Advanced Signal and Image processing

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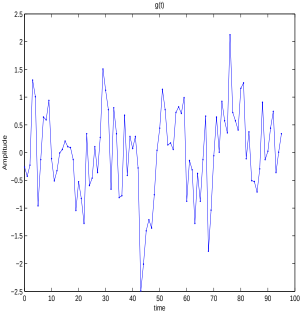
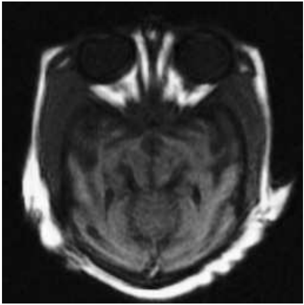
Representation of signals and images

- Signal: $f(t), f(x), f(\nu)$
  - $f(t)$ Variation of temperature in a given position as a function of time t
  - $f(x)$ Variation of temperature as a function of the position x on a line
  - $f(\nu)$ Variation of temperature as a function of the frequency ν
- Image: $f(x,y), f(x,t), f(\nu,t), f(\nu_1,\nu_2)$
  - $f(x,y)$ Distribution of temperature as a function of the position (x,y)
  - $f(x,t)$ Variation of temperature as a function of x and t
  - ...
- 3D, 3D+t, 3D+ν, ... signals: $f(x,y,z), f(x,y,t), f(x,y,z,t)$
  - $f(x,y,z)$ Distribution of temperature as a function of the position (x,y,z)
  - $f(x,y,z,t)$ Variation of temperature as a function of (x,y,z) and t
  - ...

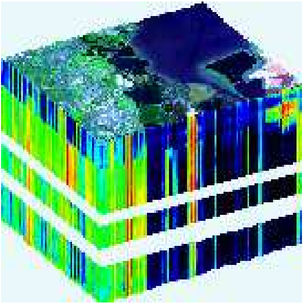
Representation of signals



1D signal



2D signal=image



3D signal

Linear Transformations

$$g(\mathbf{s}) = \int_D f(\mathbf{r})\, h(\mathbf{r},\mathbf{s})\, \mathrm{d}\mathbf{r}$$

$$f(\mathbf{r}) \longrightarrow \boxed{h(\mathbf{r},\mathbf{s})} \longrightarrow g(\mathbf{s})$$

- 1–D :

$$g(t) = \int_D f(t')\, h(t,t')\, \mathrm{d}t'$$

$$g(x) = \int_D f(x')\, h(x,x')\, \mathrm{d}x'$$

- 2–D :

$$g(x,y) = \iint_D f(x',y')\, h(x,y;x',y')\, \mathrm{d}x'\, \mathrm{d}y'$$

$$g(r,\phi) = \iint_D f(x,y)\, h(x,y;r\phi)\, \mathrm{d}x\, \mathrm{d}y$$

Linear and Invariant systems: convolution

$$h(\mathbf{r},\mathbf{r}') = h(\mathbf{r}-\mathbf{r}')$$

$$f(\mathbf{r}) \longrightarrow \boxed{h(\mathbf{r})} \longrightarrow g(\mathbf{r}) = h(\mathbf{r}) * f(\mathbf{r})$$

- 1–D :

$$g(t) = \int_D f(t')\, h(t-t')\, \mathrm{d}t'$$

$$g(x) = \int_D f(x')\, h(x-x')\, \mathrm{d}x'$$

- 2–D :

$$g(x,y) = \iint_D f(x,y)\, h(x-x',y-y')\, \mathrm{d}x'\, \mathrm{d}y'$$

- $h(t)$ impulse response
- $h(x,y)$ Point Spread Function

Linear Transformations: Separable systems

$$g(\mathbf{s}) = \int_D f(\mathbf{r})\, h(\mathbf{r},\mathbf{s})\, \mathrm{d}\mathbf{r}$$

$$h(\mathbf{r},\mathbf{s}) = \prod_j h_j(r_j, s_j)$$

Examples:

- 2D Fourier Transform

$$g(\omega_x,\omega_y) = \iint f(x,y)\, \exp\left\{-j(\omega_x x + \omega_y y)\right\}\, \mathrm{d}x\, \mathrm{d}y$$

$$h(x,y,\omega_x,\omega_y) = h_1(\omega_x x)\, h_2(\omega_y y)$$

$$\exp\left\{-j(\omega_x x + \omega_y y)\right\} = \exp\left\{-j(\omega_x x)\right\}\, \exp\left\{-j(\omega_y y)\right\}$$

- *n*D Fourier Transform

$$g(\boldsymbol{\omega}) = \int f(\mathbf{x})\, \exp\left\{-j\boldsymbol{\omega}'\mathbf{x}\right\}\, \mathrm{d}\mathbf{x}$$

Fourier Transform

[Joseph Fourier, French Mathematicien (1768-1830)]

- 1D Fourier: $\mathcal{F}_1$

$$\left\{\begin{array}{ll} g(\omega) & = \displaystyle\int f(t) \exp\{-j\omega t\}\; \mathrm{d}t \\ f(t) & = \frac{1}{2\pi}\displaystyle\int g(\omega) \exp\{+j\omega t\}\; \mathrm{d}\omega \end{array}\right.$$

- 2D Fourier: $\mathcal{F}_2$

$$\left\{\begin{array}{ll} g(\omega_x,\omega_y) & = \displaystyle\iint f(x,y) \exp\{-j(\omega_x x + \omega_y y\}\; \mathrm{d}x\, \mathrm{d}y \\ f(x,y) & = (\frac{1}{2\pi})^2 \displaystyle\iint g(\omega_x,\omega_y) \exp\{+j(\omega_x x + \omega_y y\}\; \mathrm{d}\omega_x\, \mathrm{d}\omega_y \end{array}\right.$$

- nD Fourier: $\mathcal{F}_n$

$$\left\{\begin{array}{ll} g(\boldsymbol{\omega}) & = \displaystyle\int f(\mathbf{x}) \exp\{-j\boldsymbol{\omega}'\mathbf{x}\}\; \mathrm{d}\mathbf{x} \\ f(\mathbf{x}) & = (\frac{1}{2\pi})^n \displaystyle\int g(\boldsymbol{\omega}) \exp\{+j\boldsymbol{\omega}'\mathbf{x}\}\; \mathrm{d}\boldsymbol{\omega} \end{array}\right.$$

1D Fourier Transform $\mathcal{F}_1$

$$\left\{\begin{array}{ll} g(\omega) & = \int f(t) \exp\{-j\omega t\} \, \mathrm{d}t \\ f(t) & = \frac{1}{2\pi} \int g(\omega) \exp\{+j\omega t\} \, \mathrm{d}\omega \end{array}\right.$$

- $|g(\omega)|^2$ is called the spectrum of the signal $f(t)$
- For real valued signals $f(t)$, $|g(\omega)|$ is symetric

Examples:

$f(t)$	$g(\omega)$
$\exp\{-j\omega_0 t\}$	$\delta(\omega-\omega_0)$
$\sin(\omega_0 t)$	?
$\cos(\omega_0 t)$	?
$\exp\{-t^2\}$	?
$\exp\{-\frac{1}{2}(t-m)^2/\sigma^2\}$	?
$\exp\{-t/\tau\}, t>0$	?
$1 \quad \text{if } \|t\| < T/2$	?

2D Fourier Transform: $\mathcal{F}_2$

$$\left\{\begin{array}{ll} g(\omega_x,\omega_y) & = \iint f(x,y) \exp\left\{-j(\omega_x x + \omega_y y\right\} \, \mathrm{d}x \, \mathrm{d}y \\ f(x,y) & = (\frac{1}{2\pi})^2 \iint g(\omega_x,\omega_y) \exp\left\{+j(\omega_x x + \omega_y y\right\} \, \mathrm{d}\omega_x \, \mathrm{d}\omega_y \end{array}\right.$$

- $|g(\omega_x,\omega_y)|^2$ is called the spectrum of the image $f(x,y)$
- For real valued image $f(x,y)$, $|g(\omega_x,\omega_y)|$ is symetric with respect of the two axis ω_x and ω_y.

Examples:

$f(x,y)$	$g(\omega_x,\omega_y)$
$\exp\left\{-j(\omega_{x0}x + \omega_{y0}y)\right\}$	$\delta(\omega_x - \omega_{x0})\delta(\omega_y - \omega_{y0})$
$\exp\left\{-(x^2+y^2)\right\}$	?
$\exp\left\{-\frac{1}{2}[(x-m_x)^2/\sigma_x^2 + (y-m_y)^2/\sigma_y^2]\right\}$	?
$\exp\left\{-(\|x\|+\|y\|)\right\}$	?
1 if $\|x\| < T_x/2$ & $\|y\| < T_y/2$	?
1 if $(x^2+y^2) < a$	?

nD Fourier Transform: $\mathcal{F}_n$

$$\left\{\begin{array}{ll} g(\boldsymbol{\omega}) & = \int f(\mathbf{x}) \exp\left\{-j\boldsymbol{\omega}'\mathbf{x}\right\}\, \mathrm{d}\mathbf{x} \\ f(\mathbf{x}) & = (\frac{1}{2\pi})^n \int g(\boldsymbol{\omega}) \exp\left\{+j\boldsymbol{\omega}'\mathbf{x}\right\}\, \mathrm{d}\boldsymbol{\omega} \end{array}\right.$$

- $|g(\boldsymbol{\omega})|^2$ is called the spectrum of $f(\mathbf{x})$
- For real valued image $f(\mathbf{x})$, $|g(\boldsymbol{\omega})|$ is symetric with respect of all the axis ω_j.

Examples:

$f(\mathbf{x})$	$g(\boldsymbol{\omega})$
$\exp\left\{-j(\boldsymbol{\omega}_0'\mathbf{x})\right\}$	$\delta(\boldsymbol{\omega}-\boldsymbol{\omega}_0)$
$\exp\left\{-\mathbf{x}'\mathbf{x}\right\} = \exp\left\{-\|\mathbf{x}\|^2\right\}$	$\exp\left\{-\boldsymbol{\omega}'\boldsymbol{\omega}\right\} = \exp\left\{-\|\boldsymbol{\omega}\|^2\right\}$
$\exp\left\{-\|\mathbf{D}\mathbf{x}\|^2\right\}$	?
$\exp\left\{-\frac{1}{2}[(\mathbf{x}-\mathbf{m})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\mathbf{m})\right\}$	?
$1 \ \text{ if } \|x\|^2 < R$	?
$1 \ \text{ if } \|x_j\| < R$	?

1D Fourier Transform: Classical definition and properties

Classical Definition for Integrable 1–D signals

$$f(t) \in L_1, \quad \int_{-\infty}^{+\infty} |f(t)| \,\mathrm{d}t < \infty$$

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) \exp\{-j\omega t\} \,\mathrm{d}t$$

Some important properties :

- FT is additive: $f + g \leftrightarrow F(\omega) + G(\omega)$
- FT is homogenious: $cf(t) \leftrightarrow cF(\omega)$
- The spectral amplitude is bounded:

$$|F(\omega)| \leq \int_{-\infty}^{+\infty} |f(t)| \,\mathrm{d}t$$

- $F(\omega)$ is uniformly continuous on $[-\infty, \infty]$
- Rieman–Lebesgue lemma:

$$\lim_{|\omega| \to \infty} |F(\omega)| = 0$$

Convolution

If $f(t), g(t) \in L_1$

$$h(t) = \int_{-\infty}^{+\infty} f(t-\tau)\, g(\tau)\, \mathrm{d}\tau = f * g$$

- The convolution integral $[f * g](t)$ exists almost everywhere for $t \in [-\infty, \infty]$ and, furthermore, $f * g \in L_1$.
- $$\int_{-\infty}^{+\infty} |[f * g](t)|\ \mathrm{d}t \leq \int_{-\infty}^{+\infty} |f(t)|\ \mathrm{d}t \int_{-\infty}^{+\infty} |g(t)|\ \mathrm{d}t$$
- $f * g = g * f$ (commutativity)
- $(f * g) * r = f * (g * r)$ (associativity)
- Convolution and FT: $f(t) * g(t) \leftrightarrow F(\omega) G(\omega)$

FT and Convolution properties

- if $f \in L_1 \cap L_2$, then $F(\omega) \in L_2$ and

$$\int_{-\infty}^{+\infty} |f(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\omega)|^2 \, \mathrm{d}\omega < \infty$$

Lets note by $I_{[-n,n]}(t) = 1, t \in [-n, n]$ and 0 elsewhere

Planchrel Theorem:
Let $f \in L_2$, $f_n(t) = f(t) I_{[-n,n]}(t)$ and $F_n(\omega)$ the FT of $f_n(t)$. Then
:

- There exists a function $F(\omega) \in L_2$ such that $F_n(\omega)$ converges to $F(\omega)$ in the L_2 norm, i.e.,

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} |F(\omega) - F_n(\omega)|^2 \, \mathrm{d}\omega = 0$$

FT and Convolution properties

- For any $F(\omega) \in L_2$, there exists a unique $f(t) \in L_2$ such that $f(t) \leftrightarrow F(\omega)$, where the double arrow denotes the L_2 Fourier transform.
- If $f \in L_1 \cap L_2$, then

$$F(\omega) = \lim_{n \to \infty} F_n(\omega)$$

  pointwise, and $F(\omega)$ agrees with definition of the FT for L_1 functions.
- For any $f \in L_2$, the Parseval identity holds, i.e.,

$$\int_{-\infty}^{+\infty} |f(t)|^2 \,\mathrm{d}t \frac{1}{2\pi} = \int_{-\infty}^{+\infty} |F(\omega)|^2 \,\mathrm{d}\omega < \infty.$$

FT and Convolution properties

- If $F(\omega) \in L_2$ and we note by

$$F_{[\Omega]}(\omega) = F(\omega) I_{[-\Omega,\Omega]}(\omega),$$

then $F_{[\Omega]}(\omega) \in L_1 \cap L_2$ and we can define

$$f_{[\Omega]}(t) = \mathcal{F}^{-1} F_{[\Omega]}(\omega),$$

and show that $f_{[\Omega]}(t)$ converges to $f(t)$ in the L_2 norm, i.e.,

$$\lim_{\Omega\to\infty} \int_{-\infty}^{+\infty} \left|f(t) - f_{[\Omega]}(t)\right|^2 \, \mathrm{d}t = 0$$

Note the following

$$F_n(\omega) = \int_{-n}^{n} f(t) \exp\left\{-j\omega t\right\} \, \mathrm{d}t$$

$$f_{[\Omega]}(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F(\omega) \exp\left\{j\omega t\right\} \, \mathrm{d}\omega$$

FT and Convolution properties

If $f, g \in L_2$

$$h(t) = \int_{-\infty}^{+\infty} f(t-x) g(x) \, \mathrm{d}x = f * g$$

- $h(t)$ is bounded, continuous, and converges to zero as t goes to infinity.

Convolution for 2D signals

$$F(\omega_x,\omega_y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\exp\left\{-j\left(\omega_x x+\omega_y y\right)\right\}\,\mathrm{d}x\,\mathrm{d}y$$

- Rieman–Lebesgue lemma:

$$\lim_{|\omega_x^2+\omega_y^2|\to\infty}|F(\omega_x,\omega_y)|=0$$

$$f(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\omega_x,\omega_y)\exp\left\{-j\left(\omega_x x+\omega_y y\right)\right\}\,\mathrm{d}\omega_x\,\mathrm{d}\omega_y$$

If $f,g\in L_1$

$$h(x,y)=\int_{-\infty}^{+\infty} f(x-u,y-v)\,g(u,v)\,\mathrm{d}u\,\mathrm{d}v=f*g$$

Sine and Cosine Transforms

Sine Transform: For un odd function $f(-t) = -f(t)$

$$S(\omega) = 2\int_0^\infty f(t)\sin(\omega t)\;\mathrm{d}t$$

$$f(t) = \frac{1}{\pi}\int_0^\infty S(\omega)\sin(\omega t)\;\mathrm{d}\omega$$

$$S(\omega) = \frac{1}{j}[F(\omega) - F(-\omega)]$$

Cosine Transform: For un even function $f(-t) = f(t)$

$$C(\omega) = 2\int_0^\infty f(t)\cos(\omega t)\;\mathrm{d}t$$

$$f(t) = \frac{1}{\pi}\int_0^\infty C(\omega)\cos(\omega t)\;\mathrm{d}\omega$$

$$C(\omega) = F(\omega) + F(-\omega)$$

Laplace Transforms: $\mathcal{L}$

[Pierre-Simon Laplace, French Mathematicien (1749-1827)]
Let $f(t)$ be a signal with support in $[0,\infty)$ such that $\exp\{-kt\}\, f(t) \in L_1$ for some real number k.

$$F(s) = \int_0^\infty f(t)\, \exp\{-s\, t\}\, \mathrm{d}t$$

- $F(s)$ is defined at least in the right half of the complex plane defined by $\text{Real}\,(s) > k$.
- When the inversion conditions for the FT hold, we also have an inversion for the LT given by

  $$f(t) = \frac{1}{j2\pi} \int_{a-j\infty}^{a+j\infty} F(s) \exp\{+st\}\, \mathrm{d}s, \quad \forall t > 0$$

  where $a > k$ is a real number such that $\exp\{-kt\}\, f(t) \in L_1$
- Suppose $f(t)$ and $g(t)$ have support in $[0,\infty)$, $\exp\{-k_1 t\}\, f(t)$ and $\exp\{-k_2 t\}\, g(t)$ are in L_1, $f \leftrightarrow F$, and $g \leftrightarrow G$. Then

Laplace Transform: few examples

$f(t)$	$g(s)$
t	$1/s$
$\exp\{as\}$	$1/(t-a)$
$\sin\omega t$	$\omega/(t^2+\omega^2)$

Bilateral Laplace Transform

- Definition:

$$L(s) = \int_{-\infty}^{+\infty} f(t) \exp\left\{-s\,t\right\}\,\mathrm{d}t$$

- Whereas the LT integral converges in half–planes $\text{Real}\left(s\right) > k$, the bilateral LT integral converges in infinite strips $k_1 < \text{Real}\left(s\right) < k_2$

$$f(t) = \frac{1}{j2\pi} \int_{a-j\infty}^{a+j\infty} L(s) \exp\left\{+st\right\}\,\mathrm{d}s$$

  where a is a real number $k_1 > a > k_2$, and k_1, k_2 are such that $\exp\left\{-k_1 t\right\} f(t) \in L_1$ and $\exp\left\{-k_2 t\right\} f(t) \in L_1$.

Mellin Transform (MT)

[Hjalmar Melin, Finnish mathematicien (1854-1933)]

- Definition:
  If $t^{\text{Real}(s)-1} f(t) \in L_1$ on $(0,\infty)$,

$$g(s) = \int_0^{\infty} f(t)\, t^{s-1}\, \mathrm{d}t$$

$$f(t) = \frac{1}{j2\pi} \int_{a-j\infty}^{a+j\infty} g(s)\, t^{-s}\, \mathrm{d}s$$

  where $s_1 < a < s_2$.

