Regularization and Bayesian Inference Approach for Inverse Problems in Imaging Systems and Computer Vision

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- Inversion methods :
  analytical, parametric and non parametric
- Determinitic methods:
  Data matching, Least Squares, Regularization
- Probabilistic methods:
  Probability matching, Maximum likelihood, Bayesian inference
- Bayesian inference approach
- Prior models for images
- Bayesian computation
- Applications:
  Computed Tomography, Image separation, Superresolution, SAR Imaging
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- Questions and Discussion

Inverse problems : 3 main examples

- Example 1:
  Measuring variation of temperature with a therometer
  - $f(t)$ variation of temperature over time
  - $g(t)$ variation of length of the liquid in thermometer
- Example 2:
  Making an image with a camera, a microscope or a telescope
  - $f(x, y)$ real scene
  - $g(x, y)$ observed image
- Example 3: Making an image of the interior of a body
  - $f(x, y)$ a section of a real 3D body $f(x, y, z)$
  - $g_\phi(r)$ a line of observed radiographe $g_\phi(r, z)$

- Example 1: Deconvolution
- Example 2: Image restoration
- Example 3: Image reconstruction

Measuring variation of temperature with a therometer

- $f(t)$ variation of temperature over time
- $g(t)$ variation of length of the liquid in thermometer

- Forward model: Convolution

$$g(t) = \int f(t') \, h(t - t') \, \mathrm{d}t' + \epsilon(t)$$

  $h(t)$: impulse response of the measurement system

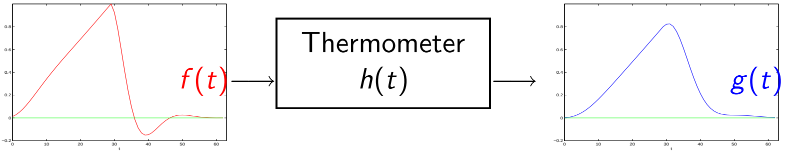
- Inverse problem: Deconvolution

  Given the forward model $\mathcal{H}$ (impulse response $h(t)$)
  and a set of data $g(t_i), i = 1, \cdots, M$
  find $f(t)$

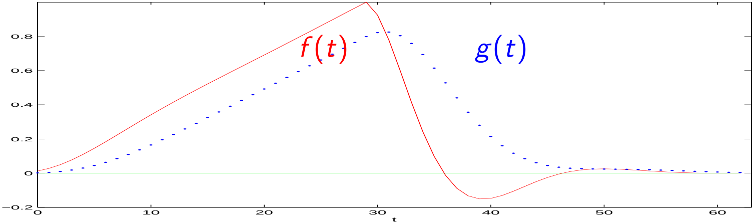
Measuring variation of temperature with a therometer

Forward model: Convolution

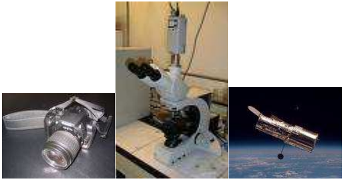
$$g(t) = \int f(t')\, h(t - t')\, \mathrm{d}t' + \epsilon(t)$$



Inversion: Deconvolution



Making an image with a camera, a microscope or a telescope



- $f(x, y)$ real scene
- $g(x, y)$ observed image
- Forward model: Convolution

$$g(x, y) = \iint f(x', y')\, h(x - x', y - y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x, y)$$

$h(x, y)$: Point Spread Function (PSF) of the imaging system

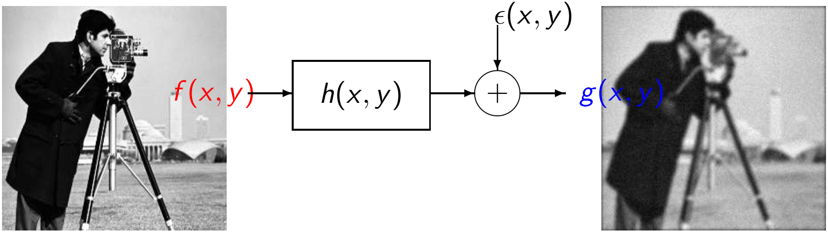
- Inverse problem: Image restoration

Given the forward model $\mathcal{H}$ (PSF $h(x, y)$))
and a set of data $g(x_i, y_i), i = 1, \cdots, M$
find $f(x, y)$

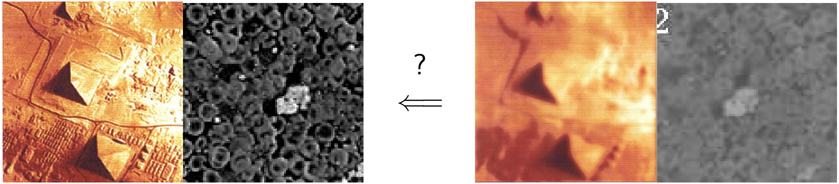
Making an image with an unfocused camera

Forward model: 2D Convolution

$$g(x,y) = \iint f(x',y')\, h(x-x', y-y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x,y)$$

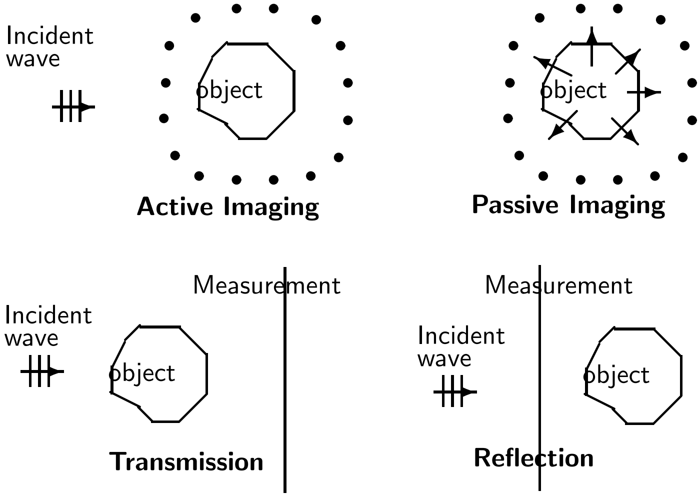


Inversion: Deconvolution



Making an image of the interior of a body

Different imaging systems:

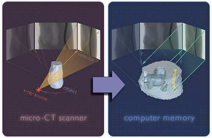


Forward problem: Knowing the object predict the data

Inverse problem: From measured data find the object

Making an image of the interior of a body

- $f(x,y)$ a section of a real 3D body $f(x,y,z)$
- $g_\phi(r)$ a line of observed radiographe $g_\phi(r,z)$



- Forward model:
  Line integrals or Radon Transform

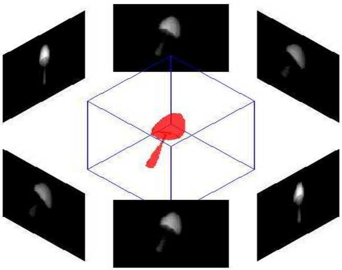
$$
\begin{aligned}
g_\phi(r) &= \int_{L_{r,\phi}} f(x,y)\,\mathrm{d}l + \epsilon_\phi(r) \\
&= \iint f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y + \epsilon_\phi(r)
\end{aligned}
$$

- Inverse problem: Image reconstruction

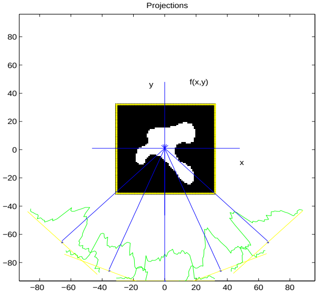
  Given the forward model $\mathcal{H}$ (Radon Transform) and
  a set of data $g_{\phi_i}(r), i = 1, \cdots, M$
  find $f(x,y)$

