Sparsity in signal and image processing with applications in biological signals and medicals images

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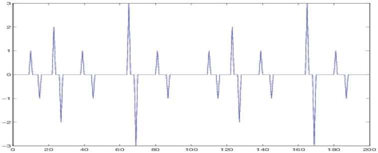
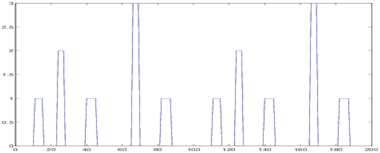
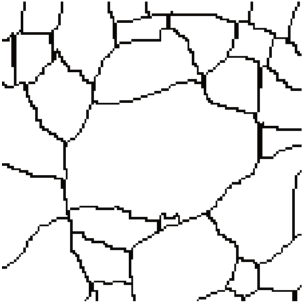
1. Sparse signals and images

- Sparse signals: Direct sparsity



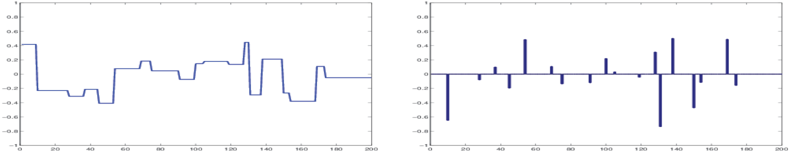


- Sparse images: Direct sparsity

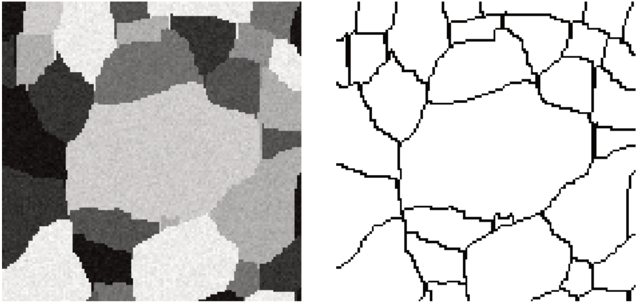


Sparse signals and images

- Sparse signals in a Transform domain



- Sparse images in a Transform domain



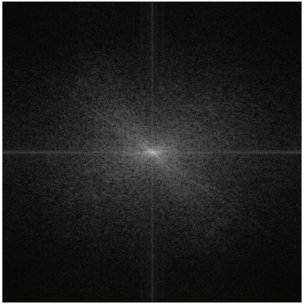
Sparse signals and images

- Sparse signals in Fourier domain

Time domain   Fourier domain

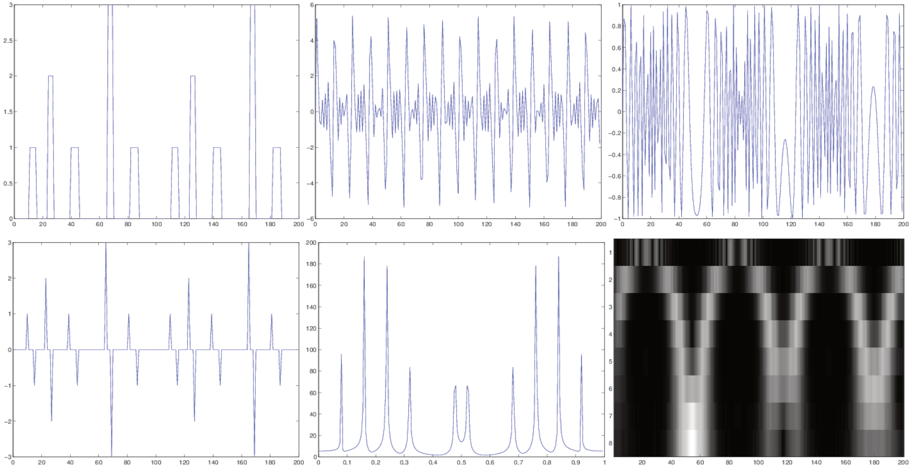
- Sparse images in wavelet domain

Space domain   Fourier domain

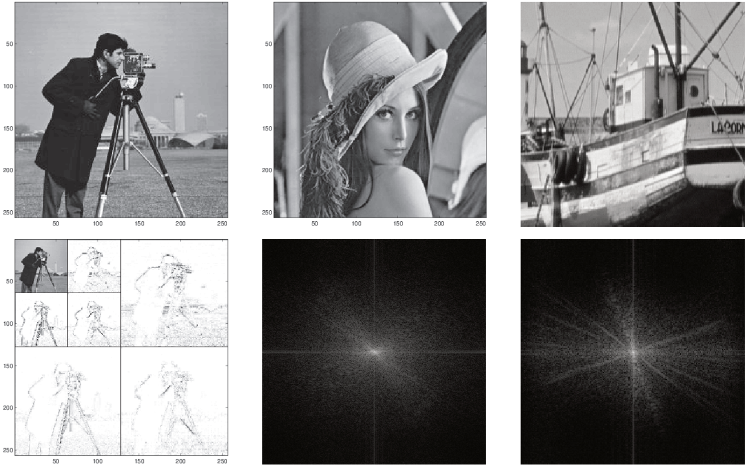


Sparse signals and images

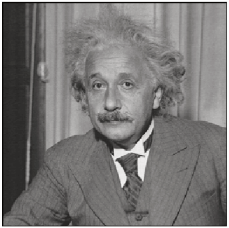
- Sparse signals: Sparsity in a Transform domaine



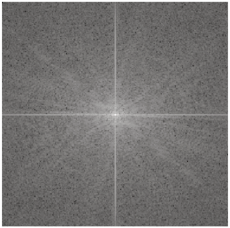
Sparse signals and images



Sparse signals and images (Fourier and Wavelets domain)



Image



Fourier

Wavelets

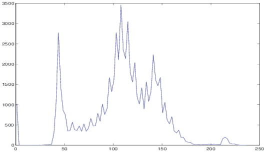
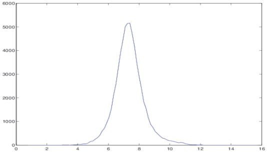
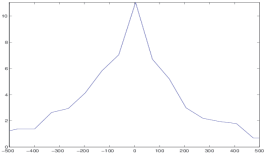


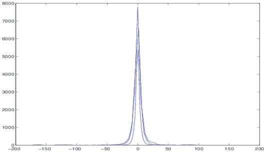
Image hist.



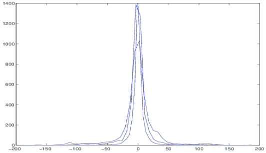
Fourier coeff. hist.



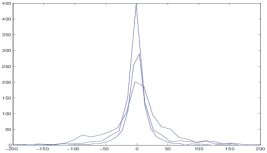
Wavelet coeff. hist.



bands 1-3



bands 4-6



bands 7-9

2. First ideas: some history

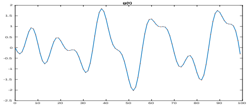
- 1948: Shannon:
  Sampling theorem and reconstruction of a band limited signal
- 1993-2007:
  - Mallat, Zhang, Candès, Romberg, Tao and Baraniuk:
    Non linear sampling, Compression and reconstruction,
  - Fuch: Sparse representation
  - Donoho, Elad, Tibshirani, Tropp, Duarte, Laska:
    Compressive Sampling, Compressive Sensing
- 2007-2012:
  Algorithms for sparse representation and compressive Sampling: Matching Pursuit (MP), Projection Pursuit Regression, Pure Greedy Algorithm, OMP, Basis Poursuit (BP), Dantzig Selector (DS), Least Absolute Shrinkage and Selection Operator (LASSO), Iterative Hard Thresholding...
- 2003-2012:
  Bayesian approach to sparse modeling
  Tipping, Bishop: Sparse Bayesian Learning,
  Relevance Vector Machine (RVM), ...

3. Modeling and representation

- Modeling via a basis
  (codebook, overcomplete dictionnary, Design Matrix)

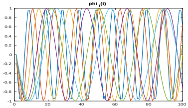
$$g(t) = \sum_{j=1}^{N} f_j \, \phi_j(t), \;\; t = 1, \cdots, T \longrightarrow \mathbf{g} = \mathbf{\Phi} \, \mathbf{f}$$

$g(t)$



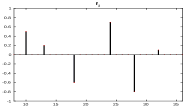
$T = 100$

$\phi_i(t)$



$[100 \times 35]$

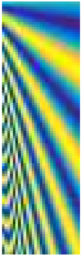
f_i



$N = 35$

Modeling and representation

$$g(t) \quad = \quad \sum_j \phi_j(t) \quad f_j$$



$$\mathbf{g} \quad = \quad \boldsymbol{\Phi} \quad \mathbf{f}$$

$T = 100$ $[100 \times 35]$ $N = 35$

Modeling and representation

- Modeling via a basis
  (codebook, overcomplete dictionnary, Design Matrix)

$$g(t)=\sum_{j=1}^{N} f_j\,\phi_j(t),\;\; t=1,\cdots,T \longrightarrow \mathbf{g}=\mathbf{\Phi}\,\mathbf{f}$$

- When $T \geq N$

$$\widehat{f}_j=\arg\min_{f_j}\left\{\sum_{t=1}^{T}\left|g(t)-\sum_{j=1}^{N} f_j\,\phi_j(t)\right|^2\right\}\longrightarrow$$

$$\widehat{\mathbf{f}}=\arg\min_{\mathbf{f}}\left\{\|\mathbf{g}-\mathbf{\Phi}\mathbf{f}\|^2\right\}=[\mathbf{\Phi}'\mathbf{\Phi}]^{-1}\mathbf{\Phi}'\mathbf{g}$$

- When orthogonal basis: $\mathbf{\Phi}'\mathbf{\Phi}=\mathbf{I}\longrightarrow\widehat{\mathbf{f}}=\mathbf{\Phi}'\mathbf{g}$

$$\widehat{f}_j=\sum_{t=1}^{N} g(t)\,\phi_j(t)=< g(t),\phi_j(t) >$$

- Application in Compression, Transmission and Decompression

Modeling and representation

- When overcomplete basis $N > T$: Infinite number of solutions for $\boldsymbol{\Phi}\mathbf{f} = \mathbf{g}$. We have to select one:

$$\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}:\ \boldsymbol{\Phi}\mathbf{f}=\mathbf{g}} \left\{\|\mathbf{f}\|_2^2\right\}$$

  or writing differently:

$$\text{minimize } \ \|\mathbf{f}\|_2^2 \ \text{ subject to } \ \boldsymbol{\Phi}\mathbf{f} = \mathbf{g}$$

  resulting to:

$$\widehat{\mathbf{f}} = \boldsymbol{\Phi}'[\boldsymbol{\Phi}\boldsymbol{\Phi}']^{-1}\mathbf{g}$$

- Again if $\boldsymbol{\Phi}\boldsymbol{\Phi}' = \mathbf{I} \longrightarrow \widehat{\mathbf{f}} = \boldsymbol{\Phi}'\mathbf{g}$.
- No real interest if we have to keep all the N coefficients:
- Sparsity:

$$\text{minimize } \ \|\mathbf{f}\|_0 \ \text{ subject to } \ \boldsymbol{\Phi}\mathbf{f} = \mathbf{g}$$

  or

$$\text{minimize } \ \|\mathbf{f}\|_1 \ \text{ subject to } \ \boldsymbol{\Phi}\mathbf{f} = \mathbf{g}$$

Sparse decomposition (MP and OMP)

- Strict sparsity and exact reconstruction

$$\text{minimize} \quad \|\mathbf{f}\|_0 \quad \text{subject to} \quad \boldsymbol{\Phi}\mathbf{f} = \mathbf{g}$$

$\|\mathbf{f}\|_0$ is the number of non-zero elements of $\mathbf{f}$

- Matching Pursuit (MP) [Mallat & Zhang, 1993]
  - MP is a greedy algorithm that finds one atom at a time.
  - Find the one atom that best matches the signal; Given the previously found atoms, find the next one to best fit, Continue to the end.
- Orthogonal Matching Pursuit (OMP) [Lin, Huang et al., 1993]
  The Orthogonal MP (OMP) is an improved version of MP that re-evaluates the coefficients after each round.

Sparse decomposition (BP,PPR,BCR,IHT,...)

- Sparsity enforcing and exact reconstruction

$$\text{minimize} \quad \|\mathbf{f}\|_1 \quad \text{subject to} \quad \boldsymbol{\Phi}\mathbf{f} = \mathbf{g}$$

  - This problem is convex (linear programming).
  - Very efficient solvers has been deployed:
    - Interior point methods [Chen, Donoho & Saunders (95)],
    - Iterated shrinkage [Figuerido & Nowak (03), Daubechies, Defrise, & Demole (04), Elad (05), Elad, Matalon, & Zibulevsky (06), Marvasti et al].
  - Basis Pursuit (BP)
  - Projection Pursuit Regression
  - Block Coordinate Relaxation
  - Greedy Algorithms
  - Iterative Hard Thresholding (IHT) [Marvasti et al]

Sparse decomposition algorithms

- Strict sparsity and exact reconstruction

$$\text{minimize} \quad \|\mathbf{f}\|_0 \quad \text{subject to} \quad \mathbf{g} = \mathbf{\Phi} \mathbf{f}$$

- Strict sparsity and approximate reconstruction

$$\text{minimize} \quad \|\mathbf{f}\|_0 \quad \text{subject to} \quad \|\mathbf{g} - \mathbf{\Phi} \mathbf{f}\|^2 < c$$

- NP-hard: Looking for approximations: BP, LASSO

$$\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \{J(\mathbf{f})\}$$

with

$$J(\mathbf{f}) = \frac{1}{2} \|\mathbf{g} - \mathbf{\Phi} \mathbf{f}\|^2 + \lambda \|\mathbf{f}\|_1$$

Sparse decomposition Applications

- Denoising: $\mathbf{g} = \mathbf{f} + \epsilon$ with $\mathbf{f} = \mathbf{\Phi z}$

$$J(\mathbf{z}) = \frac{1}{2}\|\mathbf{g} - \mathbf{\Phi z}\|^2 + \lambda\|\mathbf{z}\|_1$$

When $\widehat{\mathbf{z}}$ computed, we can compute $\widehat{\mathbf{f}} = \mathbf{\Phi}\widehat{\mathbf{z}}$.

- Compression, Compressed Sensing,
  General Linear Inverse problems:

$$\mathbf{g} = \mathbf{H f} + \epsilon \quad \text{with} \quad \mathbf{f} = \mathbf{\Phi z}$$

$$J(\mathbf{z}) = \frac{1}{2}\|\mathbf{g} - \mathbf{H\Phi z}\|^2 + \lambda\|\mathbf{z}\|_1$$

Sparse decomposition algorithms

- A closed form solution for the Unitary case

$$J(\mathbf{z}) = \frac{1}{2}\|\mathbf{g} - \boldsymbol{\Phi}\mathbf{z}\|^2 + \lambda\|\mathbf{z}\|_1$$

- When $\boldsymbol{\Phi}\boldsymbol{\Phi}' = \boldsymbol{\Phi}'\boldsymbol{\Phi} = \mathbf{I}$

$$J(\mathbf{f}) = \frac{1}{2}\|\boldsymbol{\Phi}'\mathbf{g} - \boldsymbol{\Phi}'\boldsymbol{\Phi}\mathbf{z}\|^2 + \lambda\|\mathbf{z}\|_1 = \frac{1}{2}\|\mathbf{z}_0 - \mathbf{z}\|^2 + \lambda\|\mathbf{z}\|_1$$

with $\mathbf{z}_0 = \boldsymbol{\Phi}'\mathbf{g}$, is a separable criterion:

$$J(\mathbf{f}) == \frac{1}{2}\|\mathbf{z} - \mathbf{z}_0\|^2 + \lambda\|\mathbf{z}\|_1 = \sum_j \frac{1}{2}|z_j - \mathbf{z}_{0j}|^2 + \lambda|z_j|_1$$

- Closed form solution: Shrinkage

$$z_j = \begin{cases} 0 & |z_{0j}| < \lambda \\ z_{0j} - \text{sign}(z_{0j})\lambda & \text{otherwise} \end{cases}$$

Sparse Decomposition Algorithms (Lasso and extensions)

- LASSO:

$$J(\mathbf{f}) = \|\mathbf{g} - \boldsymbol{\Phi}\mathbf{f}\|^2 + \lambda \sum_j |f_j|$$

- Other Criteria
  - L_p

$$J(\mathbf{f}) = \|\mathbf{g} - \boldsymbol{\Phi}\mathbf{f}\|^2 + \lambda_1 \sum_j |f_j|^p, \quad 1 < p \leq 2$$

  - Elastic net

$$J(\mathbf{f}) = \|\mathbf{g} - \boldsymbol{\Phi}\mathbf{f}\|^2 + \sum_j \left(\lambda_1 |f_j| + \lambda_2 |f_j|^2\right)$$

  - Group LASSO

$$J(\mathbf{f}) = \|\mathbf{g} - \boldsymbol{\Phi}\mathbf{f}\|^2 + \lambda_1 \sum_j |f_j| + \lambda_2 \sum_j |f_j - f_{j-1}|^2$$

  - Weighted L1:

$$J(\mathbf{f}) = \|\mathbf{g} - \boldsymbol{\Phi}\mathbf{f}\|^2 + \lambda \sum_j |w_j f_j|$$

$$w_j = \frac{\lambda}{}$$

Other criteria

$$J(\mathbf{f}) = \|\mathbf{g} - \boldsymbol{\Phi}\mathbf{f}\|^2 + \lambda \sum_j \phi(f_j)$$

Convex criteria:

- $\phi(f_j) = |f_j|^p, \quad p \geq 1$
- $\phi(f_j) = \begin{cases} f_j^2 & |f_j| < s \\ s^2 + |f_j - s| & \text{otherwise} \end{cases}$

Non Convex criteria:

- $\phi(f_j) = |f_j|^p, \quad p \leq 1$
- $\phi(f_j) = s \ln(1 + |f_j|/s)$
- $\phi(f_j) = |f_j| + s \ln(1 + |f_j|/s)$

Dictionary learning

- Given a set of training data $g_k(t)$ and f_{jk} related by:

$$g_k(t) = \sum_{j=1}^{N} \phi_{kj}(t)\, f_{jk},\;\; k = 1, \cdots, K,\;\; t = 1, \cdots, T$$

  or equivalently, given

$$\mathbf{g}_k = \boldsymbol{\Phi}\, \mathbf{f}_k,\;\; k = 1, \cdots, K$$

  determine $\boldsymbol{\Phi}$.

- Objective criterion:

$$J(\boldsymbol{\Phi}) = \sum_k \sum_t \left| g_k(t) - \sum_j \phi_j(t) f_{jk} \right|^2 + \lambda \sum_t \sum_j |\phi_j(t)|^2$$

$$J(\boldsymbol{\Phi}) = \sum_k \|\mathbf{g}_k - \boldsymbol{\Phi}\mathbf{f}_k\|^2 + \lambda |\boldsymbol{\Phi}\|^2$$

Dictionary learning

- Optimizing

$$J(\boldsymbol{\Phi}) = \sum_k \|\mathbf{g}_k - \boldsymbol{\Phi}\mathbf{f}_k\|^2 + \lambda|\boldsymbol{\Phi}\|^2$$

gives

$$\widehat{\boldsymbol{\Phi}} = \left[\sum_k \mathbf{g}_k \mathbf{g}_k' + \lambda \mathbf{I}\right]^{-1} \mathbf{g}_k' \mathbf{f}_k$$

- Looking for sparse dictionnary, we can use

$$J(\boldsymbol{\Phi}) = \sum_k \|\mathbf{g}_k - \boldsymbol{\Phi}\mathbf{f}_k\|^2 + \lambda\|\boldsymbol{\Phi}\|_1$$

and we can again use an iterative algorithm to find the solution.

Joint Dictionary learning and sparse reconstruction

- Given a set of data $\mathbf{g}_k$ modelled as

$$\mathbf{g}_k = \mathbf{\Phi}' \, \mathbf{f}_k, \;\; k = 1, \cdots, K$$

determine both the dictionnary $\mathbf{\Phi}$ and the compositions $\mathbf{f}_k$.

- Joint criterion

$$J(\mathbf{\Phi}, \mathbf{f}_k) = \sum_k \|\mathbf{g}_k - \mathbf{\Phi}'\mathbf{f}_k\|^2 + \lambda_0 |\mathbf{\Phi}\|^2 + \lambda_1 \sum_k \|\mathbf{f}_k\|^2$$

- Alternate optimization:

$$\begin{cases} \widehat{\mathbf{\Phi}} = [\sum_k \mathbf{g}_k \mathbf{g}_k' + \lambda_0 \mathbf{I}]^{-1} \; \mathbf{g}_k' \widehat{\mathbf{f}}_k \\ \widehat{\mathbf{f}}_k = \left[\widehat{\mathbf{\Phi}} \widehat{\mathbf{\Phi}}' + \lambda_1 \mathbf{I}\right]^{-1} \; \widehat{\mathbf{\Phi}}_k \mathbf{g}_k \end{cases}$$

- Looking for sparse dictionnary and sparse coefficients we can use

$$J(\mathbf{\Phi}) = \sum_k \|\mathbf{g}_k - \mathbf{\Phi}'\mathbf{f}_k\|^2 + \lambda_0 \|\mathbf{\Phi}\|_1 + \lambda_1 \sum_k \|\mathbf{f}_k\|_1$$

and we can again use an iterative algorithm to find the solution.

Multi Dimensional signals: PCA, SPCA, BSS, ...

$$g_i(t) = \sum_{j=1}^{N} \boldsymbol{\Phi}_{ij} f_j(t), \; i = 1, \cdots, M, \; t = 1, \cdots, T$$

$$\mathbf{g}(t) = \boldsymbol{\Phi}\, \mathbf{f}(t), \; t = 1, \cdots, T$$

$$\mathbf{G} = \boldsymbol{\Phi}\, \mathbf{F}, \quad \text{with } \mathbf{G}\ [M \times T], \quad \boldsymbol{\Phi}\ [M \times N], \quad \mathbf{F}\ [N \times T]$$

- $f_j(t)$ factors, sources, codes
- $\boldsymbol{\Phi}$ Loading matrix (Factor Analysis),
  Mixing matrix (Blind Sources Separation),
  Design matrix (Sparse coding, Compressed Sensing)
- Objective: Find $\boldsymbol{\Phi}$ and $f_j(t)$

$$J(\mathbf{f}(t), \boldsymbol{\Phi}) = \sum_t \|\mathbf{g}(t) - \boldsymbol{\Phi}\mathbf{f}(t)\|^2 + \lambda_1 \sum_i \sum_j |\boldsymbol{\Phi}_{ij}| + \lambda_2 \sum_t \sum_j |f_j(t)|$$

$$J(\mathbf{F}, \boldsymbol{\Phi}) = \|\mathbf{G} - \boldsymbol{\Phi}\mathbf{F}\|^2 + \lambda_1 \|\boldsymbol{\Phi}\|_1 + \lambda_2 \|\mathbf{F}\|_1$$

Matrix Decomposition or Approximation

- Matrix approximation:
  Find an approximate matrix $\widehat{\mathbf{G}} = \mathbf{\Phi F}$ for $\mathbf{G}$ with some degrees of sparsity in the elements of $\mathbf{\Phi}$ and $\mathbf{F}$.

$$J(\mathbf{F}, \mathbf{\Phi}) = \|\mathbf{G} - \mathbf{\Phi F}\|^2 + \lambda_1 \|\mathbf{\Phi}\|_1 + \lambda_2 \|\mathbf{F}\|_1$$

- Low rank Matrice decomposition:

$$\widehat{\mathbf{G}} = \sum_{k=1}^{K} d_k \mathbf{u}_k \mathbf{v}'_k = \mathbf{UDV}$$

with some degrees of sparsity in the elements of $\mathbf{u}$ and $\mathbf{v}$.

$$J(\mathbf{U}, \mathbf{V}) = \|\mathbf{G} - \mathbf{UDV}\|^2 + \lambda_1 \|\mathbf{U}\|_1 + \lambda_2 \|\mathbf{V}\|_1$$

4. Sparse decomposition: Bayesian MAP interpretation

- Sparsity enforcing and approximate reconstruction

$$\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \left\{ \|\mathbf{g} - \boldsymbol{\Phi}\mathbf{f}\|^2 + \lambda \|\mathbf{f}\|_1 \right\}$$

- Equivalent to MAP estimate in a Bayesian approach
  - Bayesian approach: $\mathbf{g} = \boldsymbol{\Phi}\mathbf{f} + \boldsymbol{\epsilon}$

$$p(\mathbf{f}|\mathbf{g}) = \frac{p(\mathbf{g}|\mathbf{f})\, p(\mathbf{f})}{p(\mathbf{g})} \propto p(\mathbf{g}|\mathbf{f})\, p(\mathbf{f})$$

$$\begin{cases} p(\mathbf{g}|\mathbf{f}) = \mathcal{N}(\mathbf{g}|\boldsymbol{\Phi}\mathbf{f}, \sigma_\epsilon^2) \propto \exp\left[\frac{-1}{2\sigma_\epsilon^2} \|\mathbf{g} - \boldsymbol{\Phi}\mathbf{f}\|^2\right] \\ p(\mathbf{f}) = \mathcal{DE}(\mathbf{f}|\gamma) \propto \exp\left[\gamma \|\mathbf{f}\|_1\right] \end{cases}$$

  - Maximum A Posteriori (MAP):

$$\widehat{\mathbf{f}} = \arg\max_{\mathbf{f}} \left\{ p(\mathbf{f}|\mathbf{g}) \right\} = \arg\min_{\mathbf{f}} \left\{ J(\mathbf{f}) \right\}$$

$$J(\mathbf{f}) = \|\mathbf{g} - \boldsymbol{\Phi}\mathbf{f}\|^2 + \lambda \|\mathbf{f}\|_1 \quad \text{with} \quad \lambda = 2\gamma\sigma_\epsilon^2$$

Sparse decomposition: Regularization or MAP

- Regularization: $J(\mathbf{f}) = \|\mathbf{g} - \mathbf{\Phi f}\|^2 + \lambda \|\mathbf{f}\|_1$
- With fixed λ: Find a good optimization algorithm
- How to choose λ ?
- L-Curve, Cross Validation, adhoc $\lambda = 1$, ...
- ---
- MAP: $J(\mathbf{f}) = \|\mathbf{g} - \mathbf{\Phi f}\|^2 + \lambda \|\mathbf{f}\|_1$ with $\lambda = 2\gamma\sigma_\epsilon^2$
- How to estimate γ and σ_ϵ^2?
- Bayesian: Joint MAP, Expectation-Maximization, MCMC, Variational Bayesian Approximation,...
- Advantages of the Bayesian approach:
  - More probabilistic modeling for sparsity enforcing
  - Hyperparameter estimation
  - Uncertainty handling

5. Sparsity enforcing prior models

- Simple heavy tailed models:
  - Generalized Gaussian, Double Exponential
  - Student-t, Cauchy
  - Elastic net
  - Generalized hyperbolic
  - Symmetric Weibull, Symmetric Rayleigh
- Hierarchical mixture models:
  - Mixture of Gaussians
  - Bernoulli-Gaussian
  - Mixture of Gammas
  - Bernoulli-Gamma
  - Mixture of Dirichlet
  - Bernoulli-Multinomial

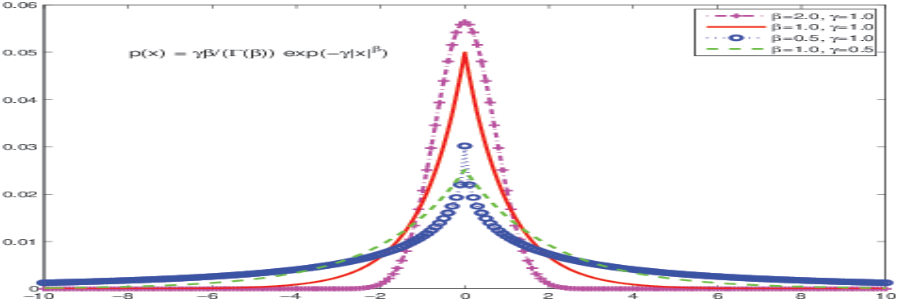
Simple heavy tailed models

- Generalized Gaussian, Double Exponential

$$p(\mathbf{f}|\gamma,\beta) = \prod_j \mathcal{GG}(f_j|\gamma,\beta) \propto \exp\left[-\gamma \sum_j |f_j|^\beta\right]$$

$\beta = 1$ Double exponential or Laplace.

$0 < \beta < 2$ are of great interest for sparsity enforcing.



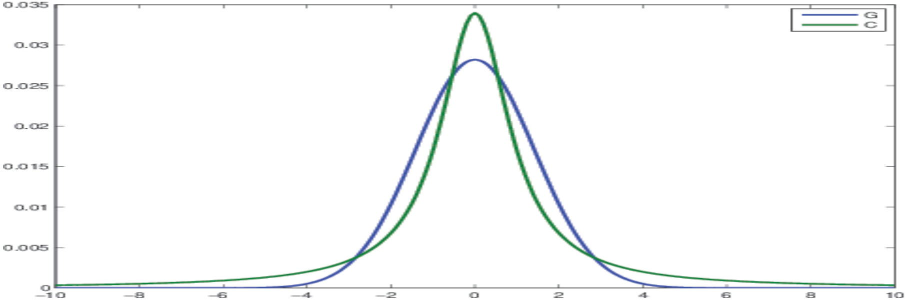
Generalized Gaussian family

Simple heavy tailed models

- Student-t and Cauchy models

$$p(\mathbf{f}|\nu) = \prod_j \mathcal{S}t(f_j|\nu) \propto \exp\left[-\frac{\nu+1}{2}\sum_j \log\left(1+f_j^2/\nu\right)\right]$$

Cauchy model is obtained when $\nu = 1$.

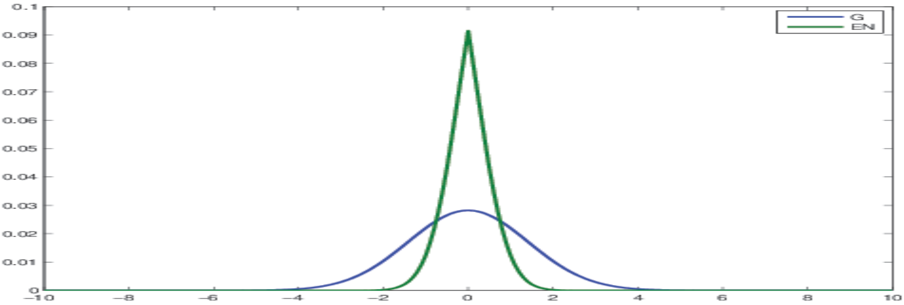


Student-t and Cauchy families

Simple heavy tailed models

- Elastic net prior model

$$p(\mathbf{f}|\nu) = \prod_j \mathcal{EN}(f_j|\nu) \propto \exp\left[-\sum_j (\gamma_1 |f_j| + \gamma_2 f_j^2)\right]$$

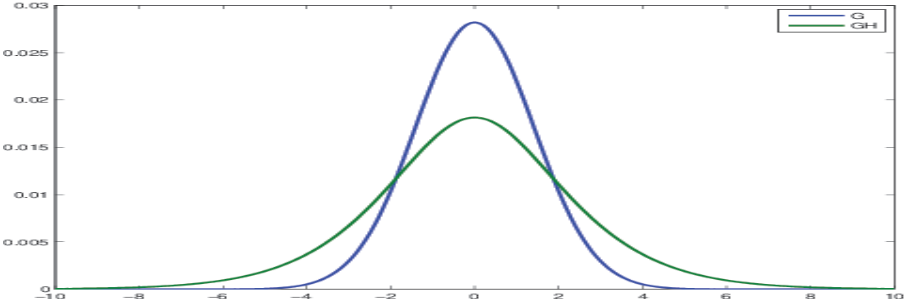


Elastic Net family

Simple heavy tailed models

- Generalized hyperbolic (GH) models

$$p(\mathbf{f}|\delta,\nu,\beta) = \prod_j (\delta^2 + f_j^2)^{(\nu-1/2)/2} \exp\left[\beta x\right)] \, K_{\nu-1/2}(\alpha\sqrt{\delta^2 + f_j^2})$$



Generalized hyperbolic family

Simple heavy tailed models

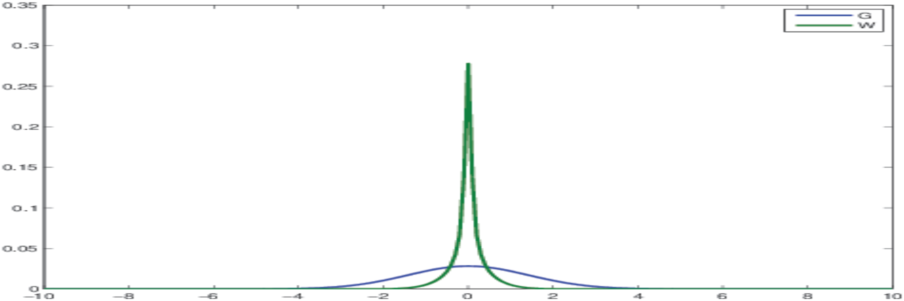
- Symmetric Weibull

$$p(\mathbf{f}|\gamma,\beta) = \prod_j \mathcal{W}(f_j|\gamma,\beta) \propto \exp\left[-\gamma \sum_j |f_j|^\beta + (\beta-1)\log|f_j|\right]$$

$\beta = 2$ is the Symmetric Rayleigh distribution.

$\beta = 1$ is the Double exponential and

$0 < \beta < 2$ are of great interest for sparsity enforcing.



Symmetric Weibull family

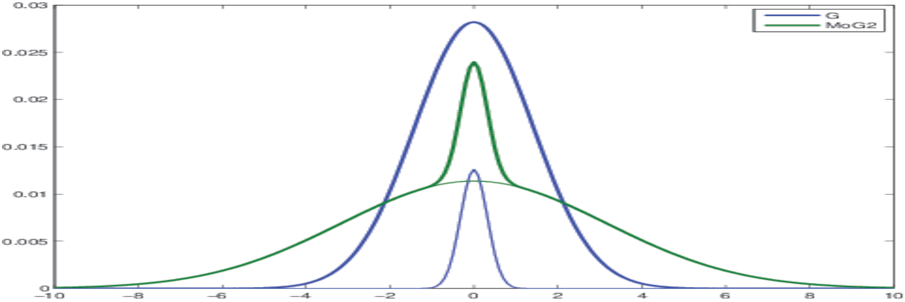
Mixture models

- Mixture of two Gaussians (MoG2) model

$$p(\mathbf{f}|\alpha, v_1, v_0) = \prod_j \left[\alpha \mathcal{N}(f_j|0, v_1) + (1-\alpha)\mathcal{N}(f_j|0, v_0)\right]$$

- Bernoulli-Gaussian (BG) model

$$p(\mathbf{f}|\alpha, v) = \prod_j p(f_j) = \prod_j \left[\alpha \mathcal{N}(f_j|0, v) + (1-\alpha)\delta(f_j)\right]$$



Mixture of 2 Gaussians families

- Mixture of Gammas

$$p(\mathbf{f}|\lambda, v_1, v_0) = \prod_j \left[\lambda \mathcal{G}(f_j|\alpha_1, \beta_1) + (1-\lambda)\mathcal{G}(f_j|\alpha_2, \beta_2)\right]$$

- Bernoulli-Gamma model

$$p(\mathbf{f}|\lambda, \alpha, \beta) = \prod_j \left[\lambda \mathcal{G}(f_j|\alpha, \beta) + (1-\lambda)\delta(f_j)\right]$$

- Mixture of Dirichlets model

$$p(\mathbf{f}|\lambda, \boldsymbol{\Phi}_1, \boldsymbol{\alpha}_1, \boldsymbol{\Phi}_2, \boldsymbol{\alpha}_2) = \prod_j \left[\lambda \mathcal{D}(f_j|\mathbf{H}_1, \boldsymbol{\alpha}_1) + (1-\lambda)\mathcal{D}(f_j|\mathbf{H}_2, \boldsymbol{\alpha}_2)\right]$$

$$\mathcal{D}(f_j|\mathbf{H}, \boldsymbol{\alpha}) = \prod_{k=1}^{K} \frac{\Gamma(\alpha)}{\Gamma(\alpha_0)\Gamma(\alpha_K)} a_k^{\alpha_k - 1}, \quad \alpha_k \geq 0, \quad a_k \geq 0$$

where $\mathbf{H} = \{a_1, \cdots, a_K\}$ and $\boldsymbol{\alpha} = \{\alpha_1, \cdots, \alpha_K\}$
with $\sum_k \alpha_k = \alpha$ and $\sum_k a_k = 1$.

- Bernoulli-Multinomial (BMultinomial) model

$$p(\mathbf{f}|\lambda, \mathbf{H}, \boldsymbol{\alpha}) = \prod_j \left[\lambda \delta(f_j) + (1-\lambda)\mathcal{M}ult(f_j|\mathbf{H}, \boldsymbol{\alpha})\right]$$

Hierarchical models and hidden variables

- All the mixture models and some of simple models can be modeled via hidden variables $\mathbf{z}$.

$$p(f)=\sum_{k=1}^{K}\alpha_k p_k(f) \longrightarrow \left\{ \begin{array}{l} p(f|z=k)=p_k(f), \\ P(z=k)=\alpha_k, \quad \sum_k \alpha_k=1 \end{array} \right.$$

- Example 1: MoG model: $p_k(f)=\mathcal{N}(f|m_k,v_k)$
  2 Gaussians: $p_0=\mathcal{N}(0,v_0), p_1=\mathcal{N}(0,v_1), \alpha_0=\lambda, \alpha_1=1-\lambda$

$$p(f_j|\lambda,v_1,v_0)=\lambda\mathcal{N}(f_j|0,v_1)+(1-\lambda)\mathcal{N}(f_j|0,v_0)$$

$$\left\{ \begin{array}{l} p(f_j|z_j=0,v_0)=\mathcal{N}(f_j|0,v_0), \\ p(f_j|z_j=1,v_1)=\mathcal{N}(f_j|0,v_1), \end{array} \right. \text{ and } \left\{ \begin{array}{l} P(z_j=0)=\lambda, \\ P(z_j=1)=1-\lambda \end{array} \right.$$

$$\left\{ \begin{array}{l} p(\mathbf{f}|\mathbf{z})=\prod_j p(f_j|z_j)=\prod_j \mathcal{N}\left(f_j|0,v_{z_j}\right) \propto \exp\left[-\frac{1}{2}\sum_j \frac{f_j^2}{v_{z_j}}\right] \\ p(\mathbf{z})=\lambda^{n_1}(1-\lambda)^{n_0}, \quad n_1=\sum_j \delta(z_j-1), \quad n_0=\sum_j \delta(z_j) \end{array} \right.$$

Hierarchical models and hidden variables

- Example 2: Student-t model

$$\mathcal{S}t(f|\nu) \propto \exp\left[-\frac{\nu+1}{2}\log\left(1+f^2/\nu\right)\right]$$

- Infinite mixture

$$\mathcal{S}t(f|\nu) \propto= \int_0^\infty \mathcal{N}(f|,0,1/z)\,\mathcal{G}(z|\alpha,\beta)\,\mathrm{d}z, \quad \text{with } \alpha=\beta=\nu/2$$

$$p(f|z)=\mathcal{N}(f|0,1/z), \quad p(z)=\mathcal{G}(z|\alpha,\beta)$$

$$\left\{\begin{array}{ll} p(\mathbf{f}|\mathbf{z}) & = \prod_j p(f_j|z_j) = \prod_j \mathcal{N}(f_j|0,1/z_j) \propto \exp\left[-\frac{1}{2}\sum_j z_j f_j^2\right] \\ p(\mathbf{z}|\alpha,\beta) & = \prod_j \mathcal{G}(z_j|\alpha,\beta) \propto \prod_j z_j^{(\alpha-1)} \exp\left[-\beta z_j\right] \\ & \propto \exp\left[\sum_j (\alpha-1)\ln z_j - \beta z_j\right] \\ p(\mathbf{f},\mathbf{z}|\alpha,\beta) & \propto \exp\left[-\frac{1}{2}\sum_j z_j f_j^2 + (\alpha-1)\ln z_j - \beta z_j\right] \end{array}\right.$$

Hierarchical models and hidden variables

$$p(f_j|z_j) = \mathcal{N}(f|0, 1/z_j), \quad p(z_j) = \mathcal{G}(z_j|\alpha, \beta)$$

$$\left\{ \begin{array}{ll} p(\mathbf{f}|\mathbf{z}) & = \prod_j p(f_j|z_j) \\ p(\mathbf{z}) & = \prod_j p(z_j) \\ p(\mathbf{g}|\mathbf{f}) & = \mathcal{N}(\mathbf{g}|\Phi'\mathbf{f}, \sigma_\epsilon^2 \mathbf{I}) \end{array} \right. \longrightarrow p(\mathbf{f}, \mathbf{z}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f}) p(\mathbf{f}|\mathbf{z}) p(\mathbf{z})$$

- With Hyperparameters $\boldsymbol{\theta}$ we have:
  - Simple priors

$$p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f}, \boldsymbol{\theta}_1)\; p(\mathbf{f}|\boldsymbol{\theta}_2)\; p(\boldsymbol{\theta})$$

  - Hierarchical priors

$$p(\mathbf{f}, \mathbf{z}, \boldsymbol{\theta}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f}, \boldsymbol{\theta}_1)\; p(\mathbf{f}|\mathbf{z}, \boldsymbol{\theta}_2)\; p(\mathbf{z}|\boldsymbol{\theta}_3)\; p(\boldsymbol{\theta})$$

Bayesian Computation and Algorithms

- When the expression of $p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g})$ or of $p(\mathbf{f}, \mathbf{z}, \boldsymbol{\theta}|\mathbf{g})$ is obtained, we have following options:
- Joint MAP: (needs optimization algorithms)

$$(\widehat{\mathbf{f}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\mathbf{f},\boldsymbol{\theta})} \{p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g})\}$$

- MCMC: Needs the expressions of the conditionals $p(\mathbf{f}|\mathbf{z}, \boldsymbol{\theta}, \mathbf{g})$, $p(\mathbf{z}|\mathbf{f}, \boldsymbol{\theta}, \mathbf{g})$, and $p(\boldsymbol{\theta}|\mathbf{f}, \mathbf{z}, \mathbf{g})$
- Variational Bayesian Approximation (VBA): Approximate $p(\mathbf{f}, \mathbf{z}, \boldsymbol{\theta}|\mathbf{g})$ by a separable one

$$q(\mathbf{f}, \mathbf{z}, \boldsymbol{\theta}|\mathbf{g}) = q_1(\mathbf{f})\, q_2(\mathbf{z})\, q_3(\boldsymbol{\theta})$$

and do any computations with these separable ones.

Joint MAP

$$p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f}, \boldsymbol{\theta}_1)\; p(\mathbf{f}|\boldsymbol{\theta}_2)\; p(\boldsymbol{\theta})$$

- Objective:

$$(\widehat{\mathbf{f}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\mathbf{f},\boldsymbol{\theta})} \{p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g})\}$$

- Alternate optimization:

$$\begin{cases} \widehat{\mathbf{f}} = \arg\max_{\mathbf{f}} \left\{p(\mathbf{f}, \widehat{\boldsymbol{\theta}}|\mathbf{g})\right\} \\ \widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \left\{p(\widehat{\mathbf{f}}, \boldsymbol{\theta}|\mathbf{g})\right\} \end{cases}$$

$$\begin{array}{ccccc} \boldsymbol{\theta}^{(0)} \longrightarrow \widehat{\boldsymbol{\theta}} \longrightarrow & \boxed{\widehat{\mathbf{f}} = \arg\max_{\mathbf{f}} \left\{p(\mathbf{f}|\widehat{\boldsymbol{\theta}}, \mathbf{g})\right\}} & \longrightarrow \widehat{\mathbf{f}} \\ \uparrow & & \downarrow \\ \widehat{\boldsymbol{\theta}} \longleftarrow & \boxed{\widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \left\{p(\boldsymbol{\theta}|\widehat{\mathbf{f}}, \mathbf{g})\right\}} & \longleftarrow \widehat{\mathbf{f}} \end{array}$$

- Uncertainties are not propagated.

MCMC based algorithm

$$p(\mathbf{f}, \mathbf{z}, \boldsymbol{\theta}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f}, \mathbf{z}, \boldsymbol{\theta})\, p(\mathbf{f}|\mathbf{z}, \boldsymbol{\theta})\, p(\mathbf{z})\, p(\boldsymbol{\theta})$$

General scheme (Gibbs Sampling):

- Generate samples from the conditionals:

$$\widehat{\mathbf{f}} \sim p(\mathbf{f}|\widehat{\mathbf{z}}, \widehat{\boldsymbol{\theta}}, \mathbf{g}) \longrightarrow \widehat{\mathbf{z}} \sim p(\mathbf{z}|\widehat{\mathbf{f}}, \widehat{\boldsymbol{\theta}}, \mathbf{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta}|\widehat{\mathbf{f}}, \widehat{\mathbf{z}}, \mathbf{g})$$

- Waite for convergency
- Compute empirical statistics (means, modes, variances) from the samples

$$\mathrm{E}\left\{\mathbf{f}\right\} \approx \frac{1}{N} \sum_{n} \mathbf{f}^{(n)}$$

Variational Bayesian Approximation

- Approximate $p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g})$ by $q(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g}) = q_1(\mathbf{f}|\mathbf{g})\, q_2(\boldsymbol{\theta}|\mathbf{g})$ and then continue computations.
- Criterion $\mathrm{KL}(q(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g}) : p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g}))$

$$\mathrm{KL}(q : p) = \iint q \ln \frac{q}{p} = \iint q_1 q_2 \ln \frac{q_1 q_2}{p}$$

- Iterative algorithm $q_1 \longrightarrow q_2 \longrightarrow q_1 \longrightarrow q_2, \cdots$

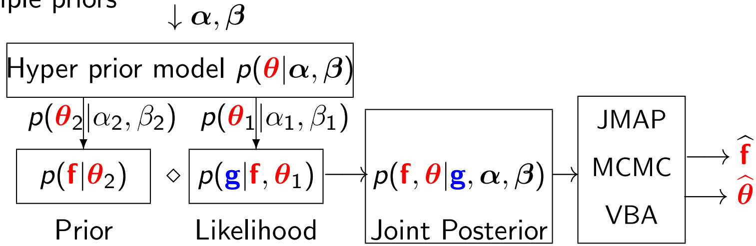
$$\left\{ \begin{array}{ll} q_1(\mathbf{f}) & \propto \exp\left[\langle \ln p(\mathbf{g}, \mathbf{f}, \boldsymbol{\theta}; \mathcal{M}) \rangle_{q_2(\boldsymbol{\theta})}\right] \\ q_2(\boldsymbol{\theta}) & \propto \exp\left[\langle \ln p(\mathbf{g}, \mathbf{f}, \boldsymbol{\theta}; \mathcal{M}) \rangle_{q_1(\mathbf{f})}\right] \end{array} \right.$$

$$\begin{array}{ccccc} \widehat{q}_2^{(0)} \longrightarrow \widehat{q}_2 \longrightarrow & \boxed{q_1(\mathbf{f}) \propto \exp\left[\langle \ln p(\mathbf{g}, \mathbf{f}, \boldsymbol{\theta}; \mathcal{M}) \rangle_{q_2}\right]} & \longrightarrow \widehat{q}_1(\mathbf{f}) \longrightarrow \widehat{\mathbf{f}} \\ \uparrow & & \downarrow \\ \widehat{\boldsymbol{\theta}} \leftarrow \widehat{q}_2(\boldsymbol{\theta}) \longleftarrow & \boxed{q_2(\boldsymbol{\theta}) \propto \exp\left[\langle \ln p(\mathbf{g}, \mathbf{f}, \boldsymbol{\theta}; \mathcal{M}) \rangle_{q_1}\right]} & \longleftarrow \widehat{q}_1 \end{array}$$

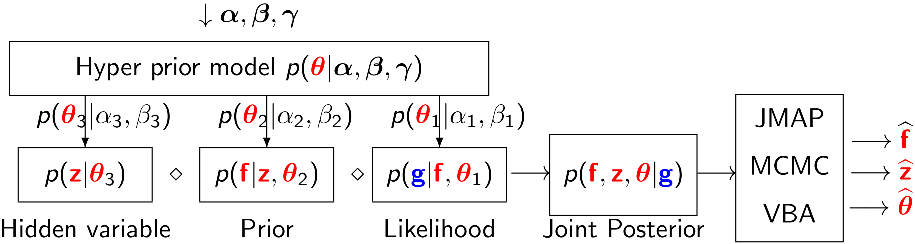
- Uncertainties are propagated (Message Passing methods)

Summary of Bayesian approach

- Simple priors



- Hierarchical priors



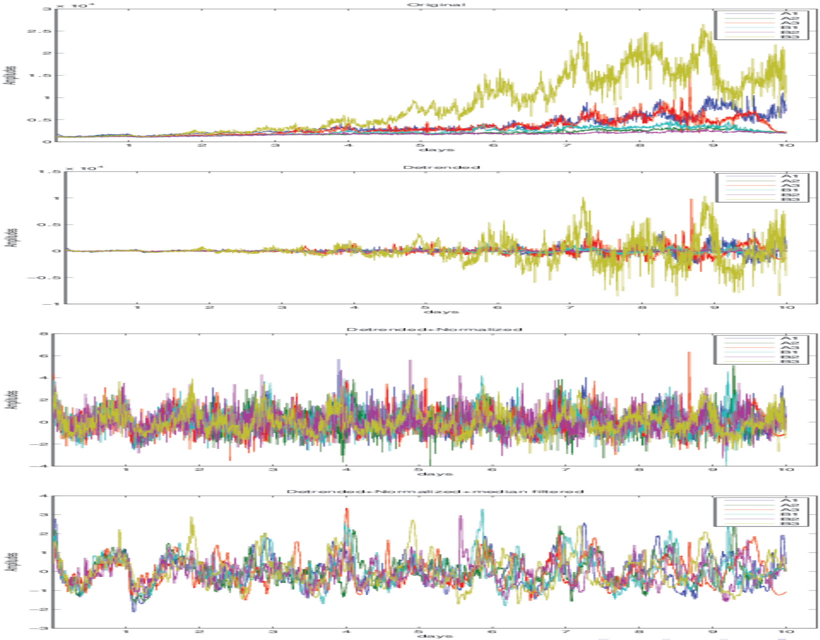
Advantages of the Bayesian Approach

- More possibilities to model sparsity
- More tools to handle hyperparameters
- More tools to account for uncertainties
- More possibilities to understand and to control many ad hoc deterministic algorithms
- Hierarchical models give still more modeling possibilities
  - Bernouilli-Gaussian: strict sparsity
  - Bernouilli-Gamma: strict sparsity + positivity
  - Bernouilli-Multinomial: strict sparsity + discrete values (finite states)
  - Independent Mixture models: sparsity enforcing
  - Mixture of multivariate models: group sparsity enforcing
  - Gauss-Markov-Potts models: sparsity in transform domains

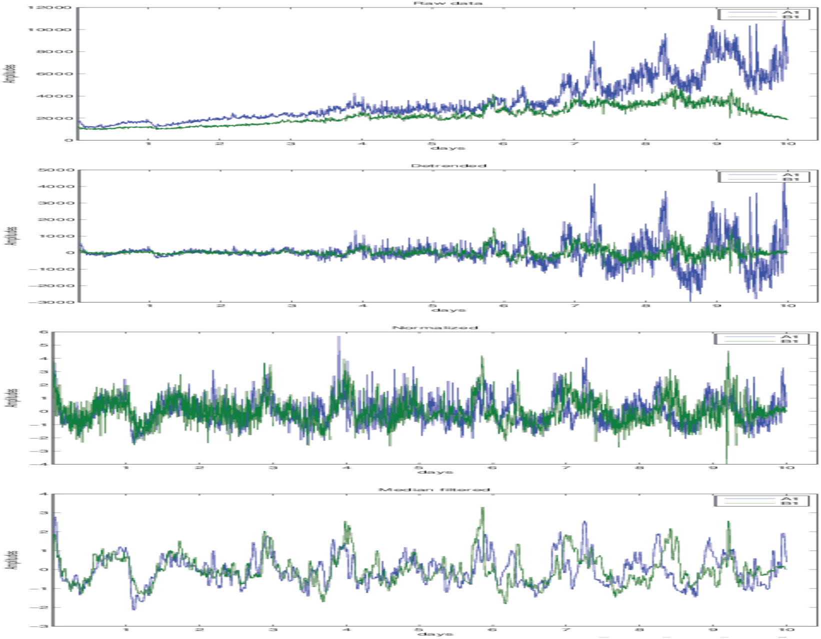
Application in biologial time series

- Cancer studies
- Genes activity phosphoresence bio-markers
- Short time (12 days) non stationary signals
- Needs for great precision and measures of uncertainties
- Changes Before-During-After some medication treatment
- multi component, multi variate data

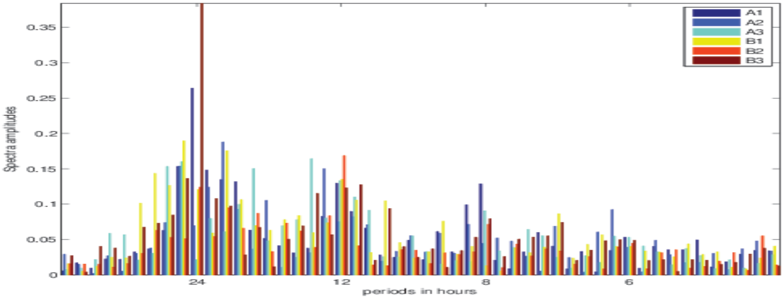
Raw, Detrended, normalized and Median filtered data

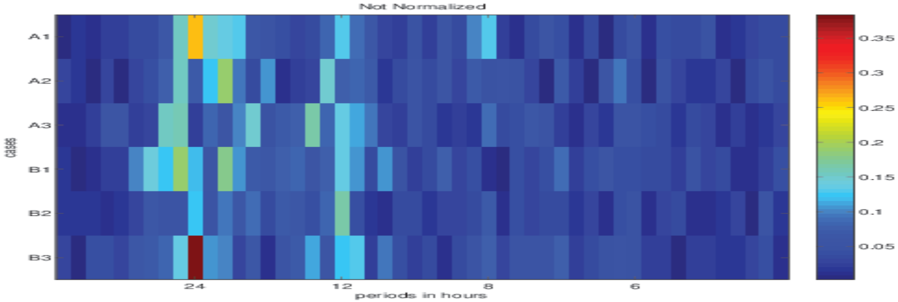


Raw, Detrended, normalized and Median filtered data

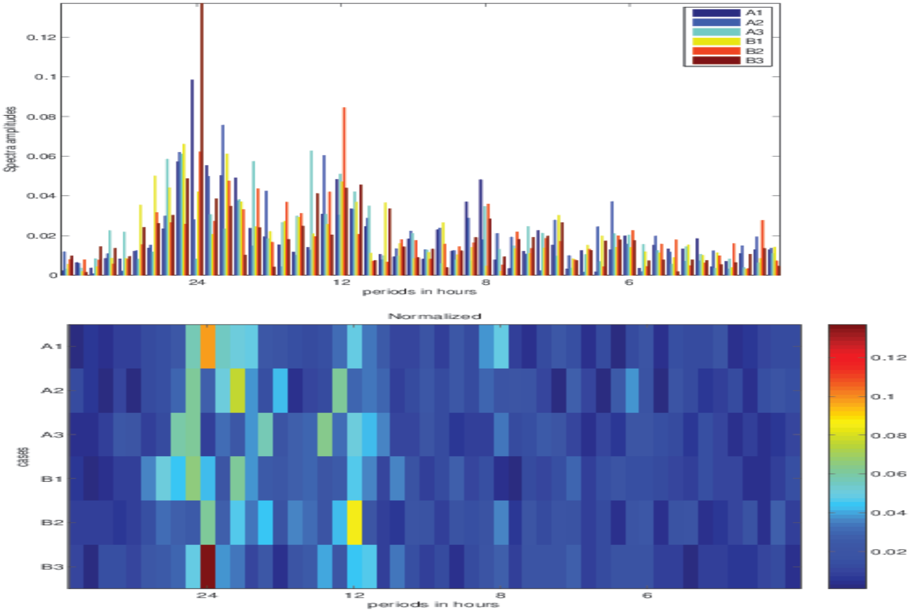


FFT over 10 days



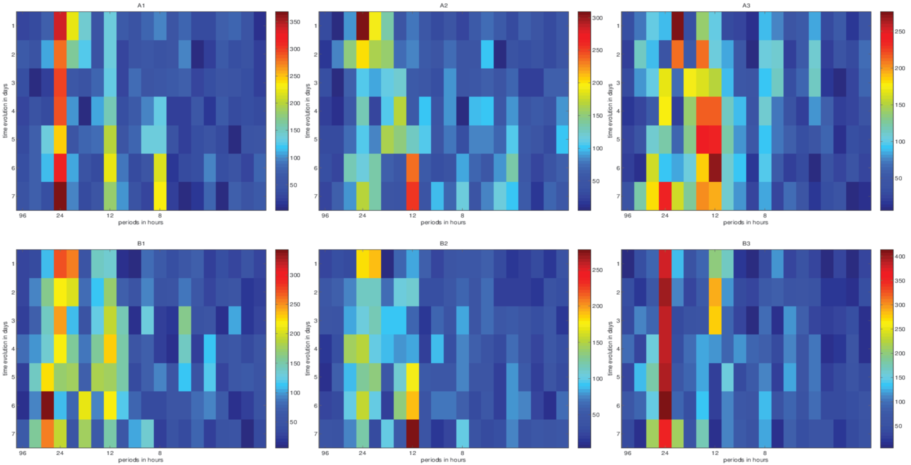


FFT over 10 days normalized



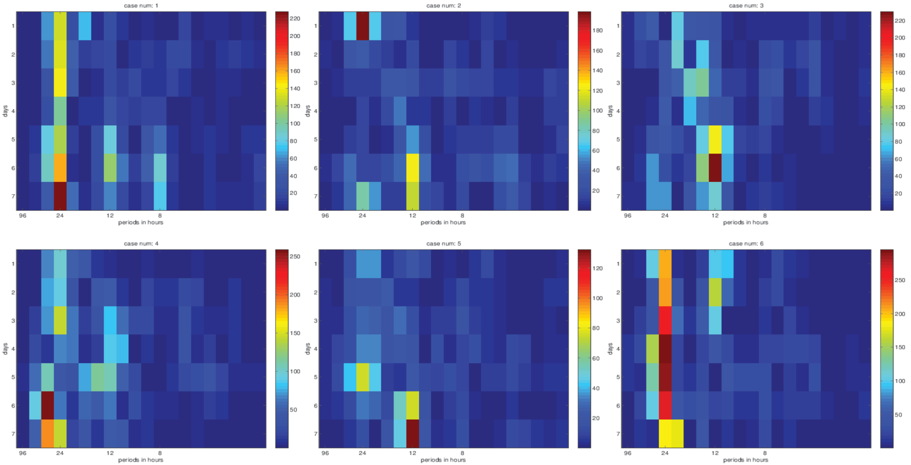
Short Time Fourier Transform (STFT)

over 4 days with shift of one day



Spectrograms

over 4 days with shift of one day



Application in Inverse Problems

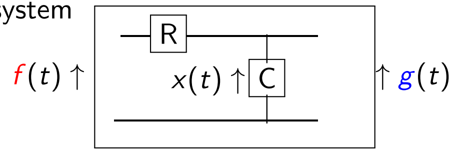
Three main examples

- Example 1:
  Instrumentation
  - $f(t)$ input of the instrument
  - $g(t)$ output of the instrument
- Example 2: Seeing outside of a body: Making an image using a camera, a microscope or a telescope
  - $f(x, y)$ real scene
  - $g(x, y)$ observed image
- Example 3: Seeing inside of a body: Computed Tomography usng X rays, US, Microwave, etc.
  - $f(x, y)$ a section of a real 3D body $f(x, y, z)$
  - $g_\phi(r)$ a line of observed radiographe $g_\phi(r, z)$

- Example 1: Deconvolution
- Example 2: Image restoration
- Example 3: Image reconstruction

Instrumentation

A simple electric system



$$f(t) = R\,i(t) + v_c(t) = RC\,\frac{\partial x(t)}{\partial t} + x(t), \quad RC = 1$$

- Differential Equation Modelling

$$\frac{\partial x(t)}{\partial t} + x(t) = f(t), \qquad x(t) = g(t)$$

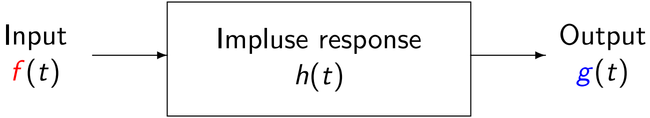
- State Space Modelling

$$\begin{cases} \frac{\partial x(t)}{\partial t} & = -x(t) + f(t) \\ g(t) & = x(t) \end{cases}$$

- Input-Output Modelling

$$\begin{cases} \frac{\partial x(t)}{\partial t} = -x(t) + f(t) \\ g(t) = x(t) \end{cases} \rightarrow \begin{cases} pX(p) = -X(p) + F(p) \rightarrow X(p) = \frac{1}{p+1} F(p) \\ g(t) = x(t) = h(t) * f(t), \quad h(t) = \exp\left[-t\right] \end{cases}$$

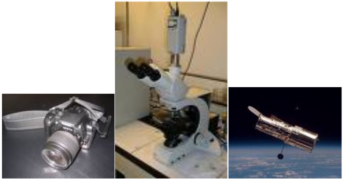
Instrumentation



- Ideal Instrument $g(t) = f(t)$ does not exist.
- A linear and time invariant instrument is characterized by its impulse response $h(t)$.
- Ideal Instrument $h(t) = \delta(t)$ does not exist.
- Forward problem: $f(t),\ h(t) \longrightarrow g(t) = h(t) * f(t)$
- Two linked problems in instrumentation:
  - Inversion: $g(t),\ h(t) \longrightarrow f(t)$
  - Identification: $g(t),\ f(t) \longrightarrow h(t)$

Seeing outside of a body: Making an image with a camera, a microscope or a telescope

- $f(x,y)$ real scene
- $g(x,y)$ observed image



- Forward model: Convolution

$$g(x,y) = \iint f(x',y')\, h(x-x',y-y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x,y)$$

  $h(x,y)$: Point Spread Function (PSF) of the imaging system

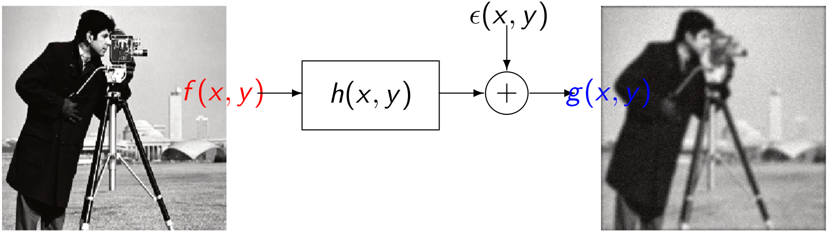
- Inverse problem: Image restoration

  Given the forward model $\mathcal{H}$ (PSF $h(x,y)$)
  and a set of data $g(x_i,y_i), i = 1,\cdots,M$
  find $f(x,y)$

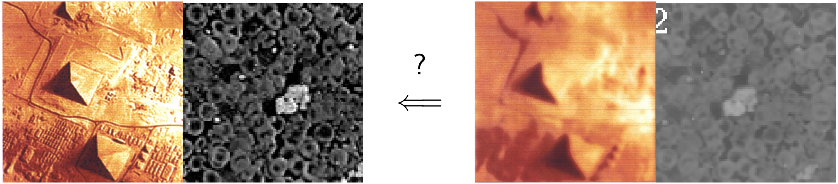
Making an image with an unfocused camera

Forward model: 2D Convolution

$$g(x,y) = \iint f(x',y')\, h(x-x', y-y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x,y)$$

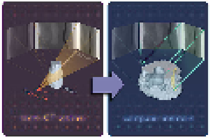


Inversion: Image Deconvolution or Restoration



Seeing inside of a body: Computed Tomography

- $f(x,y)$ a section of a real 3D body $f(x,y,z)$
- $g_\phi(r)$ a line of observed radiographe $g_\phi(r,z)$



- Forward model:
  Line integrals or Radon Transform

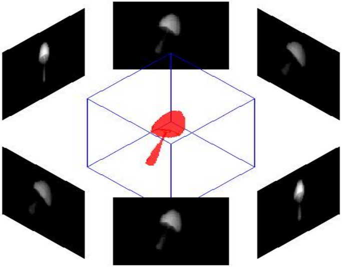
$$
\begin{aligned}
g_\phi(r) &= \int_{L_{r,\phi}} f(x,y)\,\mathrm{d}l + \epsilon_\phi(r) \\
&= \iint f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y + \epsilon_\phi(r)
\end{aligned}
$$

- Inverse problem: Image reconstruction

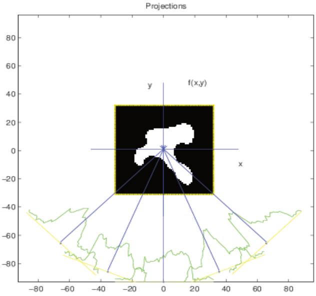
  Given the forward model $\mathcal{H}$ (Radon Transform) and
  a set of data $g_{\phi_i}(r), i = 1, \cdots, M$
  find $f(x,y)$

2D and 3D Computed Tomography

3D



2D

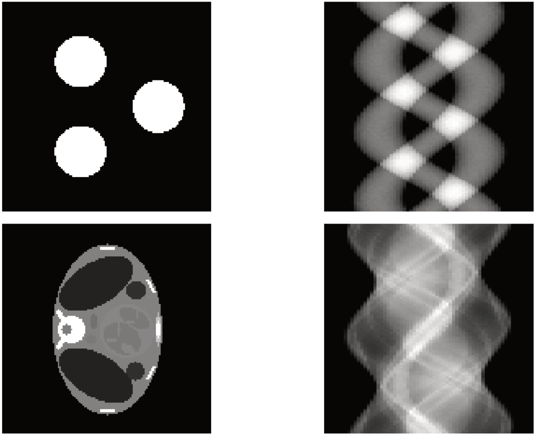


$$g_\phi(r_1, r_2) = \int_{\mathcal{L}_{r_1, r_2, \phi}} f(x, y, z)\,\mathrm{d}l \qquad g_\phi(r) = \int_{\mathcal{L}_{r, \phi}} f(x, y)\,\mathrm{d}l$$

Forward probelm: $f(x, y)$ or $f(x, y, z) \longrightarrow g_\phi(r)$ or $g_\phi(r_1, r_2)$

Inverse problem: $g_\phi(r)$ or $g_\phi(r_1, r_2) \longrightarrow f(x, y)$ or $f(x, y, z)$

Computed Tomography: Radon Transform



Forward: $\color{red}{f}(x, y) \quad \longrightarrow \quad \color{blue}{g}(r, \phi)$

Inverse: $\color{red}{f}(x, y) \quad \longleftarrow \quad \color{blue}{g}(r, \phi)$

Linear Inverse Problems

$$g(\mathbf{s}_i) = \int h(\mathbf{s}_i, \mathbf{r})\, f(\mathbf{r})\, \mathrm{d}\mathbf{r} + \epsilon(\mathbf{s}_i), \quad i = 1, \cdots, M$$

- $f(\mathbf{r})$ is assumed to be well approximated by

$$f(\mathbf{r}) \simeq \sum_{j=1}^{N} f_j\, \phi_j(\mathbf{r})$$

with $\{\phi_j(\mathbf{r})\}$ a basis or any other set of known functions

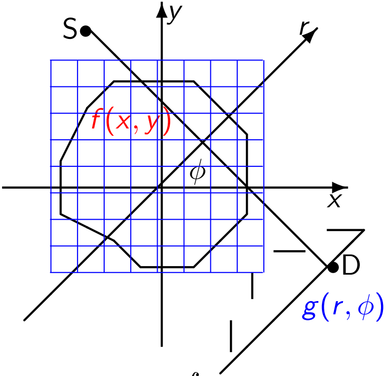
$$g(\mathbf{s}_i) = g_i \simeq \sum_{j=1}^{N} f_j \int h(\mathbf{s}_i, \mathbf{r})\, \phi_j(\mathbf{r})\, \mathrm{d}\mathbf{r}, \quad i = 1, \cdots, M$$

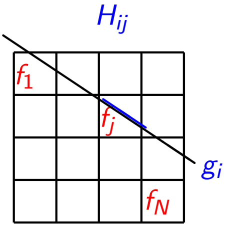
$$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon} \quad \text{with} \quad H_{ij} = \int h(\mathbf{s}_i, \mathbf{r})\, \phi_j(\mathbf{r})\, \mathrm{d}\mathbf{r}$$

- $\mathbf{H}$ is huge dimensional
- Regularization : $\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \{J(\mathbf{f})\}$ with

$$J(\mathbf{f}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda \|\mathbf{f}\|_1$$

Computed Tomography: Discretization





$$f(x,y) = \sum_j f_j \, b_j(x,y)$$

$$b_j(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \text{ pixel } j \\ 0 & \text{else} \end{cases}$$

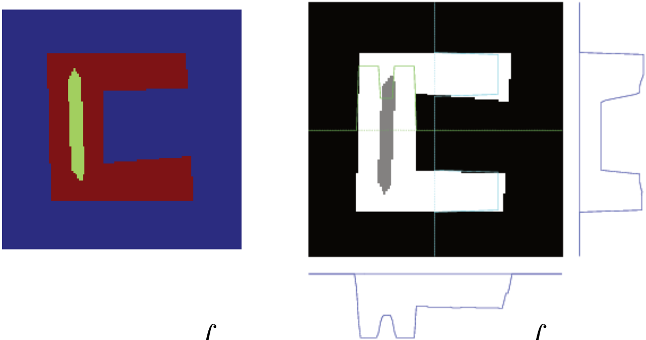
$$g(r,\phi) = \int_L f(x,y) \, \mathrm{d}l$$

$$g_i = \sum_{j=1}^{N} H_{ij} \, f_j + \epsilon_i$$

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \epsilon$$

Application of CT in NDT

Reconstruction from only 2 projections



$$g_1(x) = \int f(x,y)\, dy, \qquad g_2(y) = \int f(x,y)\, dx$$

- Given the marginals $g_1(x)$ and $g_2(y)$ find the joint distribution $f(x,y)$.
- Infinite number of solutions : $f(x,y) = g_1(x)\, g_2(y)\, \Omega(x,y)$
  $\Omega(x,y)$ is a Copula:

$$\int \Omega(x,y)\, dx = 1 \quad \text{and} \quad \int \Omega(x,y)\, dy = 1$$

Application in CT

$\mathbf{g}\vert\mathbf{f}$	$\mathbf{f}\vert\mathbf{z}$	$\mathbf{z}$	$\mathbf{c}$
$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon}$	iid Gaussian	iid	$q(\mathbf{r}) \in \{0,1\}$
$\mathbf{g}\vert\mathbf{f} \sim \mathcal{N}(\mathbf{H}\mathbf{f}, \sigma_\epsilon^2 \mathbf{I})$	or	or	$1 - \delta(z(\mathbf{r}) - z(\mathbf{r}'))$
Gaussian	Gauss-Markov	Potts	binary

$$p(\mathbf{f}, \mathbf{z}, \boldsymbol{\theta}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f}, \boldsymbol{\theta}_1)\, p(\mathbf{f}|\mathbf{z}, \boldsymbol{\theta}_2)\, p(\mathbf{z}|\boldsymbol{\theta}_3)\, p(\boldsymbol{\theta})$$

Proposed algorithms

$$p(\mathbf{f}, \mathbf{z}, \boldsymbol{\theta}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f}, \boldsymbol{\theta}_1)\, p(\mathbf{f}|\mathbf{z}, \boldsymbol{\theta}_2)\, p(\mathbf{z}|\boldsymbol{\theta}_3)\, p(\boldsymbol{\theta})$$

- MCMC based general scheme:

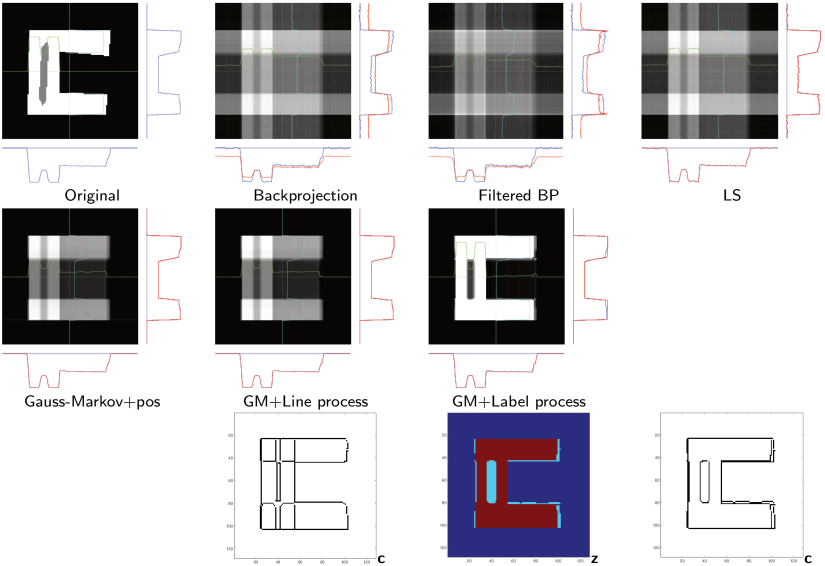
$$\widehat{\mathbf{f}} \sim p(\mathbf{f}|\widehat{\mathbf{z}}, \widehat{\boldsymbol{\theta}}, \mathbf{g}) \longrightarrow \widehat{\mathbf{z}} \sim p(\mathbf{z}|\widehat{\mathbf{f}}, \widehat{\boldsymbol{\theta}}, \mathbf{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta}|\widehat{\mathbf{f}}, \widehat{\mathbf{z}}, \mathbf{g})$$

Iterative algorithme:

- Estimate $\mathbf{f}$ using $p(\mathbf{f}|\widehat{\mathbf{z}}, \widehat{\boldsymbol{\theta}}, \mathbf{g}) \propto p(\mathbf{g}|\mathbf{f}, \boldsymbol{\theta})\, p(\mathbf{f}|\widehat{\mathbf{z}}, \widehat{\boldsymbol{\theta}})$
  Needs optimisation of a quadratic criterion.
- Estimate $\mathbf{z}$ using $p(\mathbf{z}|\widehat{\mathbf{f}}, \widehat{\boldsymbol{\theta}}, \mathbf{g}) \propto p(\mathbf{g}|\widehat{\mathbf{f}}, \widehat{\mathbf{z}}, \widehat{\boldsymbol{\theta}})\, p(\mathbf{z})$
  Needs sampling of a Potts Markov field.
- Estimate $\boldsymbol{\theta}$ using
  $p(\boldsymbol{\theta}|\widehat{\mathbf{f}}, \widehat{\mathbf{z}}, \mathbf{g}) \propto p(\mathbf{g}|\widehat{\mathbf{f}}, \sigma_\epsilon^2 \mathbf{I})\, p(\widehat{\mathbf{f}}|\widehat{\mathbf{z}}, (m_k, v_k))\, p(\boldsymbol{\theta})$
  Conjugate priors $\longrightarrow$ analytical expressions.

- Variational Bayesian Approximation
  - Approximate $p(\mathbf{f}, \mathbf{z}, \boldsymbol{\theta}|\mathbf{g})$ by $q_1(\mathbf{f})\, q_2(\mathbf{z})\, q_3(\boldsymbol{\theta})$

Results

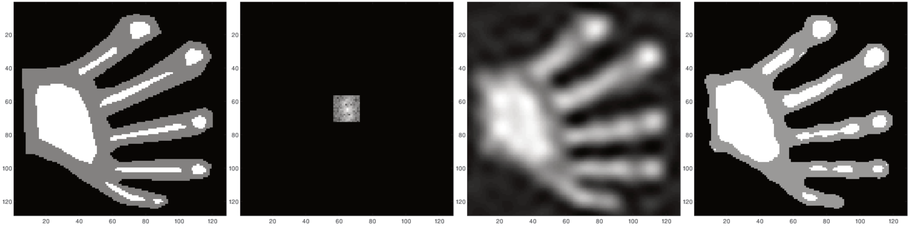

Original
Backprojection
Filtered BP
LS
Gauss-Markov+pos
GM+Line process
GM+Label process
c
z
c

Application in Microwave imaging

$$g(\boldsymbol{\omega}) = \int f(\mathbf{r}) \exp\left[-j(\boldsymbol{\omega}.\mathbf{r})\right] \,\mathrm{d}\mathbf{r} + \epsilon(\boldsymbol{\omega})$$

$$g(u,v) = \iint f(x,y) \exp\left[-j(ux+vy)\right] \,\mathrm{d}x\,\mathrm{d}y + \epsilon(u,v)$$

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon}$$



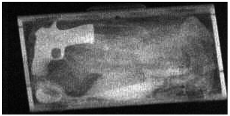
$f(x,y)$ $\quad$ $g(u,v)$ $\quad$ $\widehat{\mathbf{f}}$ IFT $\quad$ $\widehat{\mathbf{f}}$ Proposed method

Images fusion and joint segmentation

(with O. Féron)

$$\begin{cases} g_i(\mathbf{r}) = f_i(\mathbf{r}) + \epsilon_i(\mathbf{r}) \\ p(f_i(\mathbf{r})|z(\mathbf{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma^2_{i\,k}) \\ p(\underline{\mathbf{f}}|\mathbf{z}) = \prod_i p(\mathbf{f}_i|\mathbf{z}) \end{cases}$$

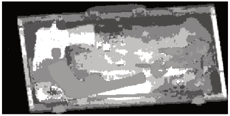
$\mathbf{g}_1$



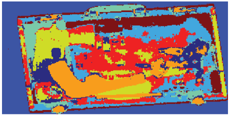
$\mathbf{g}_2$

$\longrightarrow$

$\widehat{\mathbf{f}}_1$



$\widehat{\mathbf{f}}_2$

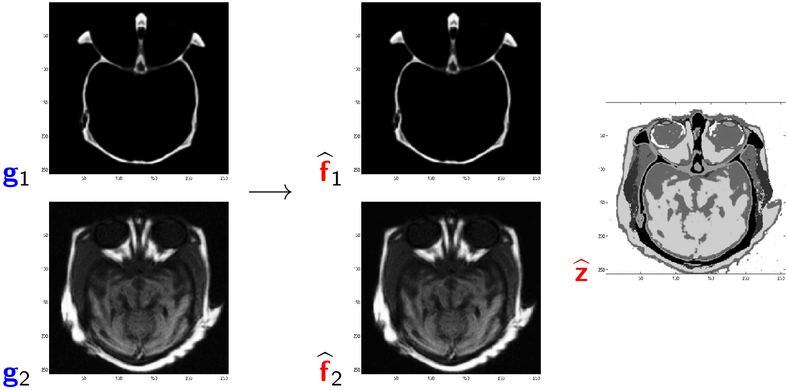


$\widehat{\mathbf{z}}$

Data fusion in medical imaging

(with O. Féron)

$$\begin{cases} g_i(\mathbf{r}) = f_i(\mathbf{r}) + \epsilon_i(\mathbf{r}) \\ p(f_i(\mathbf{r})|z(\mathbf{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma^2_{i\,k}) \\ p(\underline{\mathbf{f}}|\mathbf{z}) = \prod_i p(\mathbf{f}_i|\mathbf{z}) \end{cases}$$



Conclusions

- Sparsity: a great property to use in signal and image processing
- Origine: Sampling theory and reconstruction, modeling and representation Compressed Sensing, Approximation theory
- Deterministic Algorithms: Optimization of a two termes criterion, penalty term, regularization term
- Probabilistic: Bayesian approach
- Sprasity enforcing priors: Simple heavy tailed and Hierarchical with hidden variables.
- Gauss-Markov-Potts models for images incorporating hidden regions and contours
- Main Bayesian computation tools: JMAP, MCMC and VBA
- Application in different imaging system (X ray CT, Microwaves, PET, ultrasound and microwave imaging)

Current Projects:

- Efficient implementation in 2D and 3D cases
- Comparison between MCMC and VBA methods

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- N. Chu (2013: Acoustic sources localization)
- Th. Boulay (2013: Non Cooperative Radar Target Recognition)
- R. Prenon (2013: Proteomic and Masse Spectrometry)
- Sh. Zhu (2012: SAR Imaging)
- D. Fall (2012: Emission Positon Tomography, Non Parametric Bayesian)
- D. Pougaza (2011: Copula and Tomography)
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- N. Bali (2007: Hyperspectral imaging)
- O. Féron (2006: Microwave imaging)
- F. Humblot (2005: Super-resolution)
- M. Ichir (2005: Image separation in Wavelet domain)
- P. Brault (2005: Video segmentation using Wavelet domain)

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Older Graduated PhD students:

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- M. Nikolova (1994: Piecewise Gaussian models and GNC)
- D. Prémel (1992: Eddy current imaging)

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- Th. Rodet (L2S) (Computed Tomography)

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- D. Blacodon (ONERA) (Acoustic sources separation)
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- F. Lévi (CNRS-INSERM, Hopital Paul Brousse) (Biological rythms and Chronotherapy of Cancer)

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- M. Aminghafari (Amir Kabir University) (Independent Components Analysis)
- A. Mohammadpour (AKU) (Statistical inference)
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Questions, Discussions, Open mathematical problems

- Sparsity representation, low rank matrix decomposition
  - Sparsity and positivity or other constraints
  - Group sparsity
  - Algorithmic and implementation issues for great dimensional applications (Big Data)
  - Joint estimation of Dictionary and coefficients
- Optimization of the KL divergence for Variational Bayesian Approximation
  - Convergency of alternate optimization
  - Other possible algorithms
- Properties of the obtained approximation
  - Does the moments of q's corresponds to the moments of p?
  - How about any other statistics: entropy, ...
- Other divergency or Distance measures?
- Using Sparsity as a prior in Inverse Problems
- Applications in Medical imaging, Non Destructive Testing (NDT) Industrial Imaging, Communication, Geophysical imaging, Radio Astronomy, ...