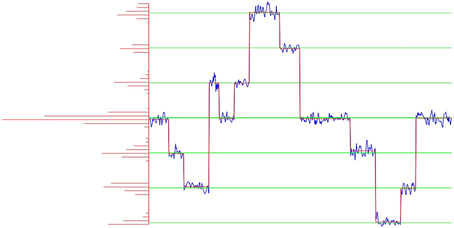
Generalized Gaussian

$p(f_j|f_{j-1}) \propto \exp\left\{-\alpha|f_j - f_{j-1}|^p\right\}$



Piecewize Gaussian

$p(f_j|q_j, f_{j-1}) = \mathcal{N}\left((1 - q_j)f_{j-1}, \sigma_f^2\right)$



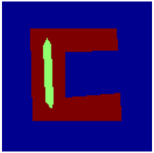
Mixture of GM

$p(f_j|z_j = k) = \mathcal{N}\left(m_k, \sigma_k^2\right)$

Gauss-Markov-Potts prior models for images

"In NDT applications of CT, the objects are, in general, composed of a finite number of materials, and the voxels corresponding to each materials are grouped in compact regions"

How to model this prior information ?



$f(\boldsymbol{r})$ $\qquad$ $z(\boldsymbol{r}) \in \{1, ..., K\}$

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k, m_k, v_k) = \mathcal{N}(m_k, v_k)$$

$$p(f(\boldsymbol{r})) = \sum_k P(z(\boldsymbol{r}) = k) \mathcal{N}(m_k, v_k) \quad \text{Mixture of Gaussians}$$

$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Four different cases

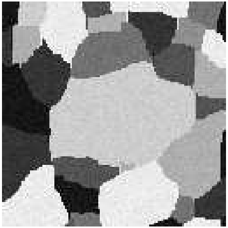
To each pixel of the image is associated 2 variables $f(\boldsymbol{r})$ and $z(\boldsymbol{r})$

- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ iid :
  Mixture of Gaussians

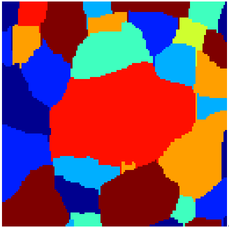
- $\boldsymbol{f}|\boldsymbol{z}$ Gauss-Markov, $\boldsymbol{z}$ iid :
  Mixture of Gauss-Markov

- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ Potts-Markov :
  Mixture of Independent Gaussians
  (MIG with Hidden Potts)

- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\boldsymbol{z}$ Potts-Markov :
  Mixture of Gauss-Markov
  (MGM with hidden Potts)