2D and 3D Computed Tomography

3D



2D



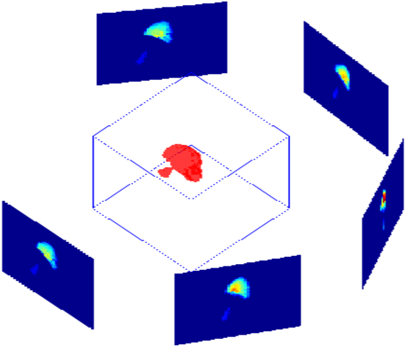
$$g_\phi(r_1, r_2) = \int_{\mathcal{L}_{r_1,r_2,\phi}} f(x, y, z) \, \mathrm{d}l \qquad g_\phi(r) = \int_{\mathcal{L}_{r,\phi}} f(x, y) \, \mathrm{d}l$$

Forward probelm: $f(x, y)$ or $f(x, y, z) \longrightarrow g_\phi(r)$ or $g_\phi(r_1, r_2)$

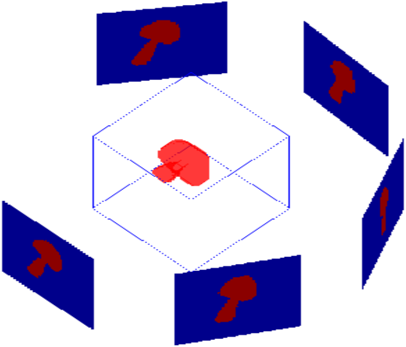
Inverse problem: $g_\phi(r)$ or $g_\phi(r_1, r_2) \longrightarrow f(x, y)$ or $f(x, y, z)$

3D Computed Tomography / 3D Shape from shadows

3D Computed Tomography

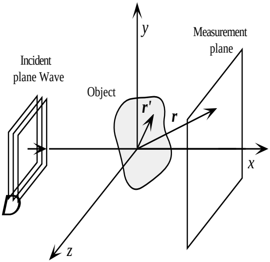


3D Shape from shadows



Microwave or ultrasound imaging

Measurs: diffracted wave by the object $g(\boldsymbol{r}_i)$
Unknown quantity: $f(\boldsymbol{r}) = k_0^2(n^2(\boldsymbol{r}) - 1)$
Intermediate quantity : $\phi(\boldsymbol{r})$

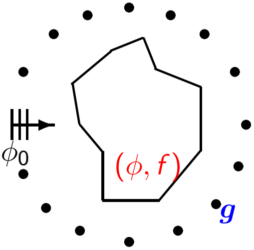


$$g(\boldsymbol{r}_i) = \iint_D G_m(\boldsymbol{r}_i, \boldsymbol{r}')\phi(\boldsymbol{r}')\, f(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}', \;\boldsymbol{r}_i \in S$$

$$\phi(\boldsymbol{r}) = \phi_0(\boldsymbol{r}) + \iint_D G_o(\boldsymbol{r}, \boldsymbol{r}')\phi(\boldsymbol{r}')\, f(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}', \;\boldsymbol{r} \in D$$

Born approximation $(\phi(\boldsymbol{r}') \simeq \phi_0(\boldsymbol{r}'))$ **):**

$$g(\boldsymbol{r}_i) = \iint_D G_m(\boldsymbol{r}_i, \boldsymbol{r}')\phi_0(\boldsymbol{r}')\, f(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}', \;\boldsymbol{r}_i \in S$$



Discretization :

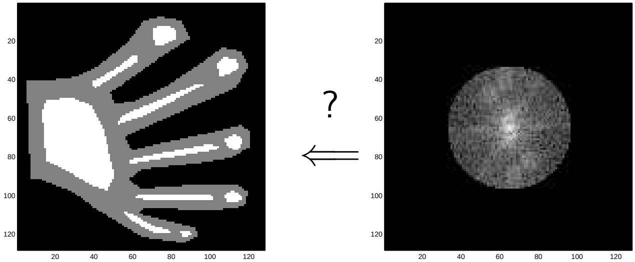
$$\begin{cases} \boldsymbol{g} = \boldsymbol{G}_m \boldsymbol{F} \boldsymbol{\phi} \\ \boldsymbol{\phi} = \phi_0 + \boldsymbol{G}_o \boldsymbol{F} \boldsymbol{\phi} \end{cases} \longrightarrow \begin{cases} \boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) \\ \text{with } \;\boldsymbol{F} = \mathrm{diag}(\boldsymbol{f}) \\ \boldsymbol{H}(\boldsymbol{f}) = \boldsymbol{G}_m \boldsymbol{F}(\boldsymbol{I} - \boldsymbol{G}_o \boldsymbol{F})^{-1} \phi_0 \end{cases}$$

Fourier synthesis in optical imaging

$$g(u,v) = \iint f(x,y) \exp\{-j(ux+vy)\}\, \mathrm{d}x\, \mathrm{d}y + \epsilon(u,v)$$

- Non coherent imaging: $\mathcal{G}(g) = |g| \longrightarrow \boldsymbol{g} = \boldsymbol{h}(\boldsymbol{f}) + \boldsymbol{\epsilon}$
- Coherent imaging: $\mathcal{G}(g) = g \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$

$$\boldsymbol{g} = \{g(\boldsymbol{\omega}),\ \boldsymbol{\omega} \in \Omega\}, \quad \boldsymbol{\epsilon} = \{\epsilon(\boldsymbol{\omega}),\ \boldsymbol{\omega} \in \Omega\} \quad \text{and}$$
$$\boldsymbol{f} = \{f(\boldsymbol{r}),\ \boldsymbol{r} \in \mathcal{R}\}$$



General formulation of inverse problems

1D convolution:

$$g(t) = \int f(t')\, h(t - t')\, \mathrm{d}t'$$

2D convolution:

$$g(x, y) = \iint f(x', y')\, h(x - x', y - y')\, \mathrm{d}x'\, \mathrm{d}y'$$

Computed Tomography:

$$g(r, \phi) = \iint f(x, y)\, \delta(r - x \cos\phi - y \sin\phi)\, \mathrm{d}x\, \mathrm{d}y$$

Fourier Synthesis:

$$g(u, v) = \iint f(x, y) \exp\{-j(ux + vy)\}\, \mathrm{d}x\, \mathrm{d}y$$

General case :

$$g(\boldsymbol{s}) = \iint f(\boldsymbol{r})\, h(\boldsymbol{r}, \boldsymbol{s})\, \mathrm{d}x\, \mathrm{d}y$$

General formulation of inverse problems

- General non linear inverse problems:

$$g(\boldsymbol{s}) = [\mathcal{H} f(\boldsymbol{r})](\boldsymbol{s}) + \epsilon(\boldsymbol{s}), \quad \boldsymbol{r} \in \mathcal{R}, \quad \boldsymbol{s} \in \mathcal{S}$$

- Linear models:

$$g(\boldsymbol{s}) = \int f(\boldsymbol{r})\, h(\boldsymbol{r}, \boldsymbol{s})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s})$$

  If $h(\boldsymbol{r}, \boldsymbol{s}) = h(\boldsymbol{r} - \boldsymbol{s}) \longrightarrow$ Convolution.