- Convolution for MT involves a product–type kernel and is given by

$$h(t) = \int_0^{t} f(x)\, g(t/x)\, x^{-1}\, \mathrm{d}x, \quad t > 0$$

$$H(s) = F(s)\, G(s)$$

Abel Transform (AT)

[Niels Henrik Abel, Norwegian mathematician (1802-1829)]

- Definition: If $f(t) \in L_1$,

$$\begin{aligned} g(s) &= 2 \int_s^\infty \frac{f(r)\, r}{r^2 - s^2} \, \mathrm{d}r \\ f(r) &= \frac{-1}{\pi} \int_r^\infty \frac{\mathrm{d}g(s)/\,\mathrm{d}s}{s^2 - r^2} \, \mathrm{d}s \end{aligned}$$

- Generalizations

$$A^\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)\,(x-t)^{\alpha-1} \, \mathrm{d}t$$

- For $\alpha = n$, a non negative integer, we have

$$A^n(x) = \int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(x_n) \, \mathrm{d}x_n \cdots \, \mathrm{d}x_1$$

Abel Transform (AT)

- For $\mathrm{Real}\,(\alpha) > 0$ and $\mathrm{Real}\,(\beta) > 0$

$$A^{\alpha}\left(A^{\beta}(f)\right) = A^{\alpha+\beta}(f)$$

$$A^{1}(f) = \int_{0}^{x} f(t)\,\mathrm{d}t$$

- If f is continuous

$$A^{0}(f) = f$$

Hilbert Transform: $\mathcal{H}$

[David Hilbert, German mathematicien (1862-1943)]

- Definition: If $f \in L_2$ on $(-\infty, \infty)$,

$$g(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{t-x} \, \mathrm{d}t$$

$$f(t) = \frac{-1}{\pi} \int_{-\infty}^{+\infty} \frac{g(x)}{x-t} \, \mathrm{d}x$$

  The integrals are interpreted in the Cauchy principal value(CPV) sense at $t = x$.

- Alternate expression useful in signal processing:

$$g(t) = \frac{1}{\pi} \lim_{\epsilon \mapsto 0} \int_{\epsilon}^{\infty} \frac{f(t+\tau) - f(t-\tau)}{\tau} \, \mathrm{d}\tau$$

Hilbert Transform: $\mathcal{H}$

- If $f \in L_2$
  - $\mathcal{H}\left(\mathcal{H}(f)\right) = f$
  - f and $\mathcal{H}(f)$ are orthogonal, i.e.,

$$\lim_{r\to\infty} \int_{-r}^{r} [f\mathcal{H}(f)](u) \, \mathrm{d}u = 0$$

  - The Hilbert transform of a constant is zero.
  - Hilbert and Fourier Transforms

$$\mathcal{H}(f) = f * \frac{-1}{\pi t} \quad \longrightarrow \mathcal{F}\{\mathcal{H}(f)\} = j\mathrm{sgn}(\omega)F(\omega)$$

Radon Transform (RT): $\mathcal{R}$

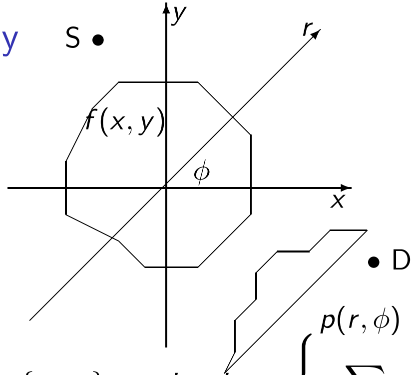
- Definition:
  This transform is defined for the functions in 2 or more dimensions. Here we give the relations only in the 2–D case.

$$
\begin{aligned}
R(r,\phi) &= \int_{L_{r,\phi}} f(x,y)\,\mathrm{d}l \\
&= \iint f(x,y)\,\delta(r - x\cos(\phi) - y\sin(\phi))\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

- The Radon transform maps the spatial domain $(x,y) \in R^2$ to the domain $(r,\phi) \in R \times [0,\pi]$. Each point in the (r,ϕ) space corresponds to a line in the spatial domain (x,y).
- Note that (r,ϕ) are not the polar coordinates of (x,y). In fact if we note the polar coordinates corresponding to the (x,y) point (ρ,θ), then we have

$$
x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad r = \rho\cos(\phi-\theta)
$$

X ray Tomography



$$I = I_0 \exp\{-\mu x\} \longrightarrow I = I_0 \exp\left\{-\sum_j \mu_j x_j\right\},$$

$$I = I_0 \exp\left\{-\int_L f(x)\,\mathrm{d}x\right\}. \longrightarrow -\ln\frac{I}{I_0} = \int_L f(x,y)\,\mathrm{d}l,$$

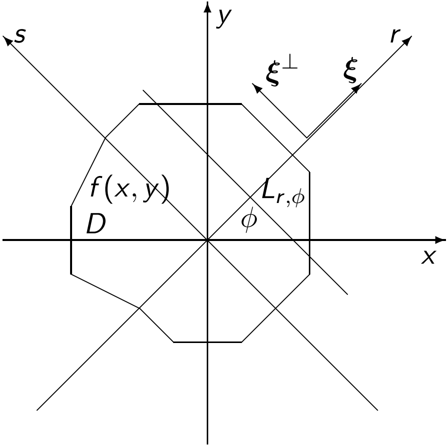
$$p(r,\phi) = \int_L f(x,y)\,\mathrm{d}l = \iint f(x,y)\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

$$f(x,y) \longrightarrow \boxed{\text{Radon Transform}} \longrightarrow p(r,\phi)$$

$$p(r,\phi) \longrightarrow \boxed{\text{Image Reconstruction}} \longrightarrow f(x,y)$$

X ray Tomography: Radon Transform

- **Deux dimensions :**



$$\boldsymbol{\xi} = [\cos\phi, \sin\phi]', \quad \boldsymbol{\xi}^{\perp} = [-\sin\phi, \cos\phi]'$$

$$\mathbf{x} = [x,y]' = [\rho\cos\theta, \rho\sin\theta], \qquad L_{r,\phi} : r = x\cos\phi + y\sin\phi = \boldsymbol{\xi}' \cdot \mathbf{x}$$

X ray Tomography: Radon Transform

$$\begin{aligned}
g(r,\phi) &= \int_L f(x,y)\,\mathrm{d}l = \iint_D f(x,y)\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y \\
g(r,\boldsymbol{\xi}) &= \iint f(x,y)\,\delta(r - \boldsymbol{\xi}'\cdot\mathbf{x})\,\mathrm{d}x\,\mathrm{d}y \\
g(r,\phi) &= \int_0^{\infty}\int_0^{2\pi} f(\rho,\theta)\delta\big(r - \rho\cos(\phi-\theta)\big)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta
\end{aligned}$$

n-D case:

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]', \quad \boldsymbol{\xi} = [\xi_1, \xi_2, \ldots, \xi_n]' \quad \text{avec} \quad |\boldsymbol{\xi}| = 1$$

$$\mathrm{d}\mathbf{x} = \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n, \quad r = \boldsymbol{\xi}'\cdot\mathbf{x} = \xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n x_n$$

$$\boxed{g(r,\boldsymbol{\xi}) = \int_{R^n} f(\mathbf{x})\,\delta(r - \boldsymbol{\xi}'\cdot\mathbf{x})\,\mathrm{d}\mathbf{x}}$$

Radon Transform: Some properties

[Johann K.A. Radon, Austrian mathematician (1887-1956)]

- Definition in cartezian coordinate system:

$$f(x,y) \xrightarrow{\mathcal{R}} g(r,\phi) = \iint f(x,y)\delta(r-x\cos(\phi)-y\sin(\phi))\,\mathrm{d}x\,\mathrm{d}y$$

- Definition in polar coordinate system:

$$f(\rho,\theta) \xrightarrow{\mathcal{R}} g(r,\phi) = \int_0^{\infty}\int_0^{2\pi} f(\rho,\theta)\delta(r-\rho\cos(\phi-\theta)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

- Inversion

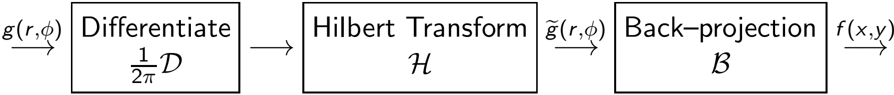
$$f(x,y) = \frac{1}{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\partial g(r,\phi)/\partial r}{r-x\cos(\phi)-y\sin(\phi)}\,\mathrm{d}r\,\mathrm{d}\phi$$

Radon Transform: Some properties

- Linearity:
  $a\,f_1(x,y) + b\,f_2(x,y) \longrightarrow a\,g_1(r,\phi) + b\,g_2(r,\phi)$
- Symetry:
  $f(x,y) \longrightarrow g(r,\phi) = g(-r,\phi \pm \pi)$
- Periodicity:
  $f(x,y) \longrightarrow g(r,\phi) = g(r,\phi \pm 2k\pi),\ k$ integer
- Shift:
  $f(x-x_0, y-y_0) \longrightarrow g(r - x_0\cos(\phi) - y_0\sin(\phi), \phi)$
- Rotation:
  $f(\rho,\theta+\theta_0) \longrightarrow g(r,\phi+\theta_0)$
- Scaling:
  $$f(a\,x, a\,y) \longrightarrow \frac{1}{|a|} g(ar,\phi), \quad a \neq 0$$
- Mass conservation:
  $$M = \iint_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x dy \longrightarrow M = \int_0^{\pi}\int_{-\infty}^{+\infty} g(r,\phi)\,\mathrm{d}r\,\mathrm{d}\phi$$

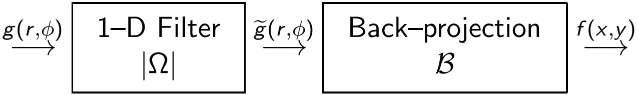
Radon Transform: Inversion

Direct Inverse Radon Transform



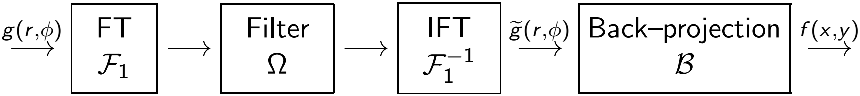
$\xrightarrow{g(r,\phi)}$ Differentiate $\frac{1}{2\pi}\mathcal{D}$ $\longrightarrow$ Hilbert Transform $\mathcal{H}$ $\xrightarrow{\widetilde{g}(r,\phi)}$ Back–projection $\mathcal{B}$ $\xrightarrow{f(x,y)}$

Convolution Back–projection method



$\xrightarrow{g(r,\phi)}$ 1–D Filter $|\Omega|$ $\xrightarrow{\widetilde{g}(r,\phi)}$ Back–projection $\mathcal{B}$ $\xrightarrow{f(x,y)}$

Filter Back–projection method



$\xrightarrow{g(r,\phi)}$ FT $\mathcal{F}_1$ $\longrightarrow$ Filter Ω $\longrightarrow$ IFT $\mathcal{F}_1^{-1}$ $\xrightarrow{\widetilde{g}(r,\phi)}$ Back–projection $\mathcal{B}$ $\xrightarrow{f(x,y)}$

Unitary Transforms: 1D case

$$\begin{aligned} F(u) &= \int f(x)\, h(u,x)\, \mathrm{d}x \\ f(x) &= \int F(u)\, h^*(x,u)\, \mathrm{d}u \end{aligned}$$

$$\begin{aligned} \langle f_1(x), f_2(x) \rangle &= \int f_1(x)\, f_2^*(x)\, \mathrm{d}x \\ \langle F_1(u), F_2(u) \rangle &= \int F_1(u)\, F_2^*(u)\, \mathrm{d}u \end{aligned}$$

$$\begin{aligned} F(u) &= \langle f(x), h_u^*(x) \rangle \\ f(x) &= \langle F(u), h_x(u) \rangle \end{aligned}$$

$$\left\{ \begin{aligned} F(u) &= \langle f(x), h_u^*(x) \rangle \\ f(x) &= \int F(u)\, h_u^*(x)\, \mathrm{d}u \end{aligned} \right.$$

Unitary Transforms: 2D case

$$
\begin{aligned}
F(u,v) &= \int f(x,y)\, h(u,v;x,y)\, \mathrm{d}x\, \mathrm{d}y \\
f(x,y) &= \int F(u,v)\, h^*(x,y;u,v)\, \mathrm{d}u\, \mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
\langle f_1(x,y), f_2(x,y)\rangle &= \iint f_1(x,y)\, f_2^*(x,y)\, \mathrm{d}x\, \mathrm{d}y \\
\langle F_1(u,v), F_2(u,v)\rangle &= \iint F_1(u,v)\, F_2^*(u,v)\, \mathrm{d}u\, \mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
F(u,v) &= \left\langle f(x,y), h_{u,v}^*(x,y)\right\rangle \\
f(x,y) &= \langle F(u,v), h_{x,y}(u,v)\rangle
\end{aligned}
$$

$$
\left\{
\begin{aligned}
F(u,v) &= \left\langle f(x,y), h_{u,v}^*(x,y)\right\rangle \\
f(x,y) &= \iint F(u,v)\, h_{u,v}(x,y)\, \mathrm{d}x\, \mathrm{d}y
\end{aligned}
\right.
$$

Discrete signals and images

Sampling theory

- Relation between a signal $f(t)$, its samples $f_n = f(n\Delta)$ and sampled signal $f_s(t) = \sum_n f_n \delta(t - n\Delta)$

$$f_s(t) = f(t) \sum_n \delta(t - n\Delta) \longrightarrow F_s(\omega) = F(\omega) * \sum_n \delta(\omega - n/\Delta)$$

- Sampling theorem:
  An Ω band limited signal can be exactly reconstructed from its samples if the sampling interval $\Delta \leq 1/(2\Omega)$.

Discrete systems

- 1–D :

$$g(k) = \sum_{m=0}^{N-1} h(k,m)\, f(m), \quad 0 \leq k \leq N-1$$

- 2–D :

$$g(k,l) = \sum\sum_{m,n=0}^{N-1} h(k,l;m,n)\, f(m,n), \quad 0 \leq k,l \leq N-1$$

◇ **Translation Invariance**

- 1–D :

$$g(k) = \sum_{m=0}^{N-1} h(k-m)\, f(m) = h*f \longrightarrow f(m) \longrightarrow \boxed{h(m)} \longrightarrow g(m)$$

- 2–D :

$$g(k,l) = \sum\sum_{m,n=0}^{N-1} h(k-m,l-n)\, f(m,n) = h*f$$

$$\longrightarrow f(m,n) \longrightarrow \boxed{h(m,n)} \longrightarrow g(m,n)$$

Discrete systems: Z transform

- 1–D :

$$U(z) = \sum_{m=-\infty}^{\infty} u(m)\, z^{-m}$$

$$u(m) = \frac{1}{j2\pi} \oint U(z)\, z^{m-1} \,\mathrm{d}z$$

- 2–D :

$$U(z_1,z_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} u(m,n)\, z_1^{-m}\, z_2^{-n}$$

$$u(m,n) = \frac{1}{(j2\pi)^2} \oint \oint U(z_1,z_2)\, z_1^{m-1}\, z_2^{n-1} \,\mathrm{d}z_1 \,\mathrm{d}z_2$$

$$g = h * f \quad \xrightarrow{\mathrm{TZ}} \quad G = H.F$$

h Impulse response
H Transfert function

Unitary Transforms: Vector-Matrix representation

$$\mathbf{v} = \mathbf{A}\mathbf{u} \longrightarrow v(k) = \sum_{m=0}^{N-1} a_k(m)\, u(m), \quad 0 \leq k \leq N-1$$

- Unitary transform: $\mathbf{A}^{-1} = \mathbf{A}^{*t}$

$$\mathbf{u} = \mathbf{A}^{*t}\mathbf{v} \longrightarrow u(m) = \sum_{k=0}^{N-1} v(k)\, a_m^*(k), \quad 0 \leq m \leq N-1$$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{N} x(k)\, y^*(k) \longrightarrow \left\{ \begin{array}{l} v(k) = \langle \mathbf{u}, \mathbf{a}_k^* \rangle \\ \mathbf{u} = \sum_{k=1}^{N} v(k)\, \mathbf{a}_k^* \end{array} \right.$$

- Basis vector:

$$\mathbf{a}_k^* = \{a_k^*(m)\} \quad \text{column of the matrix} \quad \mathbf{A}^{*t}$$

Unitary Transforms: Vector-Matrix representation

- Orthonormal Basis:

$$\langle \mathbf{a}_k, \mathbf{a}_{k'} \rangle = \sum_{m=0}^{N-1} a_k(m)\, a_{k'}^*(m) = \delta(k-k')$$

$$u_P(m) \overset{\triangle}{=} \sum_{k=0}^{P-1} v(k) a_k^*(m) \quad \text{minimise} \quad \sigma_\epsilon^2 = \sum_{m=0}^{N-1} |u(m) - u_P(m)|^2$$

- Complete Basis:

$$\sum_{k=0}^{N-1} a_k(m) a_k^*(m') = \delta(m-m')$$

$$\sigma_\epsilon^2 = \sum_{m=0}^{N-1} |u(m) - u_P(m)|^2 = 0 \quad \text{if} \quad P = N$$

Unitary Transforms in 2D

$$v(k,l)=\sum\sum_{m,n=0}^{N-1} a_{k,l}(m,n)\,u(m,n),\quad 0\leq k,l\leq N-1$$

- Unitary transforms:

$$u(m,n)=\sum\sum_{k,l=0}^{N-1} v(k,l)\,a^*_{m,n}(k,l),\quad 0\leq m,n\leq N-1$$

- Basis Images:

$$\mathbf{A}^*_{k,l}=\{a^*_{k,l}(m,n)\}$$

- Scalar product:

$$\langle\mathbf{F},\mathbf{G}\rangle=\sum\sum_{m,n=0}^{N-1} f(m,n)\,g^*(m,n)$$

$$v(k,l)=\left\langle\mathbf{U},\mathbf{A}^*_{k,l}\right\rangle,\quad \mathbf{U}=\sum\sum_{k,l=0}^{N-1} v(k,l)\mathbf{A}^*_{k,l}$$

Unitary Transforms in 2D

- Complete Basis:

$$\sum_{k,l=0}^{N-1} a_{k,l}(m,n)\, a_{k,l}^{*}(m',n') = \delta(m-m', n-n')$$

$$\sigma_{\epsilon}^{2} = \sum_{m,n=0}^{N-1} \left|u(m,m) - u_{P,Q}(m,n)\right|^{2} = 0 \quad \text{if} \quad P = Q = N$$

Separable Unitary transforms

$$a_{k,l}(m,n) = a_k(m)\,b_l(n) \overset{\triangle}{=} a(k,m)\,b(l,n)$$

$$\{a_k(m), k = 0, \ldots, N-1\}, \quad \{b_l(n), l = 0, \ldots, N-1\}$$

Two orthonormal and complete basis

$$\mathbf{A}\mathbf{A}^{*t} = \mathbf{A}^{*t}\mathbf{A} = \mathbf{I}, \qquad \mathbf{B}\mathbf{B}^{*t} = \mathbf{B}^{*t}\mathbf{B} = \mathbf{I},$$

$$v(k,l) = \sum\sum_{m,n=0}^{N-1} a(k,m)u(m,n)b(l,n) \Longrightarrow \mathbf{V} = \mathbf{A}\mathbf{U}\mathbf{B}^t$$

$$u(m,n) = \sum\sum_{k,l=0}^{N-1} a^*(k,m)v(k,l)b^*(l,n) \Longrightarrow \mathbf{U} = \mathbf{A}^{*t}\mathbf{V}\mathbf{B}^*$$