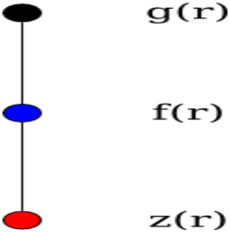


$f(\boldsymbol{r})$

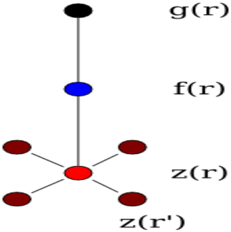


$z(\boldsymbol{r})$

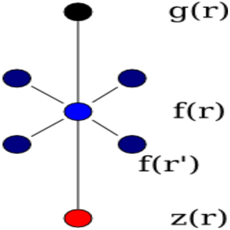
Four different cases



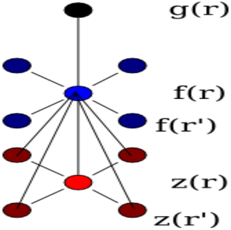
Case 1 : Mixture of Gaussians



Case 2 : Mixture of Gauss-Markov

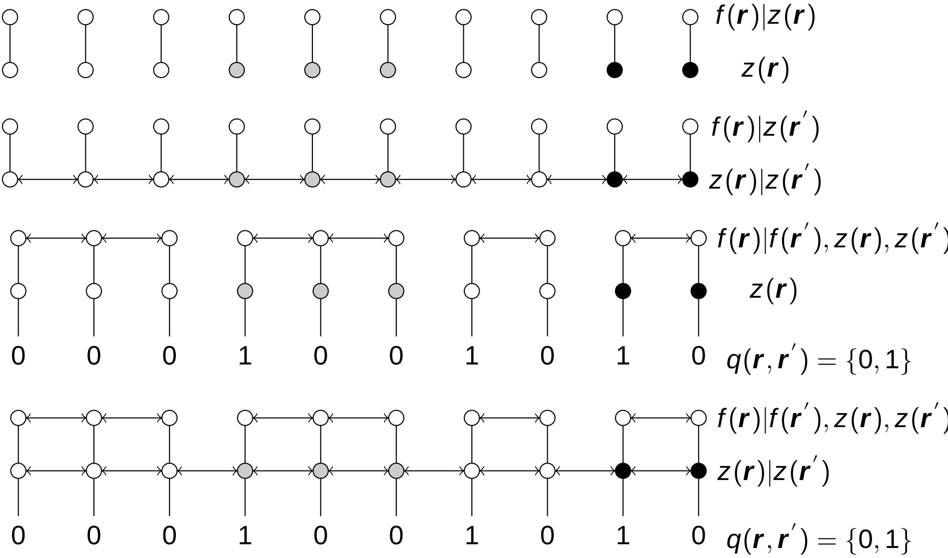


Case 3 : MIG with Hidden Potts



Case 4 : MGM with hidden Potts

Four different cases



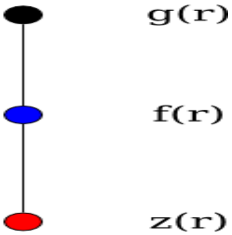
Case 1 : $f|z$ Gaussian iid, z iid

Independent Mixture of Independent Gaussiens (IMIG) :

$$p(f(\boldsymbol{r})|z(\boldsymbol{r})=k)=\mathcal{N}(m_k,v_k), \quad \forall \boldsymbol{r}\in\mathcal{R}$$

$$p(f(\boldsymbol{r}))=\sum_{k=1}^{K}\alpha_k\mathcal{N}(m_k,v_k), \text{with } \sum_k \alpha_k=1.$$

$$p(\boldsymbol{z})=\prod_{\boldsymbol{r}} p(z(\boldsymbol{r})=k)=\prod_{\boldsymbol{r}}\alpha_k=\prod_k \alpha_k^{n_k}$$



Noting $\quad \mathcal{R}_k=\{\boldsymbol{r}: z(\boldsymbol{r})=k\}, \quad \mathcal{R}=\cup_k\mathcal{R}_k,$

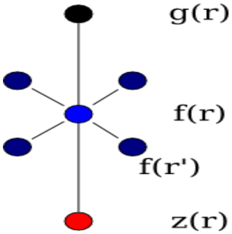
$$m_z(\boldsymbol{r})=m_k, v_z(\boldsymbol{r})=v_k, \alpha_z(\boldsymbol{r})=\alpha_k, \forall\boldsymbol{r}\in\mathcal{R}_k$$

we have :

$$\begin{aligned} p(\boldsymbol{f}|\boldsymbol{z}) &= \prod_{\boldsymbol{r}\in\mathcal{R}}\mathcal{N}(m_z(\boldsymbol{r}),v_z(\boldsymbol{r})) \\ p(\boldsymbol{z}) &= \prod_{\boldsymbol{r}}\alpha_z(\boldsymbol{r})=\prod_k \alpha_k^{\sum_{\boldsymbol{r}\in\mathcal{R}}\delta(z(\boldsymbol{r})-k)}=\prod_k\alpha_k^{n_k} \end{aligned}$$

Case 2 : $f|z$ Gauss-Markov, z iid

Independent Mixture
of Gauss-Markov (IMGM) :



$$p(f(\boldsymbol{r})|z(\boldsymbol{r}), z(\boldsymbol{r}'), f(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r}))$$

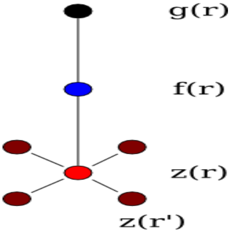
$$= \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})), \forall \boldsymbol{r} \in \mathcal{R}$$

$$\begin{aligned}
\mu_z(\boldsymbol{r}) &= \frac{1}{|\mathcal{V}(\boldsymbol{r})|} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \mu_z^*(\boldsymbol{r}') \\
\mu_z^*(\boldsymbol{r}') &= \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, f(\boldsymbol{r}') + (1 - \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, m_z(\boldsymbol{r}') \\
&= (1 - c(\boldsymbol{r}'))\quad f(\boldsymbol{r}')\ + \quad c(\boldsymbol{r}') \quad m_z(\boldsymbol{r}')
\end{aligned}$$

$$\begin{aligned}
p(\boldsymbol{f}|\boldsymbol{z}) &\propto \prod_{\boldsymbol{r}} \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})) &&\propto \prod_k \alpha_k \mathcal{N}(m_k 1, \boldsymbol{\Sigma}_k) \\
p(\boldsymbol{z}) &= \prod_{\boldsymbol{r}} v_z(\boldsymbol{r}) &&= \prod_k \alpha_k^{n_k}
\end{aligned}$$

with $1_k = 1, \forall \boldsymbol{r} \in \mathcal{R}_k$ and $\boldsymbol{\Sigma}_k$ a covariance matrix ($n_k \times n_k$).