- Discrete data:

$$g(\boldsymbol{s}_i) = \int h(\boldsymbol{s}_i, \boldsymbol{r})\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s}_i), \quad i = 1, \cdots, m$$

- Inversion: Given the forward model $\mathcal{H}$ and the data $\boldsymbol{g} = \{g(\boldsymbol{s}_i), i = 1, \cdots, m)\}$ estimate $f(\boldsymbol{r})$
- Well-posed and Ill-posed problems (Hadamard): existance, uniqueness and stability
- Need for prior information

Analytical methods (mathematical physics)

$$g(s_i) = \int h(s_i, r)\, f(r)\, dr + \epsilon(s_i), \quad i = 1, \cdots, m$$

$$g(s) = \int h(s, r)\, f(r)\, dr$$

$$\widehat{f}(r) = \int w(s, r)\, g(s)\, ds$$

$w(s, r)$ minimizing: $Q(w(s, r)) = \left\| g(s) - [\mathcal{H}\widehat{f}(r)](s) \right\|_2^2$

Example: Fourier Transform:

$$g(s) = \int f(r) \exp\{-js.r\}\, dr$$

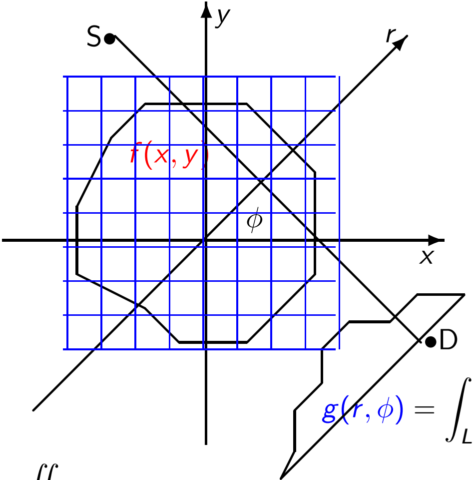
$$h(s, r) = \exp\{-js.r\} \longrightarrow w(s, r) = \exp\{+js.r\}$$

$$\hat{f}(r) = \int g(s) \exp\{+js.r\}\, ds$$

Other known classical solutions for specific expressions of $h(s, r)$:

- 1D cases: 1D Fourier, Hilbert, Weil, Melin, ...
- 2D cases: 2D Fourier, Radon, ...

X ray Tomography: Analytical Inversion methods



$$g(r,\phi) = \int_L f(x,y)\,\mathrm{d}l$$

Radon:

$$g(r,\phi) = \iint_D f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right)\int_0^{\pi}\int_{-\infty}^{+\infty}\frac{\frac{\partial}{\partial r}g(r,\phi)}{(r - x\cos\phi - y\sin\phi)}\,\mathrm{d}r\,\mathrm{d}\phi$$

Filtered Backprojection method

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right) \int_0^{\pi} \int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} g(r,\phi)}{(r - x\cos\phi - y\sin\phi)} \, \mathrm{d}r \, \mathrm{d}\phi$$

$$\text{Derivation } \mathcal{D}: \quad \bar{g}(r,\phi) = \frac{\partial g(r,\phi)}{\partial r}$$

$$\text{Hilbert Transform} \mathcal{H}: \quad g_1(r',\phi) = \frac{1}{\pi} \int_0^{\infty} \frac{\bar{g}(r,\phi)}{(r - r')} \, \mathrm{d}r$$

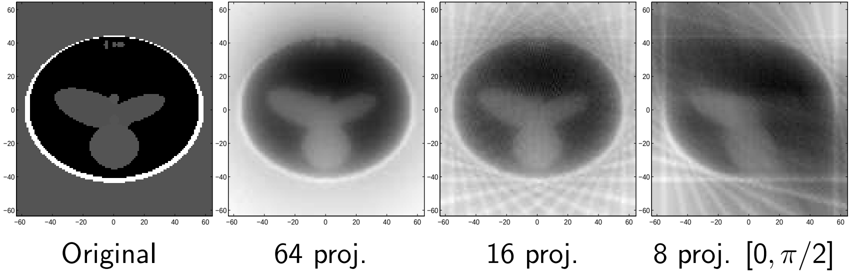
$$\text{Backprojection } \mathcal{B}: \quad f(x,y) = \frac{1}{2\pi} \int_0^{\pi} g_1(r' = x\cos\phi + y\sin\phi, \phi) \, \mathrm{d}\phi$$

$$f(x,y) = \mathcal{B}\,\mathcal{H}\,\mathcal{D}\, g(r,\phi) = \mathcal{B}\, \mathcal{F}_1^{-1}\, |\Omega|\, \mathcal{F}_1\, g(r,\phi)$$

- Backprojection of filtered projections:

$$\xrightarrow{g(r,\phi)} \boxed{\begin{array}{c}\text{FT}\\ \mathcal{F}_1\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{Filter}\\ |\Omega|\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{IFT}\\ \mathcal{F}_1^{-1}\end{array}} \xrightarrow{g_1(r,\phi)} \boxed{\begin{array}{c}\text{Backprojection}\\ \mathcal{B}\end{array}} \xrightarrow{f(x,y)}$$

Limitations : Limited angle or noisy data



- Limited angle or noisy data
- Accounting for detector size
- Other measurement geometries: fan beam, ...

Parametric methods

- $f(\boldsymbol{r})$ is described in a parametric form with a very few number of parameters $\boldsymbol{\theta}$ and one searches $\widehat{\boldsymbol{\theta}}$ which minimizes a criterion such as:
- Least Squares (LS): $Q(\boldsymbol{\theta}) = \sum_i |g_i - [\mathcal{H} f(\boldsymbol{\theta})]_i|^2$
- Robust criteria : $Q(\boldsymbol{\theta}) = \sum_i \phi\left(|g_i - [\mathcal{H} f(\boldsymbol{\theta})]_i|\right)$
  with different functions ϕ (L_1, Hubert, ...).
- Likelihood : $\mathcal{L}(\boldsymbol{\theta}) = -\ln p(\boldsymbol{g}|\boldsymbol{\theta})$
- Penalized likelihood : $\mathcal{L}(\boldsymbol{\theta}) = -\ln p(\boldsymbol{g}|\boldsymbol{\theta}) + \lambda\Omega(\boldsymbol{\theta})$

Examples:

- Spectrometry: $f(t)$ modelled as a sum og gaussians
  $f(t) = \sum_{k=1}^{K} a_k \mathcal{N}(t|\mu_k, v_k)$ $\quad \boldsymbol{\theta} = \{a_k, \mu_k, v_k\}$
- Tomography in CND: $f(x, y)$ is modelled as a superposition of circular or elliptical discs $\quad \boldsymbol{\theta} = \{a_k, \mu_k, r_k\}$

Non parametric methods

$$g(\boldsymbol{s}_i) = \int h(\boldsymbol{s}_i, \boldsymbol{r})\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s}_i), \quad i = 1, \cdots, M$$

- $f(\boldsymbol{r})$ is assumed to be well approximated by

$$f(\boldsymbol{r}) \simeq \sum_{j=1}^{N} f_j\, b_j(\boldsymbol{r})$$

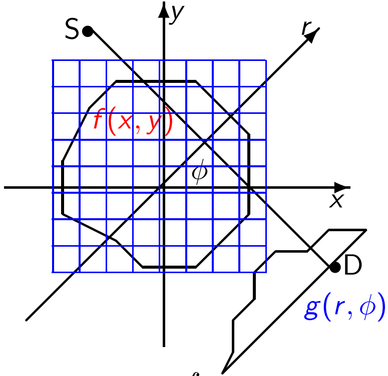
with $\{b_j(\boldsymbol{r})\}$ a basis or any other set of known functions

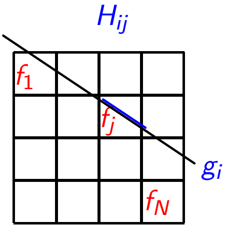
$$g(\boldsymbol{s}_i) = g_i \simeq \sum_{j=1}^{N} f_j \int h(\boldsymbol{s}_i, \boldsymbol{r})\, b_j(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}, \quad i = 1, \cdots, M$$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \quad \text{with} \quad H_{ij} = \int h(\boldsymbol{s}_i, \boldsymbol{r})\, b_j(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}$$

- $\boldsymbol{H}$ is huge dimensional
- LS solution : $\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{Q(\boldsymbol{f})\}$ with
  $Q(\boldsymbol{f}) = \sum_i |g_i - [\boldsymbol{H}\boldsymbol{f}]_i|^2 = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2$
  does not give satisfactory result.