$\diamond$ In general, we choose $\mathbf{A} = \mathbf{B}$:

$$\mathbf{V} = \mathbf{A}\mathbf{U}\mathbf{A}^t = \left[\mathbf{A}[\mathbf{A}\mathbf{U}]^t\right]^t$$

$$\mathbf{U} = \mathbf{A}^{*t}\mathbf{V}\mathbf{A}^* = \left[\mathbf{A}[\mathbf{A}\mathbf{V}]^{*t}\right]^t$$

Unitary transforms: Computational cost

- General case:

$$\mathbf{U} = \sum_{k,l=0}^{N-1}\sum v(k,l)\mathbf{A}_{k,l}^{*} \quad \Longrightarrow N^2 \times N^2 = N^4$$

- Separable case:

$$\mathbf{U} = \mathbf{A}_1^{*t}\mathbf{V}\mathbf{A}_2^{*} \quad \Longrightarrow 2N^3$$

- If $\mathbf{A}_1$ and $\mathbf{A}_2$ are separables $\quad \Longrightarrow \log_2 N \times N^2$

Discrete Fourier Transform (DFT)

- Definition:

$$v(k) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} u(m)\, W_N^{km}, \quad k = 0, \cdots, N-1 \quad \longrightarrow \quad \mathbf{v} = \mathbf{F}\mathbf{u}$$

$$u(m) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} v(k)\, W_N^{-km}, \quad m = 0, \cdots, N{-}1 \quad \longrightarrow \quad \mathbf{u} = \mathbf{F}^{-1}\mathbf{v}$$

with $\quad W_N^{km} = \exp\left\{-j2\pi km/N\right\}$

- When N is a number power of 2, DFT becomes Fast Fourier Transform (FFT).
- In `Matlab` notation: `v=fft(u)` and `u=ifft(v)`

Link between DFT and FT

$$u_e(n) \overset{\triangle}{=} \begin{cases} u(n) & n = 0, \ldots, N-1 \\ 0 & ailleurs \end{cases}$$

$$U_e(\omega) = \sum_{n=-\infty}^{\infty} u_e(n) \exp\{-j\omega n\} = \sum_{n=0}^{N-1} u_e(n) \exp\{-j\omega n\}$$

DFT of $u(n)$:

$$v(k) = \sum_{n=0}^{N-1} u(n) \exp\{-j2\pi kn/N\}$$

$$\Downarrow$$

$$v(k) = U_e(2\pi k/N)$$

Discrete Fourier Transform properties

- Symetry :

$$\mathbf{F}^{-1} = \mathbf{F}^*$$

- Periodic Extension: $u(k)$ and $v(k)$ are periodic

$$u(k+N) = u(k), \quad v(k+N) = v(k)$$

- FFT when N power of 2 : $\Longrightarrow N \log_2 N$ operations
- Real valued signals:

$$v^*(N-k) = v(k) \longrightarrow |v(N/2-k)| = |v(N/2+k)|$$

- **F** is a circulant matrice.
- For any Circulent Matrice **H**:

$$\mathbf{F}\mathbf{H}\mathbf{F}^* = \mathbf{\Lambda} = \operatorname{diag}\left[\lambda_k, k=0,\ldots,N-1\right]$$

and

$$\boldsymbol{\lambda} = \mathbf{F}\mathbf{h} = \text{DFT of the first ligne of } \mathbf{H}$$

Discrete Fourier Transform and Circular Convolution

- Circulent Convolution:

$$y(n) = \sum_{k=0}^{N-1} h_c(n-k)x(k), \quad n = 0, \ldots, N-1$$

$$h_c(n) = h(n \,\mathrm{modulo}\, N)$$

$$\mathrm{DFT}\{y(n)\}_N = \mathrm{DFT}\{h(n)\}_N \, \mathrm{DFT}\{x(n)\}_N$$

Discrete Fourier Transform and Circular Convolution

- Linear Convolution:

$$\{h(n), n = 0, \ldots, N_h - 1\}, \quad \{x(n), n = 0, \ldots, N_x - 1\},$$

$$y(n) = h(n) * x(n)$$

$$\{y(n), n = 0, \ldots, N_h + N_x - 2\}$$

1. Zero padding: $M \geq N_h + N_x - 2$,

$$\widetilde{h}(n) = \begin{cases} h(n) & \text{for} \quad n = 0, \ldots, N_h - 1 \\ 0 & n = N_h, \ldots, M \end{cases}$$

$$\widetilde{x}(n) = \begin{cases} h(n) & \text{for} \quad n = 0, \ldots, N_h - 1 \\ 0 & n = N_h, \ldots, M \end{cases}$$

2. DFT computation :

$$\mathrm{DFT}\{\widetilde{y}(n)\}_M = \mathrm{DFT}\{\widetilde{h}(n)\}_M \, \mathrm{DFT}\{\widetilde{x}(n)\}_M$$

3. Inverse DFT computation:

$$y(n) = \widetilde{y}(n), \quad n = 0, \ldots, N_x + N_h - 2$$

Some References: Signal and image processing

1. V. Smidl, A. Quinn "The Variational Bayes Method in Signal Processing", Springer, ISBN-10 3-540-28819-8 Springer Berlin Heidelberg New York
2. A. K. Jain, "Fundamentals of Digital Image Processing," Prentice Hall, Englewood Cliffs, NJ 07632, (1989).
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2. A. Blake, A. Zisserman, "Visual Reconstruction,"
3. S. R. Deans "The Radon Transform and some of its Applications,"
4. A.C. Kak, M. Slaney, "Principles of Computerized Tomographic Imaging," IEEE Press, , 1988.
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Some references: Main Journals

- IEEE transactions on :
    - Signal Processing (SI)
    - Image Processing (SI)
    - Acoustics, Speech and Signal Processing (ASSP)
    - Medical Imaging (MI)
    - Pattern Analysis and Machine Intelligency (PAMI)
- Proceedings of IEEE
- Computer Vision, Graphics, and Image Processing

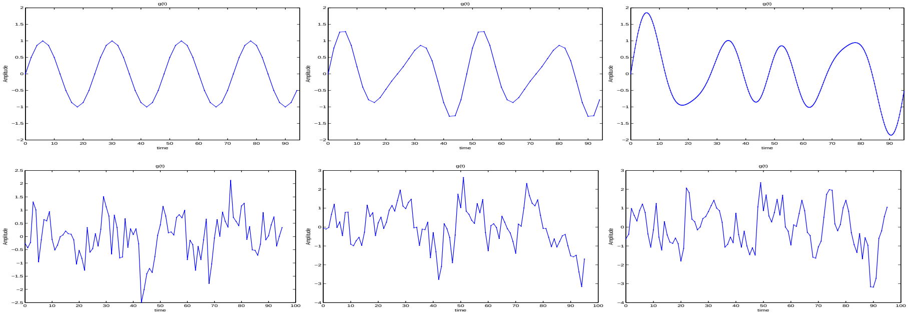
Content

1. Introduction: Signals and Images, Linear transformations (Convolution, Fourier, Laplace, Hilbert, Radon, ..., Discrete convolution, Z transform, DFT, FFT, ...)
2. **Modeling: parametric and non-parametric, MA, AR and ARMA models, Linear prediction**
3. Deconvolution and Parameter Estimation: Deterministic (LS, Regularization) and Probabilistic methods (Wiener Filtering)
4. Inverse problems, Regularization and Bayesian estimation
5. Kalman Filtering
6. Case study: Signal deconvolution
7. Case study: Image restoration
8. Case study: Image reconstruction and Computed Tomography

2. Modeling: parametric and non-parametric, MA, AR and ARMA models

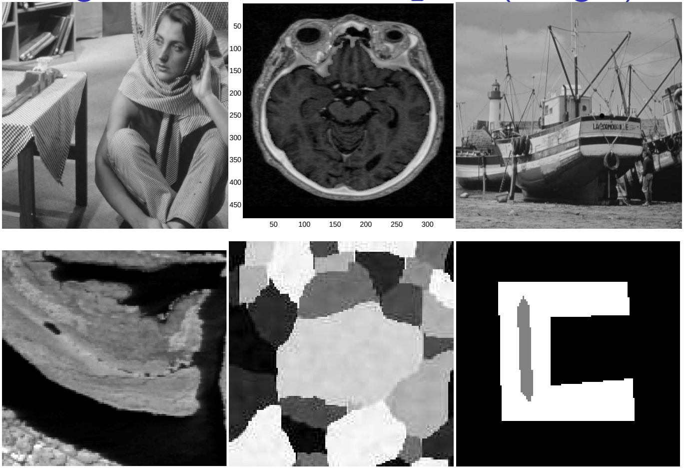
- Modeling ? for what ?
- Deterministic / Probalistic modeling
- Parametric / Non Parametric
- Moving Average (MA)
- Autoregressive (AR)
- Autoregressive Moving Average (ARMA)
- Classical methods for parameter estimation (LS, WLS)

Modeling ? What for ? 1D signals



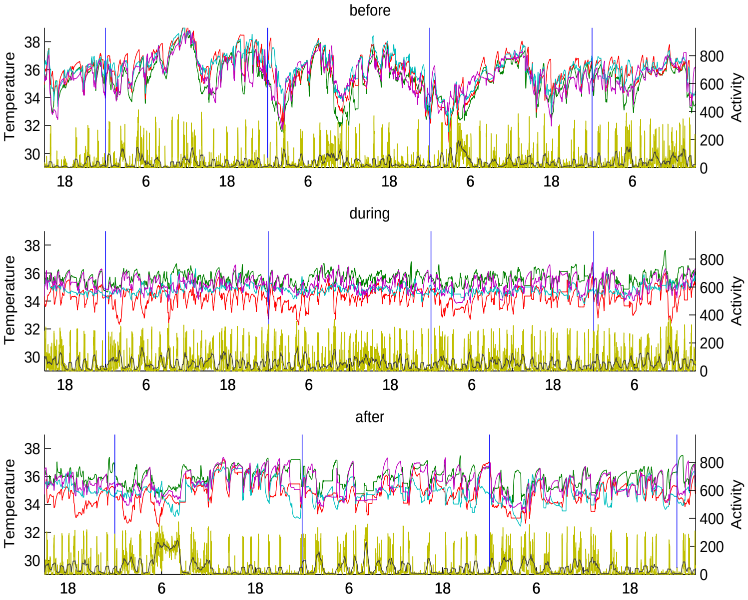
- 1D signals:
  - Is it periodic? What is the period?
  - Is there any structure?
  - Has something changed before, during and after some traitement
  - Can we compress it? How? How much?

Modeling ? What for ? 2D signals (Images)

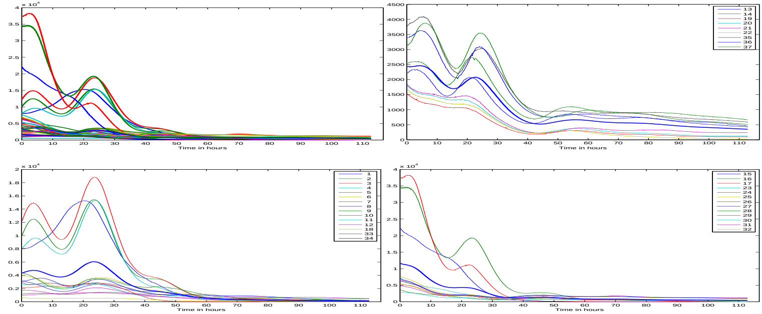


- Images:
  - Is there any structure?
  - Contours? Regions?
  - Can we compress it? How? How much?

Modeling ? What for ? multi dimensional time series



Modeling ? What for ? multi dimensional time series



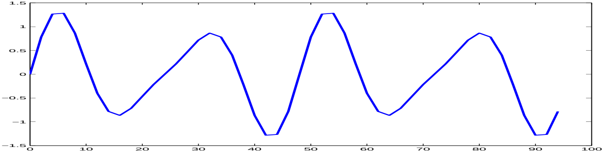
- Multi Dimentionsional signals $g_1(t), \cdots, g_n(t)$
  - Dependancy: Are they all independent?
    If not, which ones are related?
  - Dimensionality reduction: Can we reduce the dimensionality?
  - Principal Components Analysis (PCA):
    What are the principal components?
  - Independent Components Analysis (ICA):
    What are the independent components?
  - Factor Analysis (FA): What are the principal factors?

Deterministic / Probalistic modeling

- **Deterministic:**
  - The signal is a sinusoid $f(t) = a \sin(\omega t + \phi)$. We need just to determine the three parameters a, ω, ϕ.
  - The signal is a periodic $f(t) = \sum_{k=1}^{K} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$. If we know ω_0, then, we need just to determine the parameters $(a_k, b_k), k = 1, \cdots, K$.
  - The signal represents a spectra $f(t) = \sum_{k=1}^{K} a_k \mathcal{N}(m_k, v_k)$. We need just to determine the parameters $(a_k, m_k, v_k), k = 1, \cdots, K$.
  - In the last two cases, one great difficulty is determining K
- **Probabilistic:**
  - The shape of the signal is more sophisticated.
  - 1 Sinusoid + noise $f(t) = a \sin(\omega t + \phi) + \epsilon(t)$
  - K Sinusoids + noise $f(t) = \sum_{k=1}^{K} a_k \sin(\omega_k t + \phi_k) + \epsilon(t)$
  - No specific shapes: MA, AR, ARMA, ...

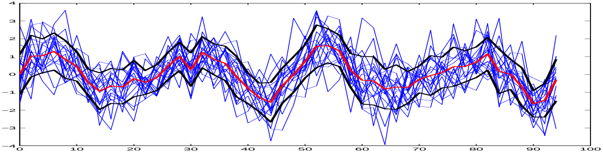
Deterministic and probabilistic modeling of signals

- Deterministe: $f(t)=sin(\omega t)$, the value of f at time t is always the same.



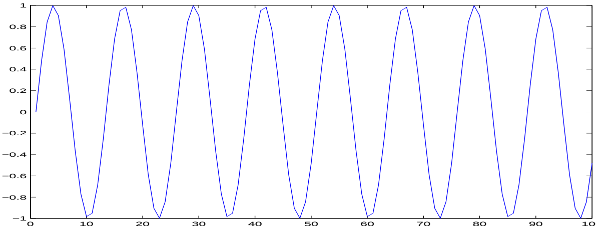
$$f_1(t)=\sin(2\omega t)+.4\sin(3\omega t),\ \omega=\pi/12$$

- Random $f(t)=sin(\omega t)+\epsilon(t)$, the value of f at time t is not always the same.
- For a random signal $f(t)$ for the value of f we can define a probability law, mean, variance, ...

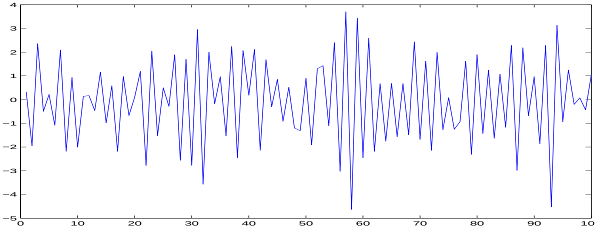


$$f2(t)=f_1(t)+\epsilon(t),\quad \epsilon(t)\sim\mathcal{N}(0,1).$$

Determinist/Probabilist

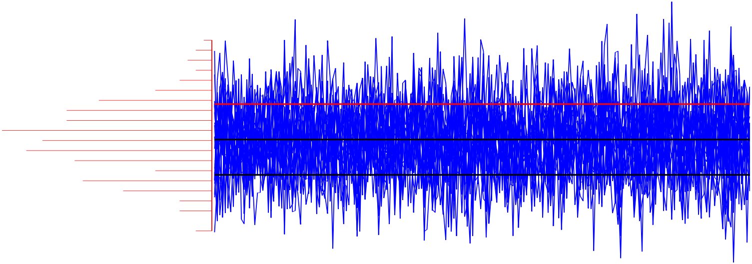


$$f(t) = a\,\sin(\omega t + \phi),\ a = 1, \omega = 2\pi, \phi = 0$$

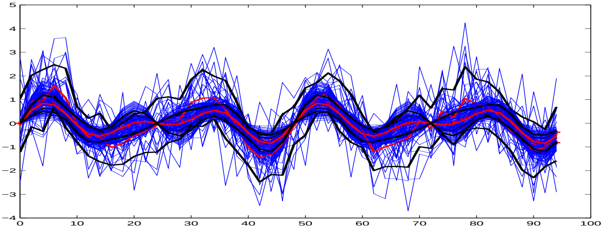


$$\text{ARMA(1,2)},\ q = 1,\ v = 1,\ b_1 = 1,\ a_1 = 1, a_2 = .8$$

Stationary/Non Stationary



$$f(t) \sim \mathcal{N}(0,1), \quad \forall t \longrightarrow \mathrm{E}\{f(t)\} = 0, \mathrm{Var}\{f(t)\} = 1, \forall t$$



$$f_3(t) = a_1 \sin(2\omega t) + a_2 \sim (3\omega t),\ \omega = \pi/12,\ a_1, a_2 \sim \mathcal{U}(0,1)$$

Parametric / Non Parametric

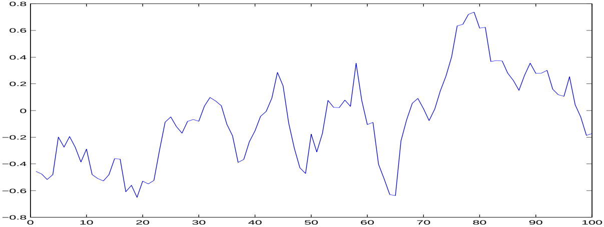
- **Parametric**
  - K Sinusoids + noise $f(t)=\sum_{k=1}^{K} a_k \sin(\omega_k t+\phi_k)+\epsilon(t)$. The parameters are $(a_k,\omega_k,\phi_k), k=1,\cdots,K$
  - K Complex exponentials + noise $f(t)=\sum_{k=1}^{K} c_k \exp\{-j\omega_k t\}+\epsilon(t)$. The parameters are $(c_k,\omega_k), k=1,\cdots,K$
  - Sum of K Gaussian shapes: $f(t)=\sum_{k=1}^{K} a_k\,\mathcal{N}(m_k,v_k)$. The parameters are $(a_k,m_k,v_k), k=1,\cdots,K$.
- **Non-Parametric**
  - The shape of the signal is more sophisticated.
  - The shape is composed of as much as the number of data of Complex exponentials + noise $f(t)=\sum_{n=1}^{N} c_n \exp\{-jn\omega_0 t\}+\epsilon(t)$. If we know ω_0, then, the parameters are $c_n, n=1,\cdots,N$
  - Sum of the Gaussian shapes: $f(t)=\sum_{k=1}^{K} a_n\,\mathcal{N}(m_n,v_n)$. The parameters are $(a_n,m_n,v_n), n=1,\cdots,N$.