Case 3 : $f|z$ Gauss iid, z Potts



Gauss iid as in Case 1 :

$$\begin{aligned} p(\boldsymbol{f}|\boldsymbol{z}) &= \prod_{\boldsymbol{r}\in\mathcal{R}} \mathcal{N}(m_z(\boldsymbol{r}), v_z(\boldsymbol{r})) \\ &= \prod_k \prod_{\boldsymbol{r}\in\mathcal{R}_k} \mathcal{N}(m_k, v_k) \end{aligned}$$

Potts-Markov :

$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

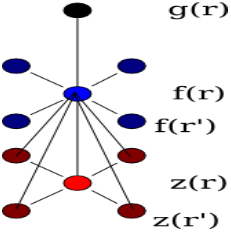
$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Case 4 : $f|z$ Gauss-Markov, z Potts

Gauss-Markov as in Case 2 :

$p(f(\boldsymbol{r})|z(\boldsymbol{r}), z(\boldsymbol{r}'), f(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) =$

$\mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})), \forall \boldsymbol{r} \in \mathcal{R}$



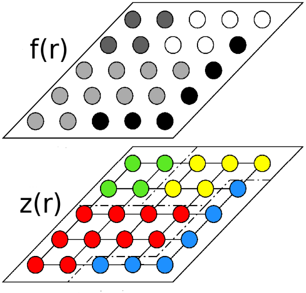
$$\begin{aligned} \mu_z(\boldsymbol{r}) &= \frac{1}{|\mathcal{V}(\boldsymbol{r})|} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \mu_z^*(\boldsymbol{r}') \\ \mu_z^*(\boldsymbol{r}') &= \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, f(\boldsymbol{r}') + (1 - \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, m_z(\boldsymbol{r}') \end{aligned}$$

$$p(\boldsymbol{f}|\boldsymbol{z}) \quad \propto \prod_{\boldsymbol{r}} \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})) \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \boldsymbol{\Sigma}_k)$$

Potts-Markov as in Case 3 :

$$p(\boldsymbol{z}) \propto \exp\left\{ \gamma \sum_{\boldsymbol{r} \in \mathcal{R}} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}')) \right\}$$

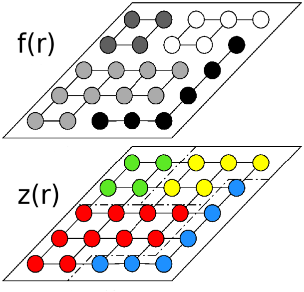
Summary of the two proposed models



$\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid

$\boldsymbol{z}$ Potts-Markov

(MIG with Hidden Potts)



$\boldsymbol{f}|\boldsymbol{z}$ Markov

$\boldsymbol{z}$ Potts-Markov

(MGM with hidden Potts)

Bayesian Computation

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{z}, v_\epsilon)\, p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{m}, \boldsymbol{v})\, p(\boldsymbol{z}|\gamma, \boldsymbol{\alpha})\, p(\boldsymbol{\theta})$$

$\boldsymbol{\theta} = \{v_\epsilon, (\alpha_k, m_k, v_k), k = 1, \cdot, K\}$ $\quad p(\boldsymbol{\theta})$ Conjugate priors

- Direct computation and use of $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})$ is too complex
- Possible approximations :
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)
- Main idea in Variational Bayesian methods :
  Approximate
  $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})$ by $q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta})$
  - Choice of approximation criterion : $KL(q : p)$
  - Choice of appropriate families of probability laws for $q_1(\boldsymbol{f})$, $q_2(\boldsymbol{z})$ and $q_3(\boldsymbol{\theta})$

MCMC based algorithm

$$p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta})\,p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta})\,p(\boldsymbol{z})\,p(\boldsymbol{\theta})$$