Algebraic methods: Discretization





$$f(x,y) = \sum_j f_j \, b_j(x,y)$$

$$b_j(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \text{ pixel } j \\ 0 & \text{else} \end{cases}$$

$$g(r,\phi) = \int_L f(x,y) \, \mathrm{d}l$$

$$g_i = \sum_{j=1}^{N} H_{ij} \, f_j + \epsilon_i$$

$$\boldsymbol{g} = \boldsymbol{H} \boldsymbol{f} + \boldsymbol{\epsilon}$$

Inversion: Deterministic methods

Data matching

- Observation model
  $g_i = h_i(\boldsymbol{f}) + \epsilon_i, \quad i = 1, \ldots, M \longrightarrow \boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) + \boldsymbol{\epsilon}$
- Misatch between data and output of the model $\Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f}))$

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{\Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f}))\}$$

- Examples:

– LS $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \|\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})\|^2 = \sum_i |g_i - h_i(\boldsymbol{f})|^2$

– L_p $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \|\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})\|^p = \sum_i |g_i - h_i(\boldsymbol{f})|^p, \quad 1 < p < 2$

– KL $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \sum_i g_i \ln \frac{g_i}{h_i(\boldsymbol{f})}$

- In general, does not give satisfactory results for inverse problems.

Regularization theory

Inverse problems = Ill posed problems

$\longrightarrow$ Need for prior information

Functional space (Tikhonov):

$$g = \mathcal{H}(f) + \epsilon \longrightarrow J(f) = ||g - \mathcal{H}(f)||_2^2 + \lambda ||\mathcal{D}f||_2^2$$

Finite dimensional space (Philips & Towmey): $\boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) + \boldsymbol{\epsilon}$

- Minimum norme LS (MNLS): $J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda ||\boldsymbol{f}||^2$
- Classical regularization: $J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda ||\boldsymbol{D}\boldsymbol{f}||^2$
- More general regularization:

$$J(\boldsymbol{f}) = \mathcal{Q}(\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})) + \lambda \Omega(\boldsymbol{D}\boldsymbol{f})$$

or

$$J(\boldsymbol{f}) = \Delta_1(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) + \lambda \Delta_2(\boldsymbol{f}, \boldsymbol{f}_0)$$

Limitations:

- Errors are considered implicitly white and Gaussian
- Limited prior information on the solution
- Lack of tools for the determination of the hyperparameters

Inversion: Probabilistic methods

Taking account of errors and uncertainties $\longrightarrow$ Probability theory

- Maximum Likelihood (ML)
- Minimum Inaccuracy (MI)
- Probability Distribution Matching (PDM)
- Maximum Entropy (ME) and Information Theory (IT)
- Bayesian Inference (BAYES)

Advantages:

- Explicit account of the errors and noise
- A large class of priors via explicit or implicit modeling
- A coherent approach to combine information content of the data and priors

Limitations:

- Practical implementation and cost of calculation

Bayesian estimation approach

$$\mathcal{M}: \qquad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Observation model $\mathcal{M}$ + Hypothesis on the noise $\boldsymbol{\epsilon} \longrightarrow$
  $$p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M}) = p_{\epsilon}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f})$$
- A priori information $\quad p(\boldsymbol{f}|\mathcal{M})$
- Bayes : $$p(\boldsymbol{f}|\boldsymbol{g};\mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M})\, p(\boldsymbol{f}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$$

Link with regularization :

Maximum A Posteriori (MAP) :

$$\begin{aligned}\widehat{\boldsymbol{f}} &= \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{g}|\boldsymbol{f})\ p(\boldsymbol{f})\} \\ &= \arg\min_{\boldsymbol{f}} \{-\ln p(\boldsymbol{g}|\boldsymbol{f}) - \ln p(\boldsymbol{f})\}\end{aligned}$$

with $\quad Q(\boldsymbol{g}, \boldsymbol{H}\boldsymbol{f}) = -\ln p(\boldsymbol{g}|\boldsymbol{f}) \quad$ and $\quad \lambda\Omega(\boldsymbol{f}) = -\ln p(\boldsymbol{f})$

Case of linear models and Gaussian priors

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Hypothesis on the noise: $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma_\epsilon^2 \boldsymbol{I}) \longrightarrow$
  $p(\boldsymbol{g}|\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2\right\}$
- Hypothesis on $\boldsymbol{f}$: $\boldsymbol{f} \sim \mathcal{N}(0, \sigma_f^2(\boldsymbol{D}^t\boldsymbol{D})^{-1}) \longrightarrow$
  $p(\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_f^2}\|\boldsymbol{D}\boldsymbol{f}\|^2\right\}$
- A posteriori:
  $$p(\boldsymbol{f}|\boldsymbol{g}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 - \frac{1}{2\sigma_f^2}\|\boldsymbol{D}\boldsymbol{f}\|^2\right\}$$
- MAP : $\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$
  with $J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2, \qquad \lambda = \frac{\sigma_\epsilon^2}{\sigma_f^2}$
- Advantage : characterization of the solution

$$\boldsymbol{f}|\boldsymbol{g} \sim \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{P}}) \text{ with } \widehat{\boldsymbol{f}} = \widehat{\boldsymbol{P}}\boldsymbol{H}^t\boldsymbol{g}, \quad \widehat{\boldsymbol{P}} = \left(\boldsymbol{H}^t\boldsymbol{H} + \lambda\boldsymbol{D}^t\boldsymbol{D}\right)^{-1}$$

MAP estimation with other priors:

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{with} \quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\Omega(\boldsymbol{f})$$

Separable priors:

- Gaussian: $p(f_j) \propto \exp\left\{-\alpha|f_j|^2\right\} \longrightarrow \Omega(\boldsymbol{f}) = \alpha \sum_j |f_j|^2$
- Gamma: $p(f_j) \propto f_j^\alpha \exp\left\{-\beta f_j\right\} \longrightarrow \Omega(\boldsymbol{f}) = \alpha \sum_j \ln f_j + \beta f_j$
- Beta:
  $p(f_j) \propto f_j^\alpha (1 - f_j)^\beta \longrightarrow \Omega(\boldsymbol{f}) = \alpha \sum_j \ln f_j + \beta \sum_j \ln(1 - f_j)$
- Generalized Gaussian:
  $p(f_j) \propto \exp\left\{-\alpha|f_j|^p\right\}, \quad 1 < p < 2 \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j |f_j|^p,$

Markovian models:

$$p(f_j|\boldsymbol{f}) \propto \exp\left\{-\alpha \sum_{i\in N_j} \phi(f_j, f_i)\right\} \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j \sum_{i\in N_j} \phi(f_j, f_i),$$

Main advantages of the Bayesian approach

- MAP = Regularization
- Posterior mean ? Marginal MAP ?
- More information in the posterior law than only its mode or its mean
- Meaning and tools for estimating hyper parameters
- Meaning and tools for model selection
- More specific and specialized priors, particularly through the hidden variables
- More computational tools:
  - Expectation-Maximization for computing the maximum likelihood parameters
  - MCMC for posterior exploration
  - Variational Bayes for analytical computation of the posterior marginals
  - ...

Full Bayesian approach

$$\mathcal{M}: \qquad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

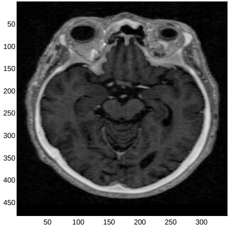
- Forward & errors model: $\longrightarrow p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1; \mathcal{M})$
- Prior models $\longrightarrow p(\boldsymbol{f}|\boldsymbol{\theta}_2; \mathcal{M})$
- Hyperparameters $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \longrightarrow p(\boldsymbol{\theta}|\mathcal{M})$
- Bayes: $\longrightarrow p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M})\, p(\boldsymbol{f}|\boldsymbol{\theta}; \mathcal{M})\, p(\boldsymbol{\theta}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$
- Joint MAP: $\qquad (\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f}, \boldsymbol{\theta})} \{p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\}$
- Marginalization: $\begin{cases} p(\boldsymbol{f}|\boldsymbol{g}; \mathcal{M}) & = \int p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f} \\ p(\boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M}) & = \int p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{\theta} \end{cases}$
- Posterior means: $\begin{cases} \widehat{\boldsymbol{f}} & = \int \boldsymbol{f}\; p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f}\; \mathrm{d}\boldsymbol{\theta} \\ \widehat{\boldsymbol{\theta}} & = \int \boldsymbol{\theta}\; p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f}\; \mathrm{d}\boldsymbol{\theta} \end{cases}$
- Evidence of the model:

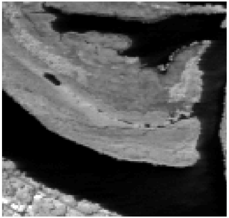
$$p(\boldsymbol{g}|\mathcal{M}) = \iint p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{f}|\boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{\theta}|\mathcal{M})\, \mathrm{d}\boldsymbol{f}\; \mathrm{d}\boldsymbol{\theta}$$

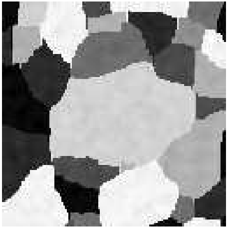
Two main steps in the Bayesian approach

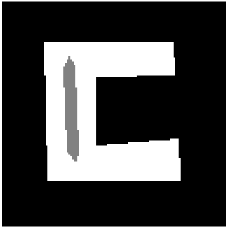
- Prior modeling
  - Separable:
    Gaussian, Generalized Gaussian, Gamma,
    mixture of Gaussians, mixture of Gammas, ...
  - Markovian: Gauss-Markov, GGM, ...
  - Separable or Markovian with hidden variables
    (contours, region labels)
- Choice of the estimator and computational aspects
  - MAP, Posterior mean, Marginal MAP
  - MAP needs optimization algorithms
  - Posterior mean needs integration methods
  - Marginal MAP needs integration and optimization
  - Approximations:
    - Gaussian approximation (Laplace)
    - Numerical exploration MCMC
    - Variational Bayes (Separable approximation)

Which images I am looking for?

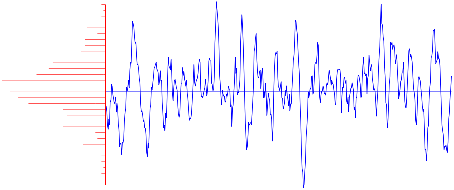




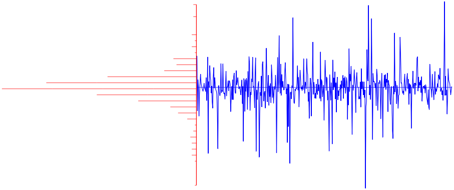




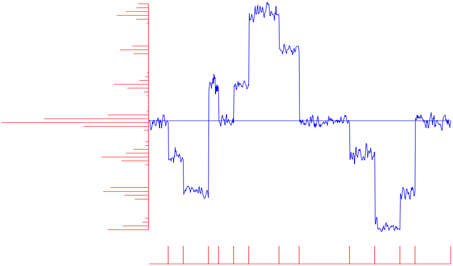
Which image I am looking for?



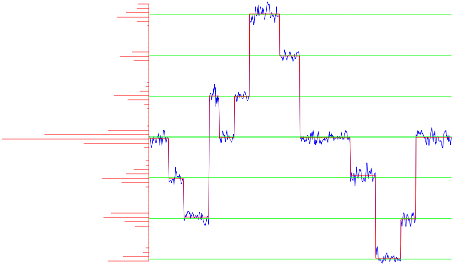
Gauss-Markov



Generalized GM



Piecewize Gaussian



Mixture of GM

Different prior models for signals and images

- Separable $p(\boldsymbol{f}) = \prod_j p_j(f_j) \propto \exp\left\{-\beta \sum_j \phi(f_j)\right\}$

$$p(\boldsymbol{f}) \propto \exp\left\{-\beta \sum_{\boldsymbol{r}\in\mathcal{R}} \phi(f(\boldsymbol{r}))\right\}$$

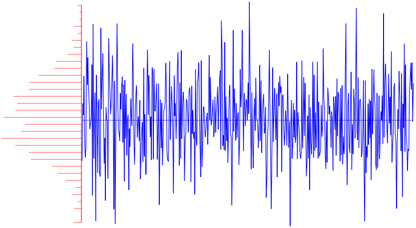
- Markoviens (simple) $p(f_j|f_{j-1}) \propto \exp\left\{-\beta\phi(f_j - f_{j-1})\right\}$

$$p(\boldsymbol{f}) \propto \exp\left\{-\beta \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \phi(f(\boldsymbol{r}), f(\boldsymbol{r}'))\right\}$$

- Markovien with hidden variables
  $z(\boldsymbol{r})$ (lines, contours, regions)

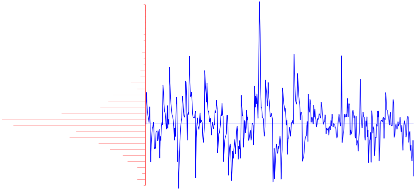
$$p(\boldsymbol{f}|\boldsymbol{z}) \propto \exp\left\{-\beta \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \phi(f(\boldsymbol{r}), f(\boldsymbol{r}'), z(\boldsymbol{r}), z(\boldsymbol{r}'))\right\}$$

Different prior models for images: Separable



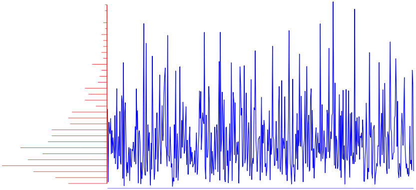
Gaussian

$p(f_j) \propto \exp\left\{-\alpha|f_j|^2\right\}$



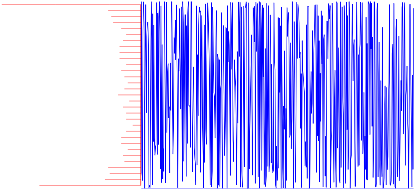
Generalized Gaussian

$p(f_j) \propto \exp\left\{-\alpha|f_j|^p\right\}, \quad 1 \leq p \leq 2$



Gamma

$p(f_j) \propto f_j^\alpha \exp\left\{-\beta f_j\right\}$



Beta

$p(f_j) \propto f_j^\alpha (1-f_j)^\beta$

Different prior models: Simple Markovian

$$p(f_j|\boldsymbol{f}) \propto \exp\left\{-\alpha \sum_{i\in v_j} \phi(f_j, f_i)\right\} \longrightarrow \quad \Phi(\boldsymbol{f}) = \alpha \sum_j \sum_{i\in V_j} \phi(f_j, f_i)$$

- 1D case and one neigbor $V_j = j - 1$:

$$\Phi(\boldsymbol{f}) = \alpha \sum_j \phi(f_j - f_{j-1})$$

- 1D Case and two neighbors $V_j = \{j-1, j+1\}$:

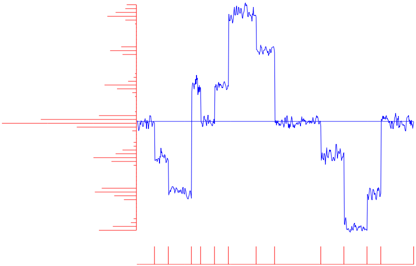
$$\Phi(\boldsymbol{f}) = \alpha \sum_j \phi\left(f_j - \beta(f_{j-1} + f_{j-1})\right)$$

- 2D case with 4 neighbors:

$$\Phi(\boldsymbol{f}) = \alpha \sum_{\boldsymbol{r}\in\mathcal{R}} \phi\left(f(\boldsymbol{r}) - \beta \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} f(\boldsymbol{r}')\right)$$

- $\phi(t) = |t|^\gamma$: Generalized Gaussian

Different prior models: Markovian with hidden variables

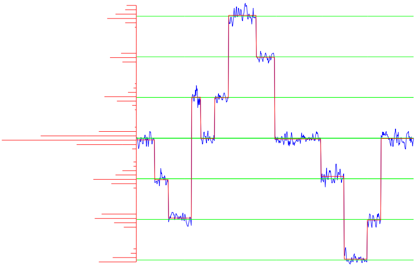


Piecewise Gaussians

(contours hidden variables)

$p(f_j|q_j, f_{j-1}) = \mathcal{N}\left((1-q_j)f_{j-1}, \sigma_f^2\right)$

$p(\boldsymbol{f}|\boldsymbol{q}) \propto \exp\left\{-\alpha \sum_j \left|f_j - (1-q_j)f_{j-1}\right|^2\right\}$



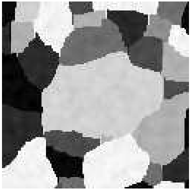
Mixture of Gaussians (MoG)

(regions labels hidden variables)

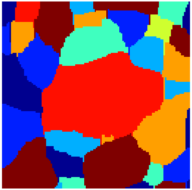
$p(f_j|z_j = k) = \mathcal{N}\left(m_k, \sigma_k^2\right)$ & z_j markovian

$p(\boldsymbol{f}|\boldsymbol{z}) \propto \exp\left\{-\alpha \sum_k \sum_{j \in \mathcal{R}_k} \left(\frac{f_j - m_k}{\sigma_k}\right)^2\right\}$

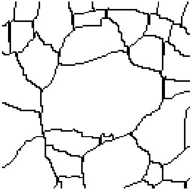
Gauss-Markov-Potts prior models for images



$f(\boldsymbol{r})$



$z(\boldsymbol{r})$



$c(\boldsymbol{r}) = 1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k, m_k, v_k) = \mathcal{N}(m_k, v_k)$$

$$p(f(\boldsymbol{r})) = \sum_k P(z(\boldsymbol{r}) = k)\, \mathcal{N}(m_k, v_k) \quad \text{Mixture of Gaussians}$$

- Separable iid hidden variables: $p(\boldsymbol{z}) = \prod_{\boldsymbol{r}} p(z(\boldsymbol{r}))$
- Markovian hidden variables: $p(\boldsymbol{z})$ Potts-Markov:

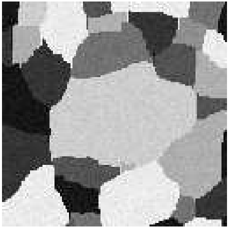
$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r} \in \mathcal{R}} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

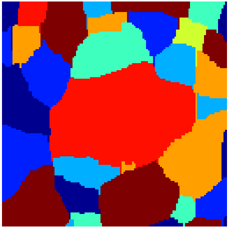
Four different cases

To each pixel of the image is associated 2 variables $f(\boldsymbol{r})$ and $z(\boldsymbol{r})$

- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ iid :
  Mixture of Gaussians
- $\boldsymbol{f}|\boldsymbol{z}$ Gauss-Markov, $\boldsymbol{z}$ iid :
  Mixture of Gauss-Markov
- $\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid, $\boldsymbol{z}$ Potts-Markov :
  Mixture of Independent Gaussians
  (MIG with Hidden Potts)
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\boldsymbol{z}$ Potts-Markov :
  Mixture of Gauss-Markov
  (MGM with hidden Potts)

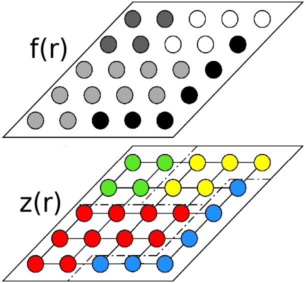


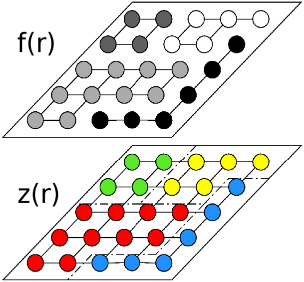
$f(\boldsymbol{r})$



$z(\boldsymbol{r})$

Summary of the two proposed models





$\boldsymbol{f}|\boldsymbol{z}$ Gaussian iid
$\boldsymbol{z}$ Potts-Markov

(MIG with Hidden Potts)

$\boldsymbol{f}|\boldsymbol{z}$ Markov
$\boldsymbol{z}$ Potts-Markov

(MGM with hidden Potts)

Bayesian Computation

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta} | \boldsymbol{g}) \propto p(\boldsymbol{g} | \boldsymbol{f}, \boldsymbol{z}, v_\epsilon)\, p(\boldsymbol{f} | \boldsymbol{z}, \boldsymbol{m}, \boldsymbol{v})\, p(\boldsymbol{z} | \gamma, \boldsymbol{\alpha})\, p(\boldsymbol{\theta})$$

$\boldsymbol{\theta} = \{v_\epsilon, (\alpha_k, m_k, v_k), k = 1, \cdot, K\}$ $p(\boldsymbol{\theta})$ Conjugate priors

- Direct computation and use of $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta} | \boldsymbol{g}; \mathcal{M})$ is too complex
- Possible approximations :
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)
- Main idea in Variational Bayesian methods:
  Approximate

  $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta} | \boldsymbol{g}; \mathcal{M})$ by $q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta})$
  - Choice of approximation criterion : $KL(q : p)$
  - Choice of appropriate families of probability laws for $q_1(\boldsymbol{f})$, $q_2(\boldsymbol{z})$ and $q_3(\boldsymbol{\theta})$

MCMC based algorithm

$$p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta})\, p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta})\, p(\boldsymbol{z})\, p(\boldsymbol{\theta})$$

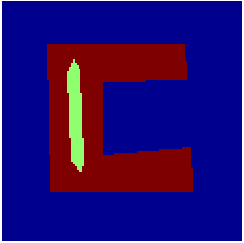
General scheme:

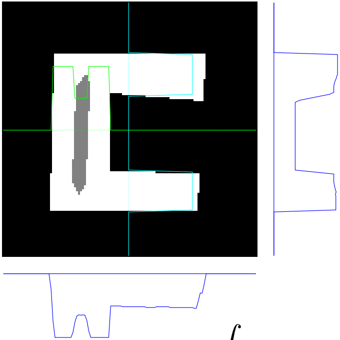
$$\widehat{\boldsymbol{f}} \sim p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{z}} \sim p(\boldsymbol{z}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\boldsymbol{g})$$

- Estimate $\boldsymbol{f}$ using $p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta})\, p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}})$
  Needs optimisation of a quadratic criterion.
- Estimate $\boldsymbol{z}$ using $p(\boldsymbol{z}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}})\, p(\boldsymbol{z})$
  Needs sampling of a Potts Markov field.
- Estimate $\boldsymbol{\theta}$ using
  $p(\boldsymbol{\theta}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}},\sigma_\epsilon^2\boldsymbol{I})\, p(\widehat{\boldsymbol{f}}|\widehat{\boldsymbol{z}},(m_k,v_k))\, p(\boldsymbol{\theta})$
  Conjugate priors $\longrightarrow$ analytical expressions.

Application of CT in NDT

Reconstruction from only 2 projections



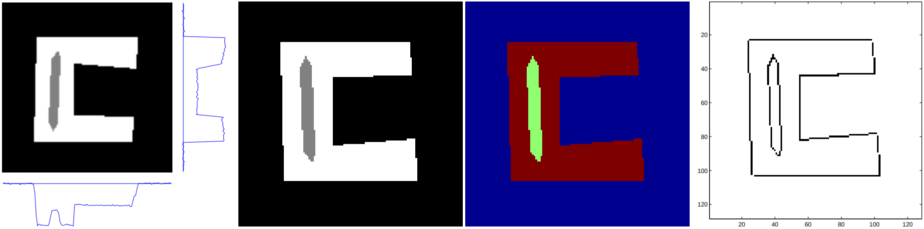


$$g_1(x) = \int f(x,y)\, dy, \qquad g_2(y) = \int f(x,y)\, dx$$

- Given the marginals $g_1(x)$ and $g_2(y)$ find the joint distribution $f(x,y)$.
- Infinite number of solutions : $f(x,y) = g_1(x)\, g_2(y)\, \Omega(x,y)$
  $\Omega(x,y)$ is a Copula:

$$\int \Omega(x,y)\, dx = 1 \quad \text{and} \quad \int \Omega(x,y)\, dy = 1$$

Application in CT



$\boldsymbol{g}|\boldsymbol{f}$

$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$

$\boldsymbol{g}|\boldsymbol{f} \sim \mathcal{N}(\boldsymbol{H}\boldsymbol{f}, \sigma_\epsilon^2 \boldsymbol{I})$

Gaussian

$\boldsymbol{f}|\boldsymbol{z}$

iid Gaussian

or

Gauss-Markov

$\boldsymbol{z}$

iid

or

Potts

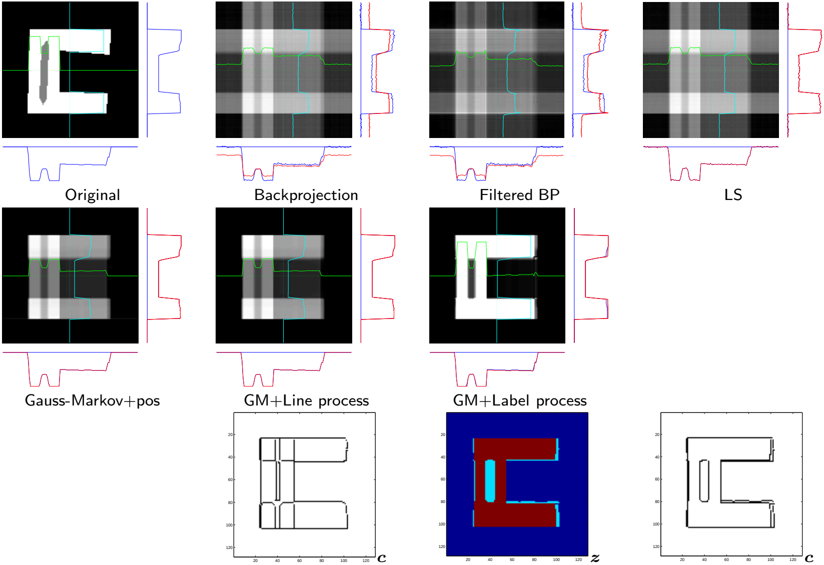
$\boldsymbol{c}$

$c(\boldsymbol{r}) \in \{0, 1\}$

$1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$

binary

Results

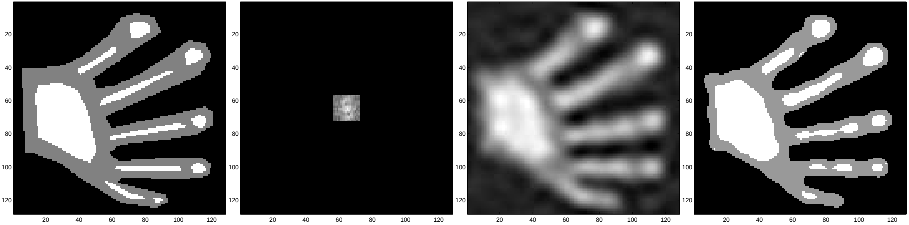

Original
Backprojection
Filtered BP
LS
Gauss-Markov+pos
GM+Line process
GM+Label process
c
z
c

Application in Microwave imaging

$$g(\omega) = \int f(\boldsymbol{r}) \exp\{-j(\boldsymbol{\omega}.\boldsymbol{r})\} \, \mathrm{d}\boldsymbol{r} + \epsilon(\omega)$$

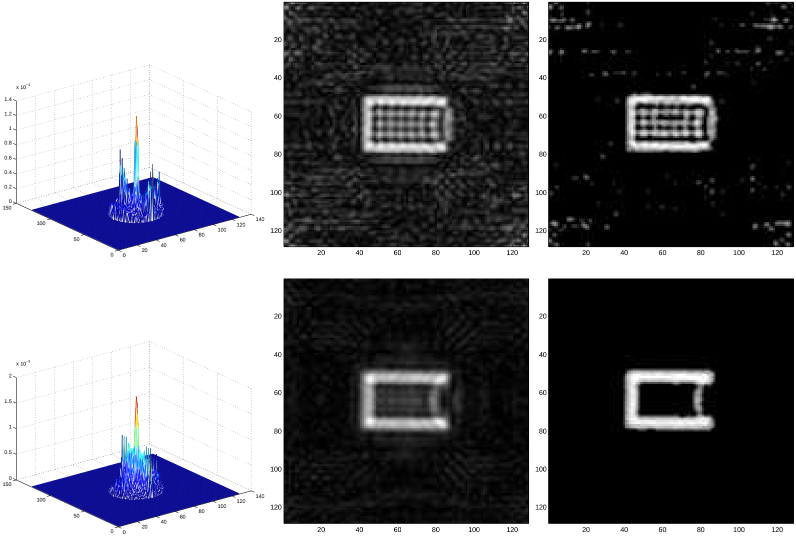
$$g(u, v) = \iint f(x, y) \exp\{-j(ux + vy)\} \, \mathrm{d}x \, \mathrm{d}y + \epsilon(u, v)$$

$$\boldsymbol{g} = \boldsymbol{H} \boldsymbol{f} + \boldsymbol{\epsilon}$$



$f(x, y)$ $g(u, v)$ $\widehat{f}$ IFT $\widehat{f}$ Proposed method

Application in Microwave imaging



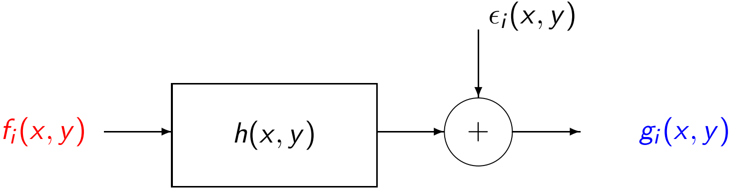
Conclusions

- Bayesian Inference for inverse problems
- Approximations (Laplace, MCMC, Variational)
- Gauss-Markov-Potts are useful prior models for images incorporating regions and contours
- Separable approximations for Joint posterior with Gauss-Markov-Potts priors
- Application in different CT (X ray, US, Microwaves, PET, SPECT)

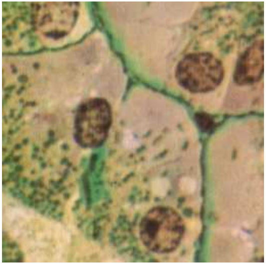
Perspectives :