Moving Average (MA)

$$f(t) = b(t) * \epsilon(t) = \int b(\tau)\epsilon(t-\tau)\,\mathrm{d}\tau$$

$$f(n) = \sum_{k=0}^{q} b(k)\epsilon(n-k), \quad \forall n$$

$$\epsilon(n) \longrightarrow \boxed{B(z) = \sum_{k=0}^{q} b(k) z^{-k}} \longrightarrow f(n)$$



MA, $q = 15$, $v = 1$, $b_k = \exp\left\{-.05 * k^2\right\}$

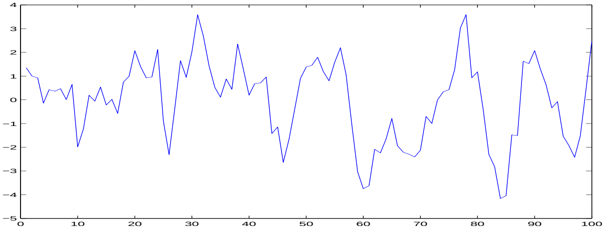
Autoregressive (AR)

$$f(t)=\sum_{k=1}^{p} a(k)\, f(t-k\Delta t)+\epsilon(t) \longrightarrow f(n)=\sum_{k=1}^{p} a(k)\, f(n-k)+\epsilon(n), \quad \forall n$$

$$\mathrm{E}\left\{\epsilon(n)\right\}=0, \quad \mathrm{E}\left\{|\epsilon(n)|^2\right\}=\beta^2,$$

$$\mathrm{E}\left\{\epsilon(n)\, f(m)\right\}=0, \quad m \neq n$$

$$\epsilon(n) \longrightarrow \boxed{H(z)=\frac{1}{A(z)}=\frac{1}{1+\sum_{k=1}^{p} a(k)\, z^{-k}}} \longrightarrow f(n)$$



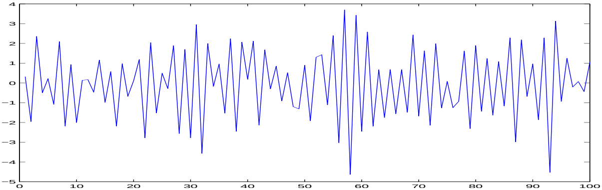
AR1, $a=.7$, $v=1$.

Autoregressive Moving Average (ARMA)

$$f(n) = \sum_{k=1}^{p} a(k)\, f(n-k) + \sum_{l=0}^{q} b(l)\, \epsilon(n-l)$$

$$\epsilon(n) \longrightarrow \boxed{H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b(k) z^{-k}}{1 + \sum_{k=1}^{p} a(k)\, z^{-k}}} \longrightarrow f(n)$$

$$\epsilon(n) \longrightarrow \boxed{B_q(z)} \longrightarrow \boxed{\frac{1}{A_p(z)}} \longrightarrow f(n)$$



ARMA(1,2), $q = 1,\ v = 1,\ b_1 = 1,\ a_1 = 1, a_2 = .8$

Linear prediction and AR modeling

- $\{f(1), \cdots, f(n-1)\}$ observed samples of a signal.
  Predict $f(n)$
- Prediction or innovation Erreur: $\epsilon(n) = f(n) - \widehat{f}(n)$
- Mean Square Errors (MSE):
  $\text{MSE} = \sum_n |\epsilon(n)|^2 = \sum_n |f(n) - \widehat{f}(n)|^2$
- Least Mean Squares (LMS) Error Linear Estimation

$$\widehat{f}(n) = \text{LMSELE}\{f(n)|f(1), \ldots, f(n-1)\} = \arg\min_{f(n)} \{\text{MSE}\}$$

- The linear predictor:

$$\widehat{f}(n) = \sum_{k=1}^{p} a(k)\, f(n-k), \quad \forall n$$

minimizes MSE.

Linear prediction and AR modeling

- $\{f(n)\}$ is a Markov chain of order p

$$\widehat{f}(n) = \mathrm{LMSELE}\{f(n)|f(n-k),\ldots,f(n-1)\} = \sum_{k=1}^{p} a(k)\, f(n-k)$$

- Wihtening Filter

$$f(n) \longrightarrow \boxed{A_p(z)} \longrightarrow \epsilon(n)$$

$$\mathrm{E}\left\{\epsilon(n)\epsilon(m)\right\} = \beta^2\, \delta(n-m)$$

- Link with AR model

$$f(n) = \sum_{k=0}^{N-1} a(k)\, f(n-k) + \epsilon(n), \quad \forall n$$

$$\epsilon(n) \longrightarrow \boxed{H(z) = \frac{1}{A(z)} = \frac{1}{1+\sum_{k=1}^{p} a(k)\, z^{-k}}} \longrightarrow f(n)$$

Minimum Variance Estimation

- $f(n)$, An n sample of a signal
- AR model:

$$\widehat{f}(n) = \sum_{k=0}^{N-1} a(k)\, f(n-k)$$

- Modeling errror

$$\epsilon(n) = f(n) - \widehat{f}(n)$$

- Criterium

$$\beta^2 = \min \mathrm{E}\left\{|\epsilon(n)|^2\right\} = \min \mathrm{E}\left\{[f(n) - \widehat{f}(n)]^2\right\}$$

- Orthogonality Condition

$$\mathrm{E}\left\{[f(n) - \sum_{k=0}^{N-1} a(k)\, f(n-k)]\, f(n-k)\right\} = \beta^2\, \delta(k),\ k = 1, \ldots, p$$

$$r(k) - \sum_{k=0}^{N-1} a(k)\, r(n-k) = \beta^2\, \delta(k)$$

Minimum Variance Estimation

- Correlation matrix

$$\mathbf{R} = \begin{bmatrix} r(0) & r(1) & r(2) & \cdots & \cdots & r(p-1) \\ r(1) & \ddots & \ddots & \ddots & & \vdots \\ r(2) & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & r(2) \\ \vdots & & \ddots & \ddots & \ddots & r(1) \\ r(p-1) & \cdots & \cdots & r(2) & r(1) & r(0) \end{bmatrix}$$

$$\mathbf{r} = [r(1), \ldots r(p)]^t, \quad \mathbf{a} = [a(1), \ldots, a(p)]^t,$$

- Normal equations

$$\begin{aligned} \mathbf{Ra} &= \mathbf{r} \\ r(0) - \mathbf{a}^t\mathbf{r} &= \beta^2 \end{aligned}$$

Causal ou Non-Causal AR models

- **Causal :**

$$f(n) = \sum_{k=1}^{q} a(k)\, f(n-k) + \epsilon(n), \quad \forall n$$

$$A(z) = 1 - \sum_{k=1}^{q} a(k)\, z^{-k} \longrightarrow \epsilon(n) \longrightarrow \boxed{H(z) = \frac{1}{A(z)}} \longrightarrow f(n)$$

- **Non–causal :**

$$f(n) = \sum_{\substack{k=-p \\ k\neq 0}}^{+q} a(k)\, f(n-k) + \epsilon(n), \quad \forall n$$

$$A(z) = 1 - \sum_{\substack{k=-p \\ k\neq 0}}^{+q} a(k)\, z^{-k} \longrightarrow \epsilon(n) \longrightarrow \boxed{H(z) = \frac{1}{A(z)}} \longrightarrow f(n)$$

2D AR Models

$$A(z_1, z_2) = 1 - \sum_{(k,l)} \sum_{\in S} a(k,l) z_1^{-k} z_2^{-k}$$

$$f(m,n) = \sum_{(k,l)} \sum_{\in S} a(k,l)\, f(m-k, n-l) + \epsilon(m,n)$$

- **Non–causal**
  $S = \{l \geq 1, \forall k\} \cup \{l = 0, k \neq 0\}$
- **Semi–ausal**
  $S = \{l \geq 1, \forall k\} \cup \{l = 0, k \geq 1\}$
- **Causal**
  $S = \{(k,l) \neq (0,0)\}$

2D AR Models

- **Causal**
  - $S = \{l \geq 1, \forall k\} \cup \{l = 0, k \neq 0\}$
  - Recursive Filtre
  - Finite Differential Equations with initial conditions
  - Hyperbolic Partial Differential Equations
- **Semi–causal**
  - $S = \{l \geq 1, \forall k\} \cup \{l = 0, k \geq 1\}$
  - Semi–recursif Filters
  - Finite Differential Equations with initial conditions in one dimention and limit conditions in other dimension
  - Parabolic Partial Differential Equations
- **Non–causal**
  - $S = \{(k, l) \neq (0, 0)\}$
  - Non-recursive Filtre
  - Finite Differential Equations with limit conditions in both dimensions
  - Elliptic Partial Differential Equations

Causal/Non-Causal Prediction

- **Causal :**

$$\widehat{f}(n) = \sum_{k=0}^{N-1} a(k)\, f(n-k)$$

- **Non-Causal :**

$$\widehat{f}(n) = \sum_{\substack{k=-p \\ k\neq 0}}^{+q} a(k)\, f(n-k)$$

2D AR models and 2D prediction

$$A(z_1,z_2) = 1 - \sum_{(k,l)} \sum_{\in S} a(k,l) z_1^{-k} z_2^{-k}$$

$$f(m,n) = \sum_{(k,l)} \sum_{\in S} a(k,l)\, f(m-k, n-l) + \epsilon(m,n)$$

- **Non–causal**
- **Semi–ausal**
- **Causal**

2D AR models and 2D prediction

- **Causal**
  - $S = \{l \geq 1, \forall k\} \cup \{l = 0, k \neq 0\}$
  - Recursive Filtering
  - Finite Difference Equation (FDE) with initial conditions
  - Partial Differential Equations (Hyperbolic)
- **Semi–causal**
  - $S = \{l \geq 1, \forall k\} \cup \{l = 0, k \geq 1\}$
  - Filtre semi–récursif
  - FDE with initial conditions in one direction and limit conditions in other direction.
  - Partial Differential Equations (Parabolic)
- **Non–causal**
  - $S = \{(k, l) \neq (0, 0)\}$
  - Non-Recursive Filtering
  - FDE with limit conditions in both directions
  - Partial Differential Equations (Elliptic)

Content

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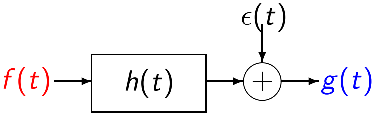
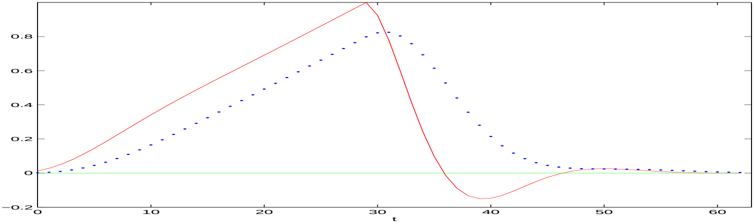
2. Modeling: parametric and non-parametric, MA, AR and ARMA models, Linear prediction
3. **Deconvolution and Parameter Estimation: Deterministic (LS, Regularization) and Probabilistic methods (Wiener Filtering)**
4. Inverse problems and Bayesian estimation
5. Kalman Filtering and smoothing
6. Case study: Signal deconvolution
7. Case study: Image restoration
8. Case study: Image reconstruction and Computed Tomography

3. Deconvolution and Parameter estimation

- Signal Deconvolution and Image Restoration
- Least Squares method (LS)
- Limitations of LS methods
- Regularization methods
- Parametric modeling
- Examples:
  - Sinusoids in the noise (MUSIC)
  - Antenna Array Processing
  - Mixture models (Gaussian and Cauchy)

Convolution in signal processing





$$g(t) = \int f(t')h(t-t')\,\mathrm{d}t' + b(t) = \int h(t')f(t-t')\,\mathrm{d}t' + b(t)$$

- $f(t)$, $g(t)$ and $h(t)$ are discretized with $\Delta T = 1$.

Convolution in signal processing: general case

- $h(t) = 0, \forall t < -q\Delta T$, et $\forall t > p\Delta T$.

$$g(m) = \sum_{k=-q}^{p} h(k)\, f(m-k) + b(m), \quad m = 0, \cdots, M \longrightarrow \mathbf{g} = \mathbf{H}\mathbf{f} + \epsilon$$

$$\mathbf{g} = [g(0), g(1), \cdots, g(M)]',$$
$$\mathbf{f} = [f(-p), \cdots, f(0), f(1), \cdots, f(M), \cdots, f(M+q)]'$$

$$\begin{bmatrix}
h(p) & \cdots & h(0) & \cdots & h(-q) & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & & \ddots & & \ddots & & & & \vdots \\
\vdots & & & & & & & & & \vdots \\
\vdots & & & h(p) & \cdots & h(0) & \cdots & h(-q) & & \vdots \\
\vdots & & & & & & & & & \vdots \\
\vdots & & & & & \ddots & & & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & h(p) & \cdots & h(0) & \cdots & h(-q)
\end{bmatrix}$$

Convolution in signal processing: Causal systems

Causal system $q = 0$:

$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix} = \begin{bmatrix} h(p) & \cdots & h(0) & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & & & & & & & \vdots \\ \vdots & & & & & & & \vdots \\ \vdots & & h(p) & \cdots & h(0) & & & \vdots \\ \vdots & & & & & & & \vdots \\ \vdots & & & & & & & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & h(p) & \cdots & h(0) \end{bmatrix} \begin{bmatrix} f(-p) \\ \vdots \\ f(0) \\ f(1) \\ \vdots \\ \vdots \\ f(M) \end{bmatrix}$$

Convolution: Causal systems and causal input

Causal system and causal input signal

$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix} = \begin{bmatrix} h(0) & & & & & \\ h(1) & \ddots & & & & \\ \vdots & & & & & \\ h(p) & \cdots & & h(0) & & \\ 0 & \ddots & & & \ddots & \\ \vdots & & & & & \\ 0 & \cdots & 0 & h(p) & \cdots & h(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f(M) \end{bmatrix}$$

if $p = M$:

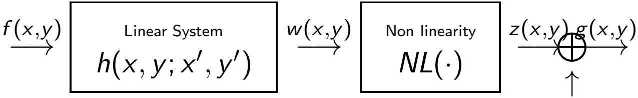
$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(M) \end{bmatrix} = \begin{bmatrix} h(0) & & & \\ h(1) & h(0) & & \\ \vdots & & \ddots & \\ h(M) & \cdots & h(1) & h(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(M) \end{bmatrix}$$

Convolution in signal processing: Circulante case

$$
\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix}
=
\begin{bmatrix}
h(0) & & & & & & h(p) & \cdots & h(1) \\
h(1) & \ddots & & & & & & \ddots & \vdots \\
\vdots & & & & & & & & h(p) \\
h(p) & \cdots & & h(0) & & & & & 0 \\
0 & \ddots & & & \ddots & & & & \vdots \\
\vdots & & & & & & & & \\
0 & \cdots & 0 & h(p) & \cdots & h(0) & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ f(M) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

Modeling imaging systems

General observation model:

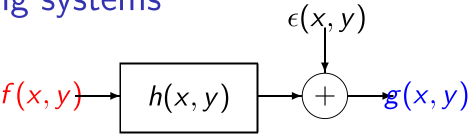


$f(x,y)$ → Linear System $h(x,y;x',y')$ → $w(x,y)$ → Non linearity $NL(\cdot)$ → $z(x,y)$ ⊕ ↑ → $g(x,y)$

If we neglect the Non Linearity, then

$$g(x,y) = \iint f(x',y')\, h(x,y;x',y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x,y)$$

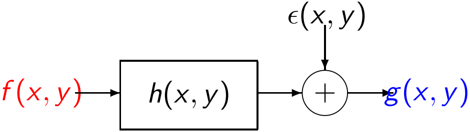
Modeling imaging systems



$$g(x, y) = \iint f(x', y')\, h(x - x', y - y')\, \mathrm{d}x'\, \mathrm{d}y' + b(x, y)$$

System	Point Spread function $h(x, y)$	Transfert Function $H(u, v)$
Diffraction+ Coherent source (Rectangular)	$ab\,\mathrm{sinc}(ax)\mathrm{sinc}(by)$	$\mathrm{rectf}\frac{u}{a}, \frac{v}{b}$
Diffraction+ Incoherent Source (Rectangular)	$\mathrm{sinc}^2(ax)\mathrm{sinc}^2(by)$	$\mathrm{trif}\frac{u}{a}, \frac{v}{b}$
Atmosphoric Turbulance	$\exp\left\{-\pi a^2(x^2 + y^2)\right\}$	$\frac{1}{a^2}\exp\left\{\frac{-\pi(u^2 + v^2)}{a^2}\right\}$
Horizontal movement	$frac1a\mathrm{rectf}\frac{x}{a} - \frac{1}{2}\delta(y)$	$\exp\{-j\pi ua\}\,\mathrm{sinc}(ua)$
CCD interaction	$\sum\sum_{k,l=-1}^{1} \alpha_{k,l}\delta(x - k\Delta, y - l\Delta)$	$\sum\sum_{k,l=-1}^{1} \alpha_{k,l}\exp\{-j2\pi\Delta(ux + vy)\}$

2D Convolution for image restoration



$$g(x,y) = \iint_D f(x',y')\, h(x-x',y-y')\, \mathrm{d}x'\, \mathrm{d}y' + b(x,y)$$

$$g(m\Delta x, n\Delta y) = \sum_{i=-I}^{+I} \sum_{j=-J}^{+J} h(i\Delta x, j\Delta y) f((m-i)\Delta x, (n-j)\Delta y)$$

$$\left\{ \begin{bmatrix} m = 1, \ldots, M \\ n = 1, \ldots, N \end{bmatrix} \right. \qquad \Delta x = \Delta y = 1$$

$$g(m,n) = \sum_{i=-I}^{+I} \sum_{j=-J}^{+J} h(i,j)\, f(m-i, n-j)$$

2D Convolution for image restoration

Two caracteristics:

$$g(m,n)=\sum_{i=-I}^{+I}\sum_{j=-J}^{+J} h(i,j)\, f(m-i,n-j)$$

- $g(m,n)$ depends on $f(k,l)$ for $(k,l)\in\mathcal{N}(k,l)$ where $\mathcal{N}(k,l)$ means the neigborhood pixels around the pixel $'(k,l) \longrightarrow$ No Causality
- The boarding effects cannot be neglected as easily as in the 1D case.

Vectorial Forme:

$$g(m,n)=\sum_{i=-I}^{+I}\sum_{j=-J}^{+J} h(i,j)\, f(m-i,n-j)$$

$$g(m,n)\left\{\begin{bmatrix} m=1,\ldots,M \\ n=1,\ldots,N\end{bmatrix}\right. \quad f(k,l)\left\{\begin{bmatrix} k=1,\ldots,K \\ l=1,\ldots,L\end{bmatrix}\right. \quad h(i,j)\left\{\begin{bmatrix} i=1,\ldots,I \\ j=1,\ldots,J\end{bmatrix}\right.$$

$$\mathbf{g}=\mathbf{H}\mathbf{f}$$

2D Convolution for image restoration

$$\mathbf{g} = [\quad \underset{---1---}{g_{(1,1)}, \ldots, g(M,1)}, \quad \underset{---2---}{g_{(1,2)}, \ldots, g_{(M,2)}}, \quad \ldots, \quad \underset{---N---}{g_{(1,N)}, \ldots, g_{(M,N)}}]^t$$

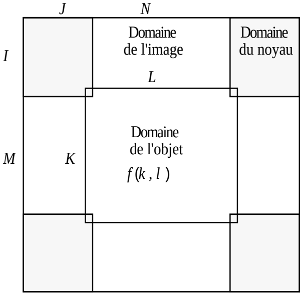
$$\mathbf{f} = [\quad \underset{---1---}{f_{(1,1)}, \ldots, f_{(K,1)}}, \quad \underset{---2---}{f_{(1,2)}, \ldots, f(K,2)}, \quad \ldots, \quad \underset{---L---}{f_{(1,L)}, \ldots, f_{(K,L)}}]^t$$

The structure of the matrix **H** depends on the domaines D_h, D_f and D_g .