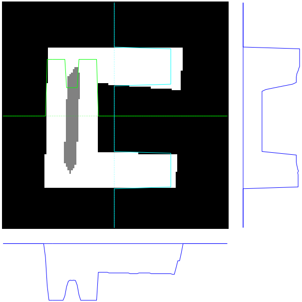
General scheme :

$$\widehat{\boldsymbol{f}} \sim p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{z}} \sim p(\boldsymbol{z}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\boldsymbol{g})$$

- Sample $\boldsymbol{f}$ from $p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta})\,p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}})$
  Needs optimisation of a quadratic criterion.
- Sample $\boldsymbol{z}$ from $p(\boldsymbol{z}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}})\,p(\boldsymbol{z})$
  Needs sampling of a Potts Markov field.
- Sample $\boldsymbol{\theta}$ from
  $p(\boldsymbol{\theta}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}},\sigma_\epsilon^2\boldsymbol{I})\,p(\widehat{\boldsymbol{f}}|\widehat{\boldsymbol{z}},(m_k,v_k))\,p(\boldsymbol{\theta})$
  Conjugate priors $\longrightarrow$ analytical expressions.

Application of CT in NDT

Reconstruction from only 2 projections



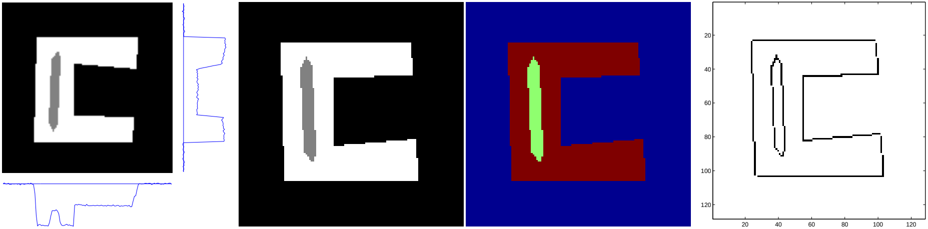
$$g_1(x) = \int f(x,y) \, dy$$

$$g_2(y) = \int f(x,y) \, dx$$

- Given the marginals $g_1(x)$ and $g_2(y)$ find the joint distribution $f(x,y)$.
- Infinite number of solutions : $f(x,y) = g_1(x)\, g_2(y)\, \Omega(x,y)$
  $\Omega(x,y)$ is a Copula :

$$\int \Omega(x,y) \, dx = 1 \quad \text{and} \quad \int \Omega(x,y) \, dy = 1$$

Application in CT



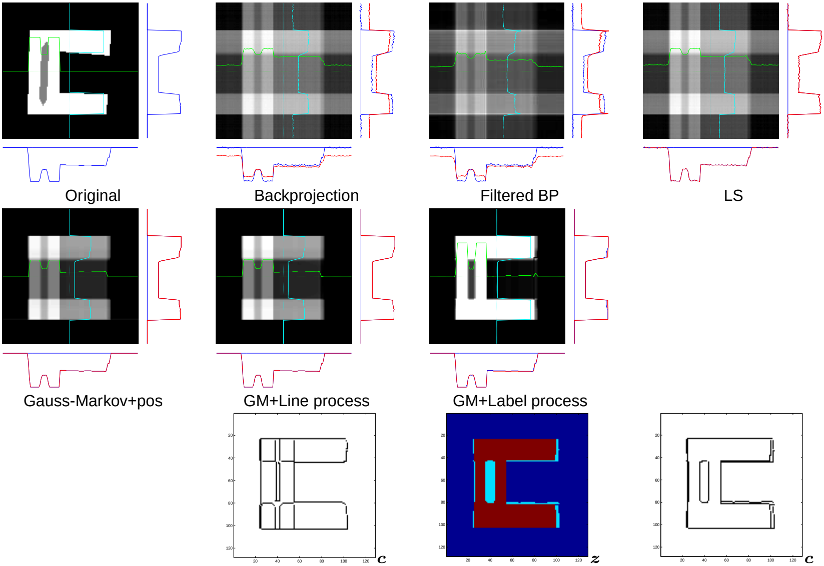
$g\|f$	$f\|z$	z	q
$g = Hf + \epsilon$	iid Gaussian	iid	$q(r) \in \{0,1\}$
$g\|f \sim \mathcal{N}(Hf, \sigma_\epsilon^2 I)$	or	or	$1-\delta(z(r)-z(r'))$
Gaussian	Gauss-Markov	Potts	binary

Forward model | Gauss-Markov-Potts Prior Model | Auxilary

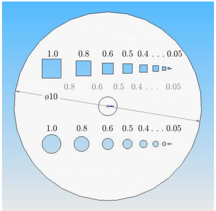
Unsupervised Bayesian estimation :

$$p(f, z, \theta | g) \propto p(g | f, z, \theta)\, p(f | z, \theta)\, p(\theta)$$

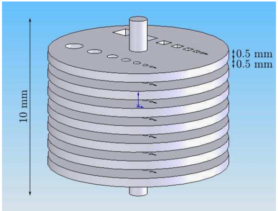
Results : 2D case



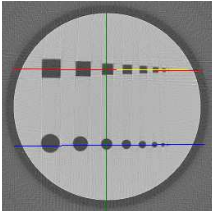
Some results in 3D case



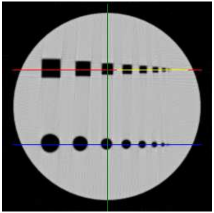
M. Defrise



Phantom

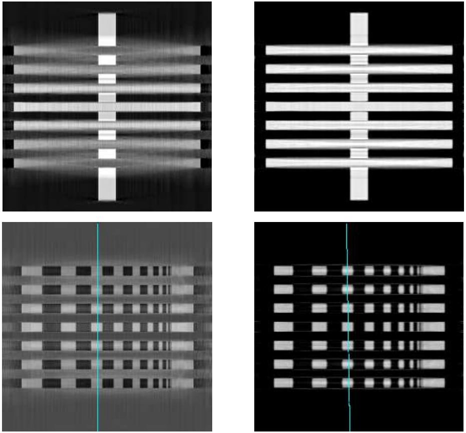


FeldKamp



Proposed method

Some results in 3D case

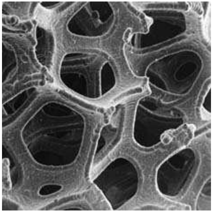


FeldKamp Proposed method

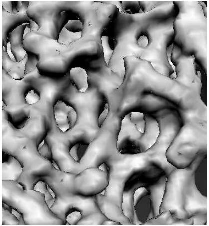
Some results in 3D case

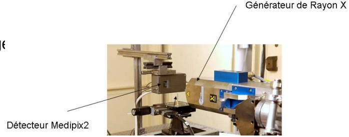
Experimental setup

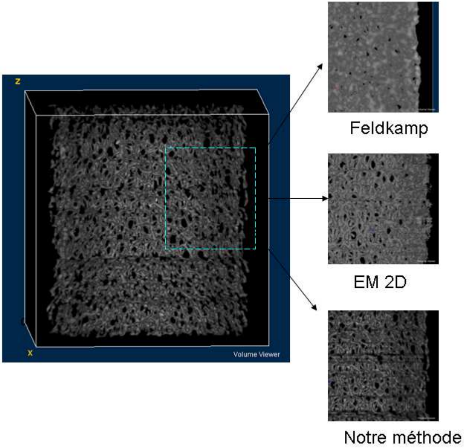
A photograpy of metalique esponge



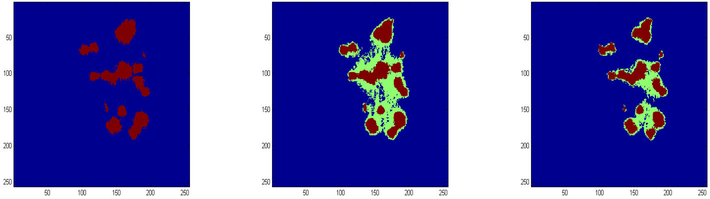
Reconstruction by proposed metho







Application : liquid evaporation in metalic esponge



Time 0 Time 1 Time 2

Conclusions

- Bayesian estimation approach gives more tools than deterministic regularization to handle inverses problems
- Gauss-Markov-Potts are useful prior models for images incorporating regions and contours
- Bayesian computation needs either multi-dimensional optimization or integration methods
- Different optimization and integration tools and approximations exist (Laplace, MCMC, Variational Bayes)

Work in Progress and Perspectives :

- Efficient implementation in 2D and 3D cases using GPU
- Application to other linear and non linear inverse problems : X ray CT, Ultrasound and Microwave imaging, PET, SPECT, ...