- Efficient implementation in 2D and 3D cases
- Evaluation of performances and comparison with MCMC methods
- Application to other linear and non linear inverse problems: (PET, SPECT or ultrasound and microwave imaging)

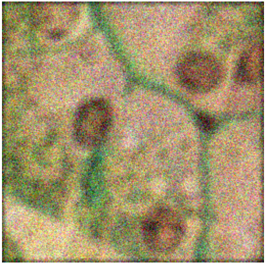
Color (Multi-spectral) image deconvolution



Observation model : $\boldsymbol{g}_i = \boldsymbol{H} \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, 2, 3$



? $\Longleftarrow$

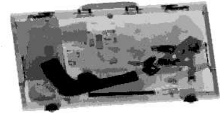


Images fusion and joint segmentation

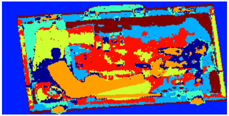
(with O. Féron)

$$\begin{cases} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma^2_{i\,k}) \\ p(\underline{f}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}) \end{cases}$$

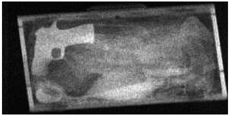
$\boldsymbol{g}_1$



$\widehat{\boldsymbol{f}}_1$

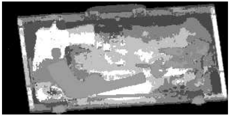


$\widehat{\boldsymbol{z}}$



$\boldsymbol{g}_2$

$\longrightarrow$

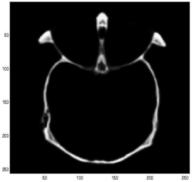


$\widehat{\boldsymbol{f}}_2$

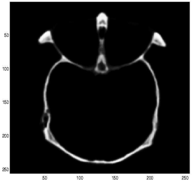
Data fusion in medical imaging

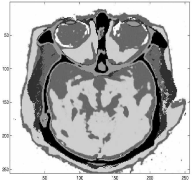
(with O. Féron)

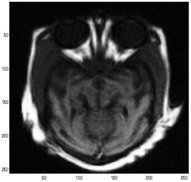
$$\begin{cases} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma^2_{i\,k}) \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}) \end{cases}$$

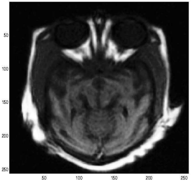
$\boldsymbol{g}_1$ 

$\longrightarrow$

$\widehat{\boldsymbol{f}}_1$ 

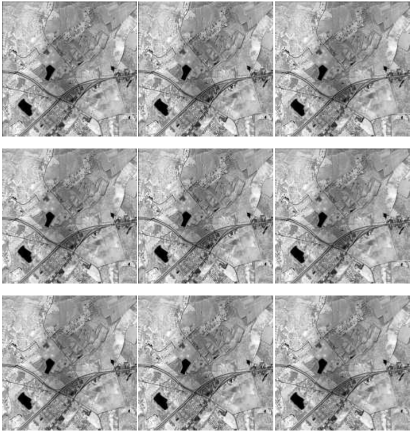
$\widehat{\boldsymbol{z}}$ 

$\boldsymbol{g}_2$ 

$\widehat{\boldsymbol{f}}_2$ 

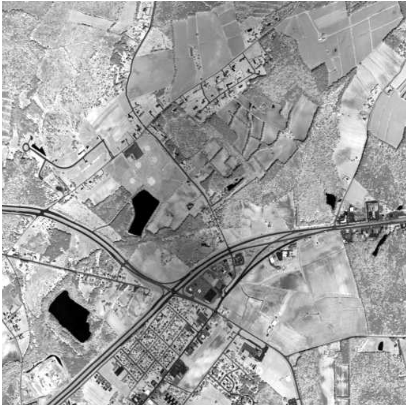
Super-Resolution

(with F. Humblot)



Low Resolution images

? ⟹

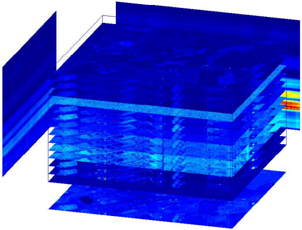


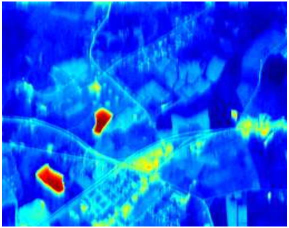
High Resolution image

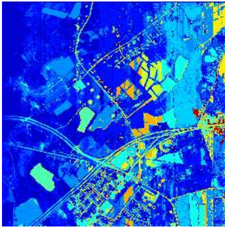
Joint segmentation of hyper-spectral images

(with N. Bali & A. Mohammadpour)

$$\begin{cases} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r}) | z(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma_{i\,k}^2), \quad k = 1, \cdots, K \\ p(\underline{\boldsymbol{f}} | \boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i | \boldsymbol{z}) \\ m_{i\,k} \; \text{follow a Markovian model along the index } i \end{cases}$$



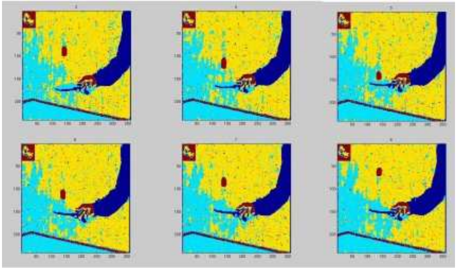




Segmentation of a video sequence of images

(with P. Brault)

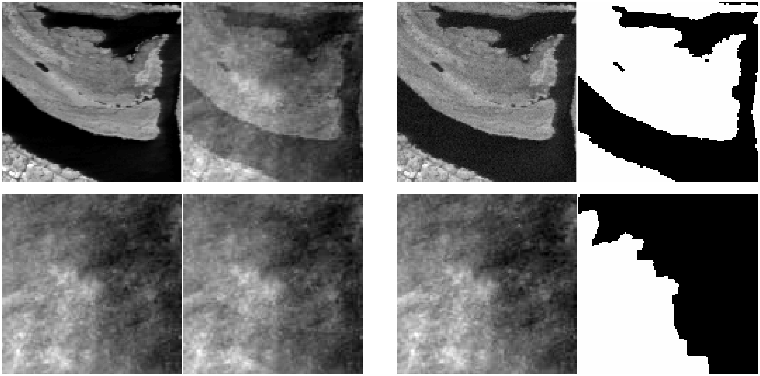
$$\begin{cases} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z_i(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma_{i\,k}^2), \quad k = 1, \cdots, K \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}_i) \\ z_i(\boldsymbol{r}) \text{ follow a Markovian model along the index } i \end{cases}$$



Source separation

(with H. Snoussi & M. Ichir)

$$
\begin{cases}
g_i(\boldsymbol{r}) = \sum_{j=1}^{N} A_{ij} f_j(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\
p(f_j(\boldsymbol{r}) | z_j(\boldsymbol{r}) = k) = \mathcal{N}(m_{j\,k}, \sigma^2_{j\,k}) \\
p(A_{ij}) = \mathcal{N}(A_{0\,ij}, \sigma^2_{0\,ij})
\end{cases}
$$



f g $\widehat{f}$ $\widehat{z}$

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- D. Pougaza (Tomography and Copula)
- ——————
- Sh. Zhu (SAR Imaging)
- D. Fall (Emission Positon Tomography, Non Parametric Bayesian)

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- N. Gac & A. Rabanal (GPU Implementation)
- Th. Rodet (Tomography)
- ——————
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- E. Barat (CEA-LIST) (Positon Emission Tomography, Non Parametric Bayesian)
- C. Comtat (SHFJ, CEA)(PET, Spatio-Temporal Brain activity)

Questions and Discussions