Matrix Form H :

- Image $>$ Object
- Image$=$Object
- Image $<$ Object

2D Convolution for image restoration: Image > Object



$$D_g > D_f \quad \left\{ \begin{bmatrix} M = K + I - 1 \\ N = L + J - 1 \end{bmatrix} \right.$$

$$g(m,n) \left\{ \begin{bmatrix} m = 1, \ldots, M \\ n = 1, \ldots, N \end{bmatrix} \right. \quad f(k,l) \left\{ \begin{bmatrix} k = 1, \ldots, K \\ l = 1, \ldots, L \end{bmatrix} \right. \quad h(i,j) \left\{ \begin{bmatrix} i = 1, \ldots, I \\ j = 1, \ldots, J \end{bmatrix} \right.$$

$$\mathbf{g} = [\quad \underset{---1---}{g_{(1,1)}, \ldots, g(M,1)}, \quad \underset{---2---}{g_{(1,2)}, \ldots, g_{(M,2)}}, \quad \ldots, \quad \underset{---N---}{g_{(1,N)}, \ldots, g_{(M,N)}}]^t$$

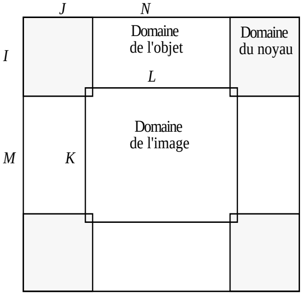
$$\mathbf{f} = [\quad \underset{---1---}{f_{(1,1)}, \ldots, f_{(K,1)}}, \quad \underset{---2---}{f_{(1,2)}, \ldots, f(K,2)}, \quad \ldots, \quad \underset{---L---}{f_{(1,L)}, \ldots, f_{(K,L)}}]^t$$

2D Convolution for image restoration: Image > Object

$$\mathbf{H} = \begin{bmatrix} H_I & \cdot & \cdots & \cdots & \cdots & \cdot \\ \vdots & H_I & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ H_1 & & & & H_I & \cdot \\ \cdot & H_1 & & & & H_I \\ \vdots & \ddots & H_1 & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \cdot & \cdots & \cdots & \cdots & \cdot & H_1 \end{bmatrix} H_i = \begin{bmatrix} h(i,J) & \cdot & \cdots & \cdots & \cdots & \cdot \\ \vdots & h(i,J) & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ h(i,1) & & & & h(i,J) & \cdot \\ \cdot & h(i,1) & & & & h(i,J) \\ \vdots & \ddots & h(i,1) & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \cdot & \cdots & \cdots & \cdots & \cdot & h(i,1) \end{bmatrix}$$

Toeplitz-Bloc-Toeplitz

2D Convolution for image restoration: Image < Object



$$D_g < D_f \quad \left\{ \begin{bmatrix} M = K - I - 1 \\ N = L - J + 1 \end{bmatrix} \right.$$

$$g(m,n) \left\{ \begin{bmatrix} m = 1, \ldots, M \\ n = 1, \ldots, N \end{bmatrix} \right. \quad f(k,l) \left\{ \begin{bmatrix} k = 1, \ldots, K \\ l = 1, \ldots, L \end{bmatrix} \right. \quad h(i,j) \left\{ \begin{bmatrix} i = 1, \ldots, I \\ j = 1, \ldots, J \end{bmatrix} \right.$$

$$\mathbf{g} = [\quad \underset{---1---}{g_{(1,1)}, \ldots, g_{(M,1)}}, \quad \underset{---2---}{g_{(1,2)}, \ldots, g_{(M,2)}}, \quad \ldots, \quad \underset{---N---}{g_{(1,N)}, \ldots, g_{(M,N)}}]^t$$

$$\mathbf{f} = [\quad \underset{---1---}{f_{(1,1)}, \ldots, f_{(K,1)}}, \quad \underset{---2---}{f_{(1,2)}, \ldots, f_{(K,2)}}, \quad \ldots, \quad \underset{---L---}{f_{(1,L)}, \ldots, f_{(K,L)}}]^t$$

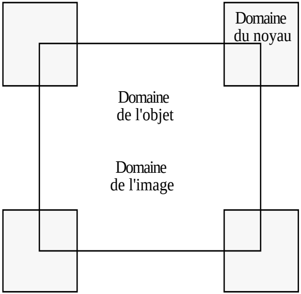
2D Convolution for image restoration: Image $<$ Object

$$\mathbf{H} = \begin{bmatrix} H_1 & H_2 & \cdots & H_I & \cdot & \cdots & \cdot \\ \cdot & H_1 & & & H_I & \ddots & \vdots \\ \vdots & \ddots & H_1 & & & H_I & \cdot \\ \vdots & & & & & & \vdots \\ \cdot & \cdots & \cdot & H_1 & H_2 & & H_I \end{bmatrix}$$

with

$$H_i = \begin{bmatrix} h(i,1) & h(i,2) & \cdots & h(i,J) & \cdot & \cdots & \cdot \\ \cdot & h(i,1) & & & h(i,J) & \ddots & \vdots \\ \vdots & \ddots & h(i,1) & & & h(i,J) & \cdot \\ \vdots & & & & & & \vdots \\ \cdot & \cdots & \cdot & h(i,1) & h(i,2) & & h(i,J) \end{bmatrix}$$

2D Convolution for image restoration: Image=Object



$$D_g = D_f \quad \left\{ \begin{bmatrix} M = K \\ N = L \end{bmatrix} \right.$$

$$g(m,n) \left\{ \begin{bmatrix} m = 1,\ldots,M \\ n = 1,\ldots,N \end{bmatrix} \right. \quad f(k,l) \left\{ \begin{bmatrix} k = 1,\ldots,K \\ l = 1,\ldots,L \end{bmatrix} \right. \quad h(i,j) \left\{ \begin{bmatrix} i = 1,\ldots,I \\ j = 1,\ldots,J \end{bmatrix} \right.$$

$$\mathbf{g} = [\quad \underset{---1---}{g_{(1,1)},\ldots,g_{(M,1)}}, \quad \underset{---2---}{g_{(1,2)},\ldots,g_{(M,2)}}, \quad \ldots, \quad \underset{---N---}{g_{(1,N)},\ldots,g_{(M,N)}}]^t$$

$$\mathbf{f} = [\quad \underset{---1---}{f_{(1,1)},\ldots,f_{(K,1)}}, \quad \underset{---2---}{f_{(1,2)},\ldots,f_{(K,2)}}, \quad \ldots, \quad \underset{---L---}{f_{(1,L)},\ldots,f_{(K,L)}}]^t$$

2D Convolution for image restoration: Circulante forme

$$g(m,n)\left\{\begin{bmatrix} m=1,\ldots,M \\ n=1,\ldots,N\end{bmatrix}\right. \quad f(k,l)\left\{\begin{bmatrix} k=1,\ldots,K \\ l=1,\ldots,L\end{bmatrix}\right. \quad h(i,j)\left\{\begin{bmatrix} i=1,\ldots,I \\ j=1,\ldots,J\end{bmatrix}\right.$$

$$\left\{\begin{bmatrix} P=K+I-1 \\ Q=L+J-1\end{bmatrix}\right.$$

$$\tilde{f}(m,n)=\begin{bmatrix} f((m,n)) & \vdots & 0 \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0 & \vdots & 0 \end{bmatrix} \qquad \dim(\tilde{f})=[P,Q]$$

$$\tilde{g}(k,l)=\begin{bmatrix} g(k,l) & \vdots & 0 \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0 & \vdots & 0 \end{bmatrix} \qquad \dim(\tilde{g})=[P,Q]$$

$$\tilde{h}(i,j)=\begin{bmatrix} h(i,j) & \vdots & 0 \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0 & \vdots & 0 \end{bmatrix} \qquad \dim(\tilde{h})=[P,Q]$$

2D Convolution for image restoration: Circulante forme

$$H=\begin{bmatrix} H_1 & H_2 & \cdots & \cdots & \cdots & H_P \\ H_P & H_1 & H_2 & \cdots & \cdots & H_{P-1} \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ H_P & H_{P-1} & \cdots & \cdots & \cdots & H_1 \end{bmatrix} \quad \text{bloc-circulante}$$

$$H_i=\begin{bmatrix} h(i,1) & h(i,2) & \cdots & \cdots & \cdots & h(i,P) \\ h(i,P) & h(i,1) & h(i,2) & \cdots & \cdots & h(i,P) \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ h(i,P) & h(i,P-1) & h(i,P-2) & \cdots & \cdots & h(i,1) \end{bmatrix} \quad \text{circulante}$$

Circulante-Bloc-Circulante

Classification of the signal and image restoration methods

- **Analytical methods**

$$f(t) \longrightarrow \boxed{\quad\mathcal{H}\quad} \longrightarrow g(t) = \mathcal{H}[f(t)]$$

$$f(x,y) \longrightarrow \boxed{\quad\mathcal{H}\quad} \longrightarrow g(x,y) = \mathcal{H}[f(x,y)]$$

$\mathcal{H}$ Linear Operator

$$g(t) \longrightarrow \boxed{\quad\mathcal{G}\quad} \longrightarrow \widehat{f}(t) = \mathcal{H}^{-1}[f(t)]$$

$$g(x,y) \longrightarrow \boxed{\quad\mathcal{G}\quad} \longrightarrow \widehat{f}(x,y) = \mathcal{H}^{-1}[f(x,y)]$$

$\mathcal{G}$ Linear Operator approximating $\mathcal{H}^{-1}$

  - Inverse Filtring
  - Pseudo–inverse Filtering
  - Wiener Filtering

Classification of the signal and image restoration methods

- **Algebraic methods**

$$g(t) = \mathcal{H}[f(t)] \longrightarrow \quad \text{Discretization} \quad \longrightarrow \mathbf{g} = \mathbf{H}\mathbf{f}$$

$$g(x,y) = \mathcal{H}[f(x,y)] \longrightarrow \quad \text{Discretization} \quad \longrightarrow \mathbf{g} = \mathbf{H}\mathbf{f}$$

Ideal case : H invertible $\longrightarrow \widehat{\mathbf{f}} = \mathbf{H}^{-1}\mathbf{g}$
More general case : H is not invertible

  - Pseudo-inverse
  - Generalized Inversion
  - Least Squares (LS) and Minimum norm LS
  - Regularization

- **Probabilistic methods**
  - Wiener Filtering
  - Kalman Filtering
  - General Bayesian approach

Analytical methods: Inverse Filtering

$$f(t) \longrightarrow \boxed{\quad h(t) \quad} \longrightarrow g(t) = h(t) * f(t)$$

$$f(x,y) \longrightarrow \boxed{\quad h(x,y) \quad} \longrightarrow g(x,y) = h(x,y) * f(x,y)$$

$$F(u,v) \longrightarrow \boxed{\quad H(u,v) \quad} \longrightarrow G(u,v) = H(u,v) \cdot F(u,v)$$

- Inverse Filtering

$$H^{-1}(u,v) = \frac{1}{H(u,v)}$$

$$G(u,v) \longrightarrow \boxed{\quad \frac{1}{H(u,v)} \quad} \longrightarrow F(u,v)$$

- **Difficulties:**
  - What to do when $H(u,v) = 0$ for sme values of (u,v)?

Analytical methods: Inverse Filtering

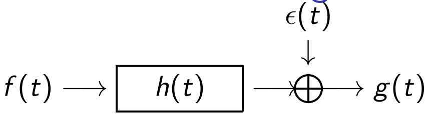
- Noise Amplification

$$\widehat{F}(u,v) = \frac{\widehat{G}(u,v)}{H(u,v)} = \frac{(H(u,v)\,F(u,v) + N(u,v))}{H(u,v)} = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

- Pseudo–inverse Filtering

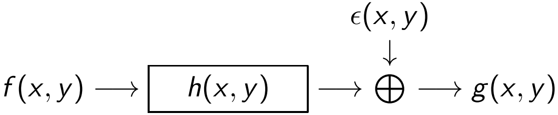
$$H^{-1}(u,v) = \begin{cases} \dfrac{1}{H(u,v)}, & H \neq 0 \\ 0, & H = 0 \end{cases}$$

Analytical methods: Wiener Filtering



$$g(t) = \iint f(t')\, h(t-t')\, \mathrm{d}t' + \epsilon(t)$$

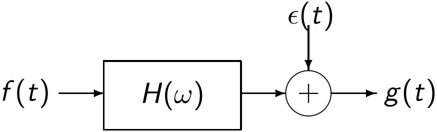
- $f(t)$, $g(t)$ and $\epsilon(t)$ are modelled as Gaussian random signal



$$g(x,y) = \iint f(x',y')\, h(x-x',y-y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x,y)$$

- $f(x,y)$, $g(x,y)$ and $\epsilon(x,y)$ are modelled as homogeneous and Gaussian random fields

Wiener Filtering



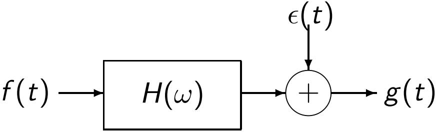
$$g(t) = h(t) * f(t) + \epsilon(t)(t)$$

$$\mathrm{E}\left\{g(t)\right\}, \quad \mathrm{E}\left\{\epsilon(t)\right\} \quad \text{and} \quad \mathrm{E}\left\{f(t)\right\}$$

$$R_{gg}(\tau) = \mathrm{E}\left\{g(t)\, g(t+\tau)\right\}$$
$$R_{ff}(\tau) = \mathrm{E}\left\{f(t)\, f(t+\tau)\right\}$$

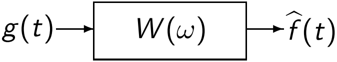
$$R_{bf}(\tau) = R_{fb}(-\tau) = \mathrm{E}\left\{\epsilon(t)\, f(t+\tau)\right\}$$
$$R_{gf}(\tau) = R_{fg}(-\tau) = \mathrm{E}\left\{g(t)\, f(t+\tau)\right\}$$

Wiener Filtering



$$
\begin{aligned}
\mathrm{E}\left\{g(t)\right\} &= h(t) * \mathrm{E}\left\{f(t)\right\} + \mathrm{E}\left\{\epsilon(t)\right\} \\
R_{gg}(\tau) &= h(t) * h(t) * R_{ff}(\tau) + R_{\epsilon\epsilon}(\tau) \\
R_{gf}(\tau) &= h(t) * R_{ff}(\tau)
\end{aligned}
$$

$$
\begin{aligned}
S_{gg}(\omega) &= |H(\omega)|^2 S_{ff}(\omega) + R_{\epsilon\epsilon}(\omega) \\
S_{gf}(\omega) &= H(\omega) S_{ff}(\omega) \\
S_{fg}(\omega) &= H^*(\omega) S_{ff}(\omega)
\end{aligned}
$$



$$
\widehat{f}(t) = w(t) * g(t)
$$

Wiener Filtering

$$EQM = \mathrm{E}\left\{[f(t) - \widehat{f}(t)]^2\right\} = \mathrm{E}\left\{[f(t) - w(t) * g(t)]^2\right\}$$

$$\frac{\partial EQM}{\partial f} = -2\mathrm{E}\left\{[f(t) - w(t) * g(t)] * g(t+\tau)\right\} = 0$$

$$\mathrm{E}\left\{[f(t) - w(t) * g(t)]\, g(t+\tau)\right\} = 0 \quad \forall t, \tau \longrightarrow$$

$$R_{fg}(\tau) = w(t) * R_{gg}(\tau)$$

$$W(\omega) = \frac{S_{fg}(\omega)}{S_{gg}(\omega)} = \frac{H^*(\omega)\, S_{ff}(\omega)}{|H(\omega)|^2\, S_{ff}(\omega) + S_{\epsilon\epsilon}(\omega)}$$

$$\boxed{W(\omega) = \frac{H^*(\omega) S_{ff}(\omega)}{|H(\omega)|^2 S_{ff}(\omega) + S_{\epsilon\epsilon}(\omega)} = \frac{1}{H(\omega)} \frac{|H(\omega)|^2}{|H(\omega)|^2 + \frac{S_{\epsilon\epsilon}(\omega)}{S_{ff}(\omega)}}}$$

Analytical methods: Wiener Filtering

- Linear Estimation: $\widehat{f}(x,y)$ is such that:
  - $\widehat{f}(x,y)$ depends on $g(x,y)$ in a linear way:

$$\widehat{f}(x,y)=\iint g(x',y')\,w(x-x',y-y')\,\mathrm{d}x'\,\mathrm{d}y'$$

    $w(x,y)$ is the impulse response of the Wiener filtre
  - minimizes MSE: $\mathrm{E}\left\{|f(x,y)-\widehat{f}(x,y)|^2\right\}$
- Orthogonality condition:

$$(f(x,y){-}\widehat{f}(x,y))\perp g(x',y') \quad\longrightarrow\quad \mathrm{E}\left\{(f(x,y)-\widehat{f}(x,y))\,g(x',y')\right\}=0$$

$$\widehat{f}=g{*}w \;\longrightarrow\; \mathrm{E}\left\{(f(x,y)-g(x,y)*w(x,y))\,g(x+\alpha_1,y+\alpha_2)\right\}=0$$

$$R_{fg}(\alpha_1,\alpha_2)=(R_{gg}{*}w)(\alpha_1,\alpha_2)\longrightarrow \mathrm{TF}\longrightarrow S_{fg}(u,v)=S_{gg}(u,v)W(u,v)$$

$$\Downarrow$$

$$\boxed{W(u,v)=\frac{S_{fg}(u,v)}{S_{gg}(u,v)}}$$

Wiener filtering

Signal	Image
$W(\omega) = \frac{S_{fg}(\omega)}{S_{gg}(\omega)}$	$W(u,v) = \frac{S_{fg}(u,v)}{S_{gg}(u,v)}$

Particular Case:

$f(x,y)$ and $b(x,y)$ are assumed to be centered and non correlated

$$S_{fg}(u,v) = H'(u,v)\,S_{ff}(u,v)$$

$$S_{gg}(u,v) = |H(u,v)|^2\,S_{ff}(u,v) + S_{\epsilon\epsilon}(u,v)$$

$$W(u,v) = \frac{H'(u,v)S_{ff}(u,v)}{|H(u,v)|^2 S_{ff}(u,v) + S_{\epsilon\epsilon}(u,v)}$$

Signal	Image
$W(\omega) = \frac{1}{H(\omega)} \frac{\|H(\omega)\|^2}{\|H(\omega)\|^2 + \frac{S_{\epsilon\epsilon}(\omega)}{S_{ff}(\omega)}}$	$W(u,v) = \frac{1}{H(u,v)} \frac{\|H(u,v)\|^2}{\|H(u,v)\|^2 + \frac{S_{\epsilon\epsilon}(u,v)}{S_{ff}(u,v)}}$

Wiener Filtering: Implementation

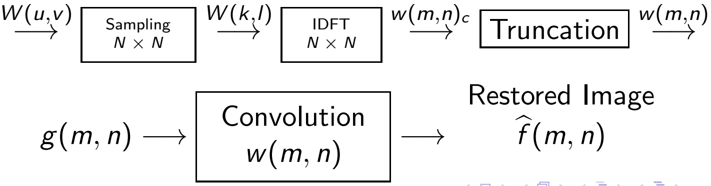
- No noise ⟶ $W(u,v) = 1/H(u,v)$
- $S_{\epsilon\epsilon}(u,v)/S_{ff}(u,v)$ have to be given.
- No blur case: $h(x,y) = \delta(x,y)$ (denoising):

$$W(u,v) = \frac{1}{1 + \frac{S_{\epsilon\epsilon}(u,v)}{S_{ff}(u,v)}}$$

Numerical implementation:

- Direct Implentation by convolution

$$W(u,v) = \frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + \frac{S_{\epsilon\epsilon}(u,v)}{S_{ff}(u,v)}}$$

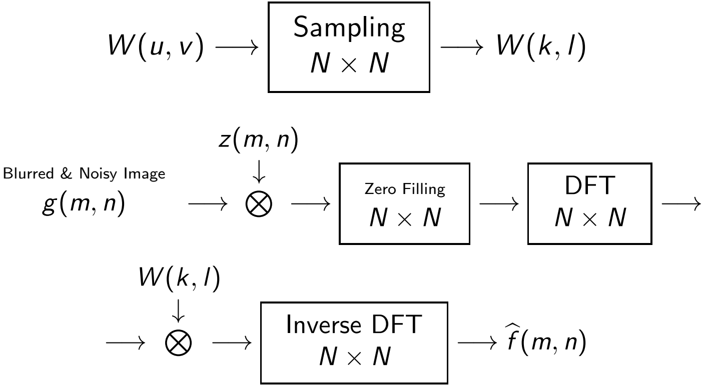


$$\xrightarrow{W(u,v)} \boxed{\begin{array}{c}\text{Sampling}\\ N \times N\end{array}} \xrightarrow{W(k,l)} \boxed{\begin{array}{c}\text{IDFT}\\ N \times N\end{array}} \xrightarrow{w(m,n)_c} \boxed{\text{Truncation}} \xrightarrow{w(m,n)}$$

$$g(m,n) \longrightarrow \boxed{\begin{array}{c}\text{Convolution}\\ w(m,n)\end{array}} \longrightarrow \begin{array}{c}\text{Restored Image}\\ \widehat{f}(m,n)\end{array}$$

Wiener Filtering: Implementation

- Fourier domaine Implentation



$W(u,v) \longrightarrow$ Sampling $N \times N$ $\longrightarrow W(k,l)$

Blurred & Noisy Image

$g(m,n)$ $\longrightarrow \otimes$ ($z(m,n)$) $\longrightarrow$ Zero Filling $N \times N$ $\longrightarrow$ DFT $N \times N$ $\longrightarrow$

$\longrightarrow \otimes$ ($W(k,l)$) $\longrightarrow$ Inverse DFT $N \times N$ $\longrightarrow \widehat{f}(m,n)$

Algebraic Approches

Signal	Image
$f(t) \longrightarrow \boxed{h(t)} \longrightarrow g(t)$	$f(x,y) \longrightarrow \boxed{h(x,y)} \longrightarrow g(x,y)$

Discretization

$$\Downarrow$$

$$\mathbf{g} = \mathbf{H}\mathbf{f}$$

- **Ideal case: H** invertible $\longrightarrow \widehat{\mathbf{f}} = \mathbf{H}^{-1}\mathbf{g}$
- $M > N$ **Least Squares**:

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \epsilon$$

$$e = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 = [\mathbf{g} - \mathbf{H}\mathbf{f}]'[\mathbf{g} - \mathbf{H}\widehat{\mathbf{f}}]$$

$$\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \{e\}$$

$$\nabla e = -2\mathbf{H}'[\mathbf{g} - \mathbf{H}\mathbf{f}] = 0 \longrightarrow \mathbf{H}'\mathbf{H}\mathbf{f} = \mathbf{H}'\mathbf{g}$$

- If $\mathbf{H}'\mathbf{H}$ is invertible $\widehat{\mathbf{f}} = (\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\mathbf{g}$

Algebraic Approches: Generalized Inversion

$$\widehat{\mathbf{f}} = \mathbf{W}\mathbf{g}$$

$\mathbf{W}$ is such that: $e = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 = [\mathbf{g} - \mathbf{H}\mathbf{f}]'[\mathbf{g} - \mathbf{H}\mathbf{f}]$ is minimum.
The solution may not be unique.

Generalized Inversion $\widehat{\mathbf{f}} = \mathbf{H}^{+}\mathbf{g}$
$\mathbf{H}^{+}$ is such that:

$$e = [\mathbf{g} - \mathbf{H}^{+}\mathbf{f}]'[\mathbf{g} - \mathbf{H}^{+}\mathbf{f}] = \|\mathbf{g} - \mathbf{H}^{+}\mathbf{f}\|^2$$

is minimum and that $\|\mathbf{f}\|^2$ minimum too.

- $\widehat{\mathbf{f}} = \mathbf{H}^{+}\mathbf{g}$ also minimizes

$$e = [\mathbf{f} - \widehat{\mathbf{f}}]'[\mathbf{f} - \widehat{\mathbf{f}}] = \text{Tr}\left\{[\mathbf{f} - \widehat{\mathbf{f}}][\mathbf{f} - \widehat{\mathbf{f}}]'\right\} = \text{Tr}\left\{\mathbf{f}\mathbf{f}'[\mathbf{I} - \mathbf{H}^{+}\mathbf{H}]'\right\}$$

- Estimation is perfect if $\mathbf{H}^{+}\mathbf{H} = \mathbf{I}$.

Algebraic Approches: Generalized Inversion

General case of $[M, N]$ matrix $\mathbf{H}$:

- if $M = N$ and $\operatorname{rang}\{\mathbf{H}\} = N$ then $\mathbf{H}^{+} = \mathbf{H}^{-1}$
- if $M > N$ and $\operatorname{rang}\{\mathbf{H}\} = N$ then $\mathbf{H}^{+} = (\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'$
- if $M < N$ and $\operatorname{rang}\{\mathbf{H}\} = M$ then $\mathbf{H}^{+} = \mathbf{H}'(\mathbf{H}\mathbf{H}')^{-1}$
- if $\operatorname{rang}\{\mathbf{H}\} = K < \inf M, N$ then
  - Singular Value Decomposition (SVD)
  - Iterative methods
  - Recursive methods

Generalized inversion by Singular Value Decomposition (SVD)

$$\mathbf{g} = \mathbf{H}\,\mathbf{f}$$

$$\begin{array}{lll} \mathbf{H}'\mathbf{H}\,\mathbf{v}_j = \lambda_j^2\mathbf{v}_j, \quad j = 1,\ldots,n & \mathbf{v}_i \text{ Eigen Vectors of } \mathbf{H}'\mathbf{H} \\ \mathbf{H}\mathbf{H}'\,\mathbf{u}_i = \lambda_i^2\mathbf{u}_k, \quad i = 1,\ldots,m & \mathbf{u}_j \text{ Eigen Vectors of } \mathbf{H}\mathbf{H}' \end{array}$$

$$\begin{array}{l} \mathbf{H}\,\lambda_j\,\mathbf{u}_j, \quad j = 1,\ldots,n \\ \mathbf{H}'\,\mathbf{u}_i = \lambda_i\,\mathbf{v}_i, \quad i = 1,\ldots,m \end{array}$$

$$\mathbf{H} = \mathbf{U}\,\boldsymbol{\Lambda}\,\mathbf{V}'$$

$$\mathbf{U} = \left[\mathbf{u}_1, \vdots, \mathbf{u}_2, \vdots, \cdots\right], \quad \mathbf{V} = \left[\mathbf{v}_1, \vdots, \mathbf{v}_2, \vdots, \cdots\right], \quad \boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \cdots)$$

- Truncation order r

$$\mathbf{H} = \sum_{m=1}^{r} \lambda_m \mathbf{v}_m \mathbf{u}_m' \longrightarrow \mathbf{H}^+ = \sum_{m=1}^{r} \frac{1}{\lambda_m} \mathbf{v}_m\,\mathbf{u}_m' \longrightarrow \mathbf{f}^+ = \sum_{m=1}^{r} \frac{\mathbf{v}_m'\,\mathbf{g}}{\lambda_m} \mathbf{u}_m$$

Generalized inversion: Iterative Algorithms

- **Bialy Algorithm** If $M = N$

$$\begin{cases} \mathbf{f}_0 = \mathbf{0} \\ \mathbf{f}_{n+1} = \mathbf{f}_n + \alpha(\mathbf{g} - \mathbf{H}\mathbf{f}_n), \quad \text{with} \;\; 0 < \alpha < 2/\|\mathbf{H}\| \end{cases}$$

- **Landweber Algorithm**

$$\begin{cases} \mathbf{f}_0 = \mathbf{0} \\ \mathbf{f}_{n+1} = \mathbf{f}_n + \alpha\mathbf{H}'(\mathbf{g} - \mathbf{H}\mathbf{f}_n), \quad \text{with} \;\; 0 < \alpha < 2/\|\mathbf{H}'\mathbf{H}\| \end{cases}$$

- **Van–Cittert Algorithm**

$$\begin{cases} \mathbf{f}_0 = \mathbf{0} \\ \mathbf{f}_{n+1} = \mathbf{f}_n + \alpha\mathbf{H}'(\mathbf{g} - \mathbf{H}\mathbf{T}\mathbf{f}_n), \quad \text{with} \;\; 0 < \alpha < 2/\|\mathbf{T}'\mathbf{H}'\mathbf{H}\mathbf{T}\| \end{cases}$$

where $\mathbf{T}$ is a truncation operator

$$\mathbf{f}_n = \alpha \sum_{k=0}^{n} [\mathbf{I} - \alpha\mathbf{H}'\mathbf{H}\mathbf{T}]^k \mathbf{H}'\mathbf{g}$$

n filters $[\mathbf{I} - \alpha\mathbf{H}'\mathbf{H}\mathbf{T}]$

Generalized inversion: Iterative Algorithms

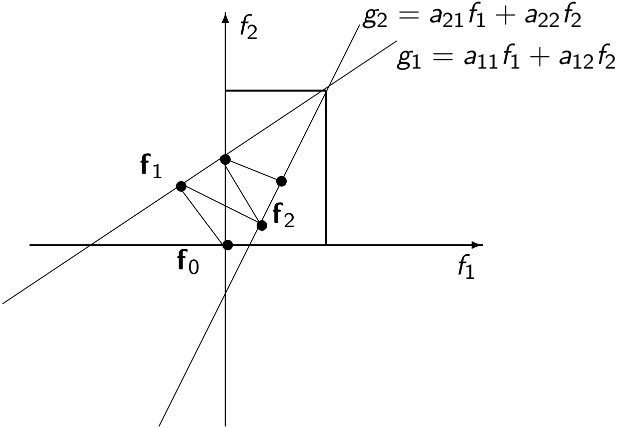
- **Kaczmarz Algorithm**
  - uses the data one by one as they arrived
  - is based on the more general Projection On Convex Sets (POCS)
  - $\mathbf{g} = \mathbf{H}\,\mathbf{f} \longrightarrow y_i = [\mathbf{Hf}] = \sum_j H_{ij}\, f_j =< \mathbf{h}_i, \mathbf{f} >$
  - each data is considered as a constraint
  - in Compted Tomography, it is called Algebraic Reconstruction Technics (ART)

$$\begin{cases} \mathbf{f}_0 = \mathbf{0} \\ \mathbf{f}_{n+1} = C(\mathbf{g}, \mathbf{f}_n) = c_m(\cdots c_2(c_1(\mathbf{f}))\cdots) \\ c_i(\mathbf{f}^{k+1}) = \mathbf{f}^{k+1} + \frac{g_i - \langle \mathbf{f}, \mathbf{h}_i \rangle}{\langle \mathbf{h}, \mathbf{h}_i \rangle}\, \mathbf{h}_i \quad i = 1, 2, \ldots, M \end{cases}$$

where $\mathbf{h}_i$ is the ith line of the matrix $\mathbf{H}$.

C is a set of other constraints such as positivity

Kaczmarz or ART Algorithms



Regularization

Regularization of an **ill-posed** inverse problem is its transformation to a **well-posed** problem, *i.e.*defining a **unique solution** for **any possible data** and insure that this solution is **stable** with respect to **errors** in those data.

A **regularizer** of $\mathbf{g} = \mathbf{H}\,\mathbf{f}$ is a family of of operators

$$\{R_\alpha; \alpha \in \Lambda \subset \mathbf{R}^+\}$$

such that:

- $\forall \mathbf{f} \in \mathcal{F}, \lim_{\alpha\to 0} \|R_\alpha \mathbf{H}\mathbf{f} - \mathbf{f}\|^2 = 0$
- $\forall \alpha \in \Lambda, R_\alpha$ is a continuous operator of $\mathcal{G}$ to $\mathcal{F}$

$$\Downarrow$$

Approximate Solution

$$\mathbf{f}_\epsilon = R_\alpha \mathbf{g}_\epsilon, \quad \text{with} \quad \mathbf{g}_\epsilon = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon}$$

$$\|\mathbf{f}_\epsilon - \mathbf{H}^{-1}\mathbf{g}\| \leq \underbrace{\|R_\alpha \mathbf{g} - \mathbf{H}^{-1}\mathbf{g}\|}_{\text{regularization error}} + \underbrace{\|R_\alpha(\mathbf{g}_\epsilon - \mathbf{g})\|}_{\text{error due to the noise}}$$

Regularization by Truncated SV Decomposition

$$\mathbf{g} = \mathbf{H}\,\mathbf{f}$$

$$\mathbf{H}'\,\mathbf{H}\,\mathbf{v}_j = \lambda_j^2 \mathbf{v}_j, \quad j = 1, \ldots, n \qquad \{\mathbf{u}_i\} \text{ eigen values of } \mathbf{H}'\mathbf{H}$$
$$\mathbf{H}\,\mathbf{H}'\,\mathbf{u}_i = \lambda_i^2 \mathbf{u}_k, \quad i = 1, \ldots, m \qquad \{\mathbf{v}_j\} \text{ eigen values of } \mathbf{H}\mathbf{H}'$$

$$\mathbf{H}\,\mathbf{v}_j = \lambda_j\,\mathbf{u}_j, \quad j = 1, \ldots, n$$
$$\mathbf{H}'\,\mathbf{u}_i = \lambda_i\,\mathbf{v}_i, \quad i = 1, \ldots, m$$

$$\mathbf{H} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}'$$

$$\mathbf{U} = \left[\mathbf{u}_1, \vdots, \mathbf{u}_2, \vdots, \ldots\right], \quad \mathbf{V} = \left[\mathbf{v}_1, \vdots, \mathbf{v}_2, \vdots, \ldots\right], \quad \boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \cdots)$$

- Truncation order k

$$\Lambda^+ = \{\alpha_i\}, \quad \begin{cases} \alpha_i = \frac{1}{\lambda_i} & \text{si} \quad \lambda_i \neq 0 (\text{significatif}) \\ \alpha_i = 0 & \text{si} \quad \lambda_i \simeq 0 (\text{non–significatif}) \end{cases}$$

$$\mathbf{g} + \delta\mathbf{g} = H(\mathbf{f} + \delta\mathbf{f}) \longrightarrow \frac{||\delta\mathbf{f}||}{||\mathbf{f}||} \leq \left(\frac{\lambda_{max}}{\lambda_{min}}\right)^2 \frac{||\delta\mathbf{g}||}{||\mathbf{g}||}$$

Regularization

- Stopping rule in Iterative method
  - Establish a stopping rule based on the residual error

$$\|\mathbf{H}\widehat{\mathbf{f}}^{k} - \mathbf{g}_{\epsilon}\| > \delta \text{ pour } i < k$$

$$\|\mathbf{H}\widehat{\mathbf{f}}^{k} - \mathbf{g}_{\epsilon}\| < \delta \text{ pour } i > k$$

  - $\frac{1}{k}$ plays the role of the regularization parameter
- Introduction of additional constraints: $\mathbf{C}$

$$\mathbf{f} = \mathbf{C}\mathbf{f} \quad \text{iff} \quad \mathbf{f} \quad \text{satisfait la contrainte}$$

Example : Positivity

$$\mathbf{C}\mathbf{f} = \begin{cases} \mathbf{f} & \text{if} \quad \mathbf{f} > 0 \\ 0 & \text{sinon} \end{cases}$$

$$\mathbf{g} = \mathbf{H}\,\mathbf{f} \longrightarrow \mathbf{g} = \mathbf{H}\mathbf{C}\mathbf{f}$$

At each iteration replace **H** by **HC**

Regularization for Continuous / Discretized problems

- **Tikhonov** (Regularization for continuous problems)
- **Philipps and Twomey** (Regularization for discretized problems)

Continuous case:

$$J_\alpha(f) = \|\mathcal{H}f - g\|_Y^2 + \alpha\Omega(f) \quad \alpha > 0$$

$$\Omega(f) = \int \sum_i w_i(t)[f^{(i)}(t)]^2 \,\mathrm{d}t$$

where $f^{(i)}(t)$ is the ith derivative of $f(t)$ and $w_i(t)$ postive, continuous and derivable functions.

Example : $i = 1$: first derivative

$$J_\alpha(f) = \|\mathcal{H}f - g\|_Y^2 + \alpha \int |f'(t)|^2 \,\mathrm{d}t$$

Regularization for Continuous / Discretized problems

Discrete case:

$$J_\alpha(\mathbf{f}) = [\mathbf{Hf}-\mathbf{g}]'[\mathbf{Hf}-\mathbf{g}] + \alpha[\mathbf{Df}]'[\mathbf{Df}] = \|\mathbf{Hf}-\mathbf{g}\|^2 + \alpha\|\mathbf{Df}\|^2$$

$$DCb = \begin{bmatrix} 1 & 0 & \cdots & & \cdots & 0 \\ -1 & 1 & \ddots & & & \vdots \\ 0 & -1 & 1 & \ddots & & \vdots \\ & 0 & -1 & 1 & \ddots & \\ & & & & & \\ 0 & & & 0 & -1 & 1 \end{bmatrix} \text{ or } \mathbf{D} = \begin{bmatrix} 1 & 0 & \cdots & & \cdots & 0 \\ -2 & 1 & \ddots & & & \vdots \\ 1 & -2 & 1 & \ddots & & \vdots \\ & 1 & -2 & 1 & \ddots & \\ & & & & & \\ 0 & & & 1 & -2 & 1 \end{bmatrix}$$

$$\nabla J_\alpha = 2\mathbf{H}'[\mathbf{Hf}-\mathbf{g}]' + 2\alpha\mathbf{D}'\mathbf{Df} = 0$$

$$[\mathbf{H}'\mathbf{H} + \alpha\mathbf{D}'\mathbf{D}]\widehat{\mathbf{f}} = \mathbf{H}'\mathbf{g} \longrightarrow \widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \alpha\mathbf{D}'\mathbf{D}]^{-1}\mathbf{H}'\mathbf{g}$$

Regularization Algorithmes

$$\text{minimize} \quad J(\mathbf{f}) = Q(\mathbf{f}) + \lambda\Omega(\mathbf{f})$$

$$\text{with} \quad Q(\mathbf{f}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 = [\mathbf{g} - \mathbf{H}\mathbf{f}]'[\mathbf{g} - \mathbf{H}\mathbf{f}]$$

$$=$$

minimize $\Omega(\mathbf{f})$ subj. to the constraint

$$e = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 = [\mathbf{g} - \mathbf{H}\mathbf{f}]'[\mathbf{g} - \mathbf{H}\mathbf{f}] < \epsilon$$

A priori Information:

- Smoothnesse

$$\Omega(\mathbf{f}) = [\mathbf{D}\mathbf{f}]'[\mathbf{D}\mathbf{f}] = \|\mathbf{D}\mathbf{f}\|^2$$

$$\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda\mathbf{D}'\mathbf{D}]^{-1}\mathbf{H}'\mathbf{g}$$

- Positivity:
  $\Omega(\mathbf{f}) =$ a nonquadratique function of $\mathbf{f}$
  No explicite solution

Regularization Algorithmes: 3 main approaches

$$\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda \mathbf{D}'\mathbf{D}]^{-1}\mathbf{H}'\mathbf{g}$$

Computation of $\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda \mathbf{D}'\mathbf{D}]^{-1}\mathbf{H}'\mathbf{g}$

- Circulante matrix approximation:
  when **H** and **D** are Toeplitz, they can be approximated by the circulant matrices
- Iterative methods:

$$\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \left\{ \| J(\mathbf{f}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda \mathbf{D}\mathbf{f}\|^2 \right\}$$

- Recursive methods:
  $\widehat{\mathbf{f}}$ at iteration k is computed as a function of $\widehat{\mathbf{f}}$ at previous iteration with one less data.

Regularization algorithms: Circulant approximation

1D Deconvolution:

$$\mathbf{g} = \mathbf{H}\,\mathbf{f} + \epsilon$$

H Toeplitz matrix

$$\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \{\mathbf{f}\}\, J(\mathbf{f}) = Q(\mathbf{f}) + \lambda\Omega(\mathbf{f})$$

$Q(\mathbf{f}) = \|\mathbf{g}{-}\mathbf{H}\mathbf{f}\|^2 = [\mathbf{g}{-}\mathbf{H}\mathbf{f}]'[\mathbf{g}{-}\mathbf{H}\mathbf{f}]$ and $\Omega(\mathbf{f}) = \|\mathbf{C}\mathbf{f}\|^2 = [\mathbf{C}\mathbf{f}]'[\mathbf{C}\mathbf{f}]$

C a convolution matrix with the following impulse response

$$\mathbf{h}_1 = [1, -2, 1] \quad \longrightarrow \quad x(i) = x(i+1) - 2x(i) + x(i-1)$$

$$\Omega(\mathbf{f}) = \sum_{j=1}^{N} (x(i+1) - 2x(i) + x(i-1))^2 = \|\mathbf{C}\mathbf{f}\|^2 = \mathbf{f}'\mathbf{C}'\mathbf{C}\mathbf{f}$$

Solution :

$$\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda\mathbf{C}'\mathbf{C}]^{-1} H'\mathbf{g}$$

Regularization algorithms: Circulant approximation

Main Idea : expand the vectors $\mathbf{f}$, $\mathbf{h}$ and $\mathbf{g}$ by the zeros to obtain $\mathbf{g}_e = \mathbf{H}_e \mathbf{f}_e$ with $\mathbf{H}_e$ a circulante matrix

$$\mathbf{f}_e(i) = \begin{cases} x(i) & i = 1, \ldots, Nx \\ 0 & i = Nx + 1, \ldots, P \geq Nx + Nh - 1 \end{cases}$$

$$y_e(i) = \begin{cases} y(i) & i = 1, \ldots, Ny \\ 0 & i = Ny + 1, \ldots, P \end{cases}$$

$$h_e(i) = \begin{cases} h(i) & i = 1, \ldots, Nh \\ 0 & i = Nh + 1, \ldots, P \end{cases}$$

$$y_e(k) = \sum_{i=0}^{Nh-1} x_e(k-i) h_e(i) \quad \longrightarrow \quad \mathbf{g}_e = \mathbf{H}_e \mathbf{f}_e$$

with $\mathbf{H}_e$ a circulante matrix whcich can diagonalized by FFT

Regularization algorithms: Circulant approximation

$$\mathbf{H}_e = \mathbf{F}\Lambda\mathbf{F}^{-1} \text{ with } \mathbf{F}[k,l] = \exp\left\{j2\pi\frac{kl}{P}\right\} \quad \mathbf{F}^{-1}[k,l] = \frac{1}{P}\exp\left\{-j2\pi\frac{kl}{P}\right\}$$

$$\boldsymbol{\Lambda} = \text{diag}[\lambda_1, \ldots, \lambda_P] \text{ and } [\lambda_1, \ldots, \lambda_P] = \text{TFD } [h_1, \ldots, h_{Nh}, 0, \ldots, 0]$$

$$c = [1, -2, 1] \quad c_e(i) = \begin{cases} c(i) & i = 1, \ldots, 3 \\ 0 & i = 4, \ldots, P \end{cases}$$

$$\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda\mathbf{C}'\mathbf{C}]^{-1}\mathbf{H}'\mathbf{g} \longrightarrow \mathbf{F}\widehat{\mathbf{f}}_e = [\Lambda_h'\Lambda_h + \lambda\Lambda_c'\Lambda_c]^{-1}\Lambda_h'\mathbf{F}\mathbf{g}$$

$$\text{TFD } \{\mathbf{f}_e\} = [\Lambda_h'\Lambda_h + \lambda\Lambda_c'\Lambda_c]^{-1}\Lambda_h'\text{TFD } \{\mathbf{g}\}$$

$$\widehat{\mathbf{f}}(\omega) = \frac{1}{H(\omega)}\frac{|H(\omega)|^2}{|H(\omega)|^2 + \lambda|C(\omega)|^2}\, y(\omega)$$

Link with Wiener filter: $C(\omega) = N(\omega)/X(\omega)$

Image Restoration

C Convolution matrix with the following impulse response:

$$H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned}\Omega(\mathbf{f}) = \sum\sum \quad & (f(i+1,j) + f(i-1,j) \\ & + f(i+1,j+1) + f(i-1,j+1) - 4f(i,j))^2\end{aligned}$$

$$x_e(k,l) = \begin{cases} x(k,l) & k = 1,\ldots,K \qquad l = 1,\ldots,L \\ 0 & k = K+1,\ldots,P \quad l = L+1,\ldots,P \end{cases}$$

Regularization: Iterative methods: Gradient based

$$\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \{J(\mathbf{f}) = Q(\mathbf{f}) + \lambda\Omega(\mathbf{f})\}$$

Let note : $\mathbf{g}^k = \nabla J\left(\mathbf{f}^k\right)$ gradient, $\mathbf{H}^k = \nabla^2 J\left(\mathbf{f}^k\right)$ Hessien.

First order gradient methods

- fixed step:

$$\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \alpha \mathbf{g}^{(k)} \qquad \alpha \quad \text{fixe}$$

- Optimal or steepest descente step:

$$\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \alpha^{(k)} \mathbf{g}^{(k)}$$

$$\alpha^{(k)} = -\frac{\mathbf{g}^{(k)t}\mathbf{g}^{(k)}}{\mathbf{g}^{(k)t}\mathbf{H}^k\mathbf{g}^{(k)}} = \frac{||\mathbf{g}^k\|^2}{\|\mathbf{g}^k\|_H^2}$$

Regularization: Iterative methods: Conjugate Gradient

- **Conjugate Gradient (CG)**

$$\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \alpha^{(k)} \mathbf{d}^{(k)} \quad \alpha^{(k)} = -\frac{\mathbf{d}^{(k)t} \mathbf{g}^{(k)}}{\mathbf{d}^{(k)t} \mathbf{H}^{k} \mathbf{d}^{(k)}}$$

$$\mathbf{d}^{(k+1)} = \mathbf{d}^{(k)} + \beta^{(k)} \mathbf{g}^{(k)} \quad \beta^{(k)} = -\frac{\mathbf{g}^{(k)t} \mathbf{g}^{(k)}}{\mathbf{g}^{(k-1)t} \mathbf{g}^{(k-1)}}$$

- **Newton method**

$$\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + (\mathbf{H}^{(k)})^{-1} \mathbf{g}^{(k)}$$

- Advantages : $\Omega(\mathbf{f})$ can be any convexe function
- Limitations : Computational cost

Regularization: Recursive algorithms

$$\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda \mathbf{D}]^{-1}\mathbf{H}'\mathbf{g}$$

Main idea: Express $\mathbf{f}_{i+1}$ as a function of $\mathbf{f}_i$

$$\mathbf{f}_{i+1} = (\mathbf{H}'_{i+1}\mathbf{H}_{i+1} + \alpha \mathbf{D})^{-1}\mathbf{H}'_{i+1}\mathbf{g}_{i+1}$$

$$\mathbf{f}_i = (\mathbf{H}_i^t\mathbf{H}_i + \alpha \mathbf{D})^{-1}\mathbf{H}_i^t\mathbf{g}_i$$

$$\Downarrow$$

$$\mathbf{f}_{i+1} = (\mathbf{H}_i^t\mathbf{H}_i + \mathbf{h}_{i+1}\mathbf{h}'_{i+1} + \alpha \mathbf{D})^{-1}(\mathbf{H}_i^t\mathbf{g}_i - \mathbf{h}_{i+1}\mathbf{g}_i + 1)$$

Noting:

$$\mathbf{P}_i = (\mathbf{H}_i^t\mathbf{H}_i + \alpha \mathbf{D})^{-1} \quad \text{and} \quad \mathbf{P}_{i+1}^t = \mathbf{P}_i^t + \mathbf{h}_{i+1}\mathbf{h}'_{i+1}$$

$$\Downarrow$$

$$\mathbf{f}_{i+1} = \mathbf{f}_i + \mathbf{P}_{i+1}\mathbf{h}_{i+1}(\mathbf{g}_{i+1} - \mathbf{h}'_{i+1}\mathbf{f}_i)$$

$$\mathbf{P}_{i+1} = \mathbf{P}_i - \mathbf{P}_i\mathbf{h}_{i+1}(\mathbf{h}'_{i+1}\mathbf{P}_i\mathbf{H}_{i+1} + \alpha)^{-1}\mathbf{h}'_{i+1}\mathbf{P}_i$$

Regularization: Wiener Filtering with FIR

$$g(m,n) = \sum_{i,j=-N}^{N} h(i,j)\, f(m-i,n-j) + b(m,n)$$

$$\widehat{f}(m,n) = \sum_{i,j=-M}^{M} w(i,j)\, f(m-i,n-j)$$

$$\mathrm{EQM} = \mathrm{E}\left\{(\widehat{f}(m,n) - f(m,n))^2\right\}$$

Orthogonality condition:

$$\mathrm{E}\left\{(\widehat{f}(m,n) - f(m,n))\, g(m-k,n-l)\right\} = 0, \quad \forall k,l = -M,\ldots,M$$

$(2M+1)^2$ linear equations

$$R_{fg}(k,l) - \sum_{i,j=-M}^{M} w(i,j)\, R_{ff}(k-i,l-j) = 0, \quad \forall k,l = -M,\ldots,M$$

Regularization: Wiener Filtering

Linearity and white Gaussian noise variance σ_b^2 :

$$R_{gg}(k,l) = R_{ff}(k,l) * a(k,l) + \sigma_b^2 \delta(k,l)$$

$$a(k,l) \overset{\triangle}{=} h(k,l) \star h(k,l) = \sum_{i,j=-\infty}^{\infty}\sum h(i,j)\, h(i+k, j+k)$$

$$R_{fg}(k,l) = h(k,l) \star R_{ff}(k,l) = \sum_{i,j=-\infty}^{\infty}\sum h(i,j) R_{ff}(i+k, j+k)$$

$$r_0(k,l) \overset{\triangle}{=} \frac{R_{ff}(k,l)}{R_{ff}(0,0)} = \frac{R_{ff}(k,l)}{\sigma_f^2}$$

$$r_0(-k,-l) = r_0(k,l)$$

$$\left[\frac{\sigma_b^2}{\sigma_f^2}\delta(k,l) + r_0(k,l) \star a(k,l)\right] * w(k,l) = h(k,l) * r_0(k,l), \ \forall k,l = -M, \ldots, M$$

Regularization: Wiener Filtering

- Equation

$$\left[\frac{\sigma_b^2}{\sigma_f^2}\delta(k,l) + r_0(k,l) \star a(k,l)\right] * w(k,l) = h(k,l) * r_0(k,l)$$

- Matrix-Vector form:

$$\left[\frac{\sigma_b^2}{\sigma_f^2}\mathbf{I} + \mathbf{R}\right]\mathbf{g} = \mathbf{r}$$

- **R** is a Bloc–Toeplitz of $(2M+1) \times (2M+1)$ blocs, each bloc has the dimensions $(2M+1) \times (2M+1)$.
- **g** and **r** are two vectors of dimensions $(2M+1)^2 \times 1$ containing $g(k,l)$ and $h(k,l) * r_0(k,l)$.

A more direct presentation

$$g(m,n)=\sum_{i,j=-N}^{N}\sum h(i,j)\,f(m-i,n-j)+\epsilon(m,n)\longrightarrow \mathbf{g}=\mathbf{H}\mathbf{f}+\boldsymbol{\epsilon}$$

$$\widehat{f}(m,n)=\sum_{i,j=-M}^{M}\sum w(i,j)\,f(m-i,n-j)\longrightarrow \widehat{\mathbf{f}}=\mathbf{W}\mathbf{f}$$

$$\mathrm{EQM}=\mathrm{E}\left\{\sum_{i,j=-M}^{M}\sum(\widehat{f}(m,n)-f(m,n))^2\right\}=\mathrm{E}\left\{[\widehat{\mathbf{f}}-\mathbf{f}]'[\widehat{\mathbf{f}}-\mathbf{f}]\right\}$$

Orthogonality:

$$\mathrm{E}\left\{[\widehat{\mathbf{f}}-\mathbf{f}]\mathbf{g}'\right\}=0$$

$$\mathbf{W}=\mathrm{E}\left\{\mathbf{f}\mathbf{g}'\right\}[\mathrm{E}\left\{\mathbf{g}\mathbf{g}'\right\}]^{-1}=\mathbf{R}_{ff}\mathbf{H}'[\mathbf{H}\mathbf{R}_{ff}\mathbf{H}'+\mathbf{R}_{\epsilon\epsilon}]^{-1}$$

$$\mathbf{R}_{ff}=\mathrm{E}\left\{\mathbf{f}\mathbf{f}'\right\},\quad \mathbf{R}_{\epsilon\epsilon}=\mathrm{E}\left\{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'\right\}\quad \mathbf{R}_{f\epsilon}=\mathrm{E}\left\{\mathbf{f}\boldsymbol{\epsilon}'\right\}=0$$

- Matrix Inversion Lemma:

$$\mathbf{W} = \mathbf{R}_{\mathit{ff}}\mathbf{H}'[\mathbf{H}\mathbf{R}_{\mathit{ff}}\mathbf{H}' + \mathbf{R}_{\epsilon\epsilon}]^{-1} = [\mathbf{H}'\mathbf{R}_{\epsilon\epsilon}^{-1}\mathbf{H} + \mathbf{R}_{\mathit{ff}}^{-1}]^{-1}\mathbf{H}'\mathbf{R}_{\epsilon\epsilon}^{-1}$$

- $\mathbf{R}_{\epsilon\epsilon} = \boldsymbol{\sigma}^2\mathbf{I}$

$$\mathbf{W} = \mathbf{R}_{\mathit{ff}}\mathbf{H}'[\mathbf{H}\mathbf{R}_{\mathit{ff}}\mathbf{H} + \sigma_b^2\mathbf{I}]^{-1} = [\mathbf{H}\mathbf{H}' + \sigma_b^2\mathbf{R}_{\mathit{ff}}^{-1}]^{-1}\mathbf{H}'$$

- Particular Case: $\sigma_b^2 = 0$

$$\mathbf{W} = \begin{cases} [\mathbf{H}'\mathbf{H}]^{-1}\mathbf{H}' & \longrightarrow \widehat{[\mathbf{f}} = \mathbf{H}'\mathbf{H}]^{-1}\mathbf{H}'\mathbf{g} \\ \mathbf{R}_{\mathit{ff}}\mathbf{H}'[\mathbf{H}\mathbf{R}_{\mathit{ff}}\mathbf{H}']^{-1} \longrightarrow & \longrightarrow \widehat{\mathbf{f}} = \mathbf{R}_{\mathit{ff}}\mathbf{H}'[\mathbf{H}\mathbf{R}_{\mathit{ff}}\mathbf{H}']^{-1}\mathbf{g} \end{cases}$$

Generalized Wiener filter using unitary transforms

$$\mathbf{W} = [\mathbf{H}'\mathbf{H} + \sigma_b^2 \mathbf{R}_{ff}^{-1}]^{-1}\mathbf{H}'$$

$$\mathbf{P} \overset{\triangle}{=} [\mathbf{H}'\mathbf{H} + \sigma_b^2 \mathbf{R}_{ff}^{-1}]^{-1}$$

Consider a unitary transform **F**, *i.e.* $\mathbf{F}'\mathbf{F} = \mathbf{F}\mathbf{F}' = \mathbf{I}$

$$\widehat{\mathbf{f}} = \mathbf{F}'[\mathbf{F}\mathbf{P}\mathbf{F}']\mathbf{F}\mathbf{H}'\mathbf{g} \overset{\triangle}{=} \mathbf{F}'\bar{\mathbf{P}}\mathbf{z}$$

$$\bar{\mathbf{P}} \overset{\triangle}{=} [\mathbf{F}\mathbf{P}\mathbf{F}'], \quad \mathbf{z} \overset{\triangle}{=} \mathbf{F}\mathbf{H}'\mathbf{g}$$

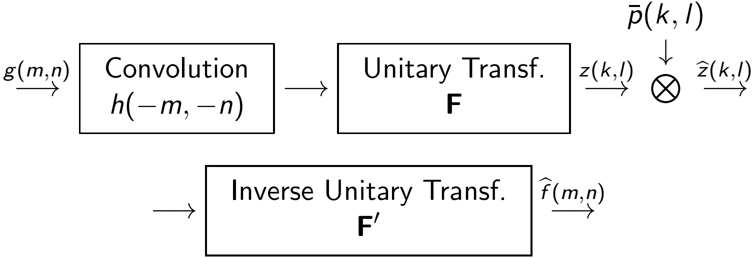
$$\mathbf{g} \longrightarrow \boxed{\mathbf{H}'} \longrightarrow \boxed{\mathbf{F}} \longrightarrow \mathbf{z} \longrightarrow \boxed{\bar{\mathbf{P}} = [\mathbf{F}\mathbf{P}\mathbf{F}']} \longrightarrow \widehat{\mathbf{z}} \longrightarrow \boxed{\mathbf{F}'} \longrightarrow \widehat{\mathbf{f}}$$

For an appropriate unitary transforms $\bar{\mathbf{P}}$ becomes an almost diagonal matrix

$$\widehat{\mathbf{z}} = \bar{\mathbf{P}}\mathbf{z} \quad \Longrightarrow \quad \widehat{z}(k,l) \simeq \bar{p}(k,l)\, z(k,l)$$

Generalized Wiener filter using unitary transforms

If **H** corresponds to convolution matrix $h(m,n)$, $\mathbf{H}'$ is also a convolution matrix with $h(-m,-n)$



$\bar{p}(k,l)$

$\xrightarrow{g(m,n)}$ Convolution $h(-m,-n)$ $\longrightarrow$ Unitary Transf. $\mathbf{F}$ $\xrightarrow{z(k,l)}$ $\bigotimes$ $\xrightarrow{\widehat{z}(k,l)}$

$\longrightarrow$ Inverse Unitary Transf. $\mathbf{F}'$ $\xrightarrow{\widehat{f}(m,n)}$

Content

1. Introduction: Signals and Images, Linear transformations (Convolution, Fourier, Laplace, Hilbert, Radon, ..., Discrete convolution, Z transform, DFT, FFT, ...)
2. Modeling: parametric and non-parametric, MA, AR and ARMA models, Linear prediction
3. Deconvolution and Parameter Estimation: Deterministic (LS, Regularization) and Probabilistic methods (Wiener Filtering)
4. **Inverse problems, Regularization and Bayesian estimation**
5. Kalman Filtering
6. Case study: Signal deconvolution
7. Case study: Image restoration
8. Case study: Image reconstruction and Computed Tomography

4. Inverse problems and Bayesian estimation

- Inverse problems? Why need for Bayesian approach?
- Probability ?
- Discrete and Continuous variables
- Bayes rule and Bayesian inference
- Maximum Likelihood method (ML) and its limitations
- Bayesian estimation theory
- Bayesian inference for inverse problems in signal and image processing
- Prior modeling
- Bayesian computation (Laplace approximation, MCMC, BVA)

Inverse problems : 3 main examples

- Example 1:
  Measuring variation of temperature with a therometer
    - $f(t)$ variation of temperature over time
    - $g(t)$ variation of length of the liquid in thermometer
- Example 2:
  Making an image with a camera, a microscope or a telescope
    - $f(x,y)$ real scene
    - $g(x,y)$ observed image
- Example 3: Making an image of the interior of a body
    - $f(x,y)$ a section of a real 3D body $f(x,y,z)$
    - $g_\phi(r)$ a line of observed radiographe $g_\phi(r,z)$

- Example 1: Deconvolution
- Example 2: Image restoration
- Example 3: Image reconstruction

Measuring variation of temperature with a therometer

- $f(t)$ variation of temperature over time
- $g(t)$ variation of length of the liquid in thermometer

- Forward model: Convolution

$$g(t) = \int f(t')\, h(t-t')\, \mathrm{d}t' + \epsilon(t)$$

$h(t)$: impulse response of the measurement system

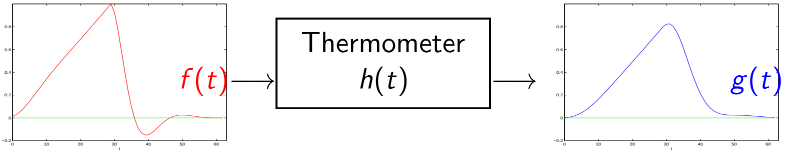
- Inverse problem: Deconvolution

Given the forward model $\mathcal{H}$ (impulse response $h(t)$))
and a set of data $g(t_i), i = 1, \cdots, M$
find $f(t)$

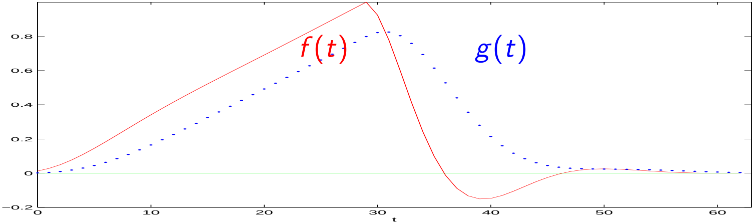
Measuring variation of temperature with a therometer

Forward model: Convolution

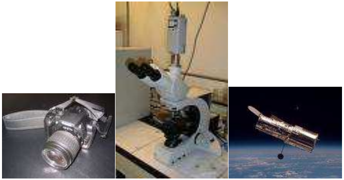
$$g(t) = \int f(t') \, h(t - t') \, \mathrm{d}t' + \epsilon(t)$$



Inversion: Deconvolution



Making an image with a camera, a microscope or a telescope



- $f(x,y)$ real scene
- $g(x,y)$ observed image
- Forward model: Convolution

$$g(x,y) = \iint f(x',y')\, h(x-x',y-y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x,y)$$

  $h(x,y)$: Point Spread Function (PSF) of the imaging system

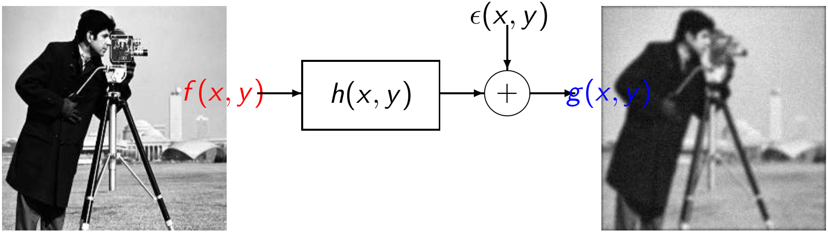
- Inverse problem: Image restoration

  Given the forward model $\mathcal{H}$ (PSF $h(x,y)$))
  and a set of data $g(x_i,y_i), i=1,\cdots,M$
  find $f(x,y)$

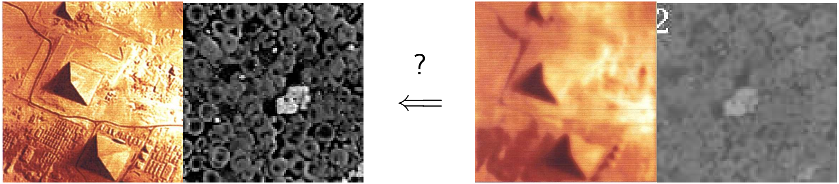
Making an image with an unfocused camera

Forward model: 2D Convolution

$$g(x,y)=\iint f(x',y')\,h(x-x',y-y')\,\mathrm{d}x'\,\mathrm{d}y'+\epsilon(x,y)$$

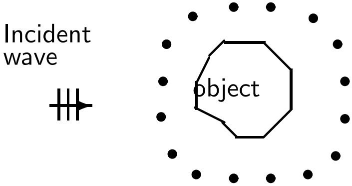


Inversion: Deconvolution

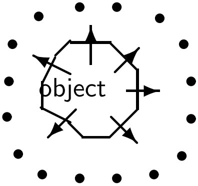


Making an image of the interior of a body

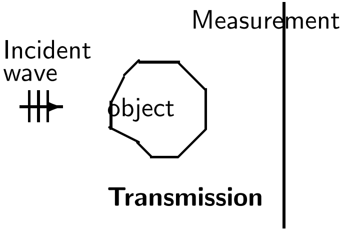
Different imaging systems:



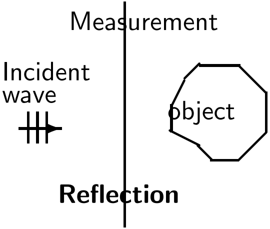
Active Imaging



Passive Imaging



Transmission



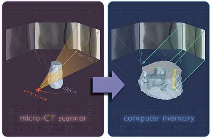
Reflection

Forward problem: Knowing the object predict the data

Inverse problem: From measured data find the object

Making an image of the interior of a body

- $f(x,y)$ a section of a real 3D body $f(x,y,z)$
- $g_\phi(r)$ a line of observed radiographe $g_\phi(r,z)$



- Forward model:
  Line integrals or Radon Transform

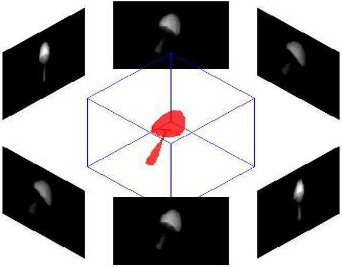
$$
\begin{aligned}
g_\phi(r) &= \int_{L_{r,\phi}} f(x,y)\,\mathrm{d}l + \epsilon_\phi(r) \\
&= \iint f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y + \epsilon_\phi(r)
\end{aligned}
$$

- Inverse problem: Image reconstruction

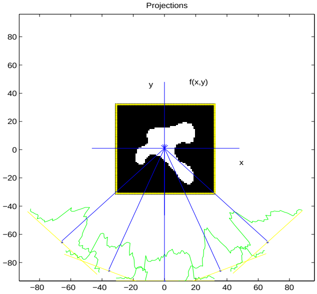
  Given the forward model $\mathcal{H}$ (Radon Transform) and
  a set of data $g_{\phi_i}(r), i = 1, \cdots, M$
  find $f(x,y)$

2D and 3D Computed Tomography

3D



2D



$$g_\phi(r_1, r_2) = \int_{\mathcal{L}_{r_1,r_2,\phi}} f(x, y, z) \, \mathrm{d}l \qquad g_\phi(r) = \int_{\mathcal{L}_{r,\phi}} f(x, y) \, \mathrm{d}l$$

Forward probelm: $f(x, y)$ or $f(x, y, z) \longrightarrow g_\phi(r)$ or $g_\phi(r_1, r_2)$

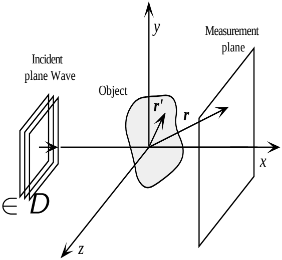
Inverse problem: $g_\phi(r)$ or $g_\phi(r_1, r_2) \longrightarrow f(x, y)$ or $f(x, y, z)$

Microwave or ultrasound imaging

Measurs: diffracted wave by the object $g(\mathbf{r}_i)$

Unknown quantity: $f(\mathbf{r}) = k_0^2(n^2(\mathbf{r}) - 1)$

Intermediate quantity : $\phi(\mathbf{r})$

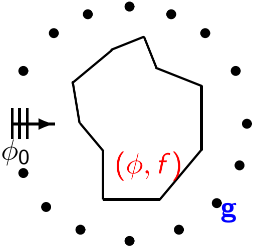


$$g(\mathbf{r}_i) = \iint_D G_m(\mathbf{r}_i, \mathbf{r}')\phi(\mathbf{r}') f(\mathbf{r}') \, \mathrm{d}\mathbf{r}', \; \mathbf{r}_i \in S$$

$$\phi(\mathbf{r}) = \phi_0(\mathbf{r}) + \iint_D G_o(\mathbf{r}, \mathbf{r}')\phi(\mathbf{r}') f(\mathbf{r}') \, \mathrm{d}\mathbf{r}', \; \mathbf{r} \in D$$

Born approximation $(\phi(\mathbf{r}') \simeq \phi_0(\mathbf{r}'))$ **):**

$$g(\mathbf{r}_i) = \iint_D G_m(\mathbf{r}_i, \mathbf{r}')\phi_0(\mathbf{r}') f(\mathbf{r}') \, \mathrm{d}\mathbf{r}', \; \mathbf{r}_i \in S$$



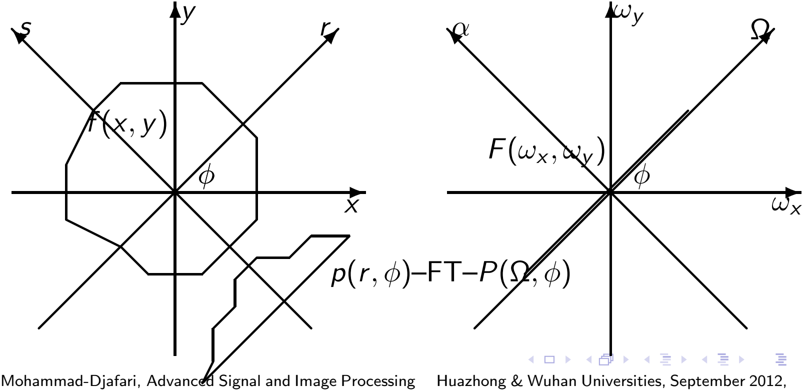
Discretization :

$$\begin{cases} \mathbf{g} = \mathbf{G}_m \mathbf{F} \phi \\ \phi = \phi_0 + \mathbf{G}_o \mathbf{F} \phi \end{cases} \longrightarrow \begin{cases} \mathbf{g} = \mathbf{H}(\mathbf{f}) \\ \text{with} \quad \mathbf{F} = \mathrm{diag}(\mathbf{f}) \\ \mathbf{H}(\mathbf{f}) = \mathbf{G}_m \mathbf{F} (\mathbf{I} - \mathbf{G}_o \mathbf{F})^{-1} \phi_0 \end{cases}$$

Fourier Synthesis in X ray Tomography

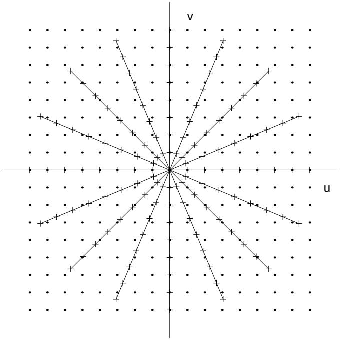
$$\begin{aligned} g(r,\phi) &= \iint f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y \\ G(\Omega,\phi) &= \int g(r,\phi)\exp\{-j\Omega r\}\;\mathrm{d}r \\ F(\omega_x,\omega_y) &= \iint f(x,y)\exp\{-j\omega_x x,\omega_y y\}\;\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

$$F(\omega_x,\omega_y) = G(\Omega,\phi) \qquad \text{for} \quad \omega_x = \Omega\cos\phi \quad \text{and} \quad \omega_y = \Omega\sin\phi$$



Fourier Synthesis in X ray tomography

$$G(\omega_x, \omega_y) = \iint f(x, y) \exp\left\{-j\left(\omega_x x + \omega_y y\right)\right\} \, \mathrm{d}x \, \mathrm{d}y$$



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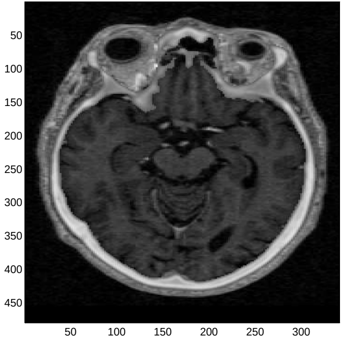
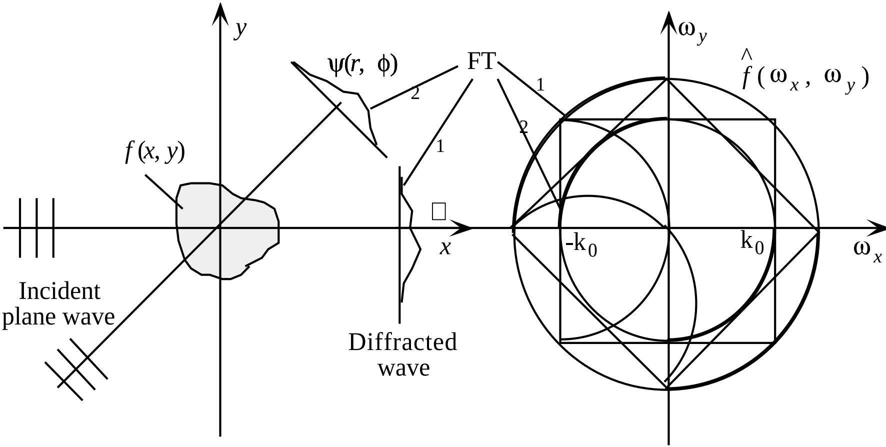
$\Longrightarrow$



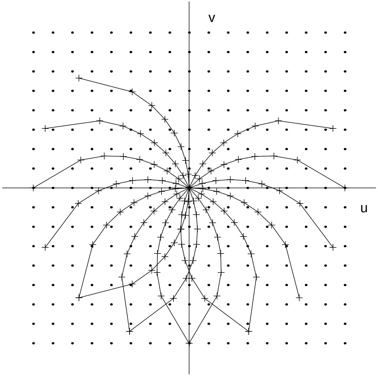
Forward problem: Given $f(x, y)$ compute $G(\omega_x, \omega_y)$
Inverse problem: Given $G(\omega_x, \omega_y)$ on those lines estimate $f(x, y)$

Fourier Synthesis in Diffraction tomography



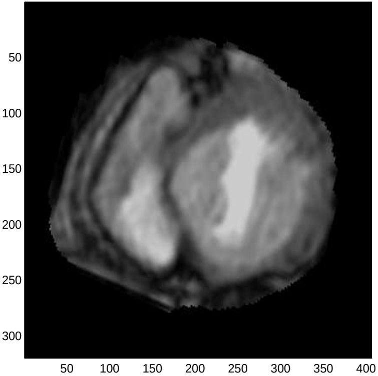
Fourier Synthesis in Diffraction tomography

$$G(\omega_x, \omega_y) = \iint f(x, y) \exp\left\{-j\left(\omega_x x + \omega_y y\right)\right\} \, \mathrm{d}x \, \mathrm{d}y$$



?

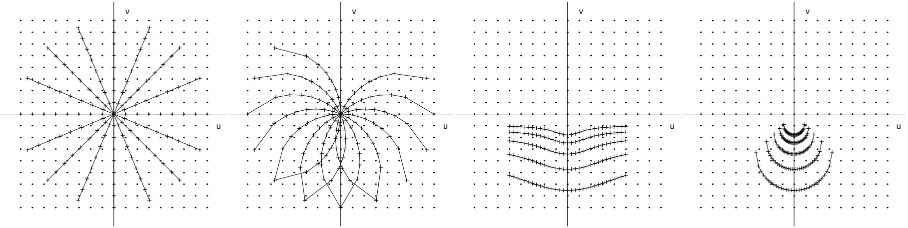
$\Longrightarrow$



Forward problem: Given $f(x, y)$ compute $G(\omega_x, \omega_y)$
Inverse problem : Given $G(\omega_x, \omega_y)$ on those semi cercles estimate $f(x, y)$

Fourier Synthesis in different imaging systems

$$G(\omega_x,\omega_y) = \iint f(x,y) \exp\left\{-j\left(\omega_x x + \omega_y y\right)\right\}\, \mathrm{d}x\, \mathrm{d}y$$



X ray Tomography   Diffraction   Eddy current   SAR & Radar

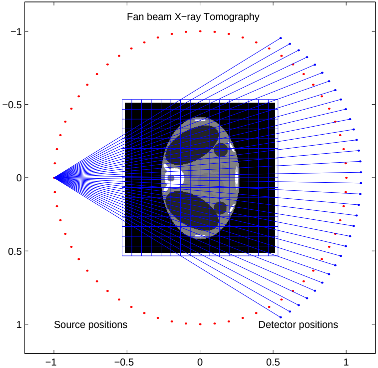
Forward problem: Given $f(x,y)$ compute $G(\omega_x,\omega_y)$
Inverse problem : Given $G(\omega_x,\omega_y)$ on those algebraic lines, cercles or curves, estimate $f(x,y)$

Invers Problems: other examples and applications

- X ray, Gamma ray Computed Tomography (CT)
- Microwave and ultrasound tomography
- Positron emission tomography (PET)
- Magnetic resonance imaging (MRI)
- Photoacoustic imaging
- Radio astronomy
- Geophysical imaging
- Non Destructive Evaluation (NDE) and Testing (NDT) techniques in industry
- Hyperspectral imaging
- Earth observation methods (Radar, SAR, IR, ...)
- Survey and tracking in security systems

Computed tomography (CT)

A Multislice CT Scanner



$$g(s_i) = \int_{L_i} f(\mathbf{r}) \; \mathrm{d}l_i + \epsilon(s_i)$$

Discretization

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \epsilon$$

Positron emission tomography (PET)

Magnetic resonance imaging (MRI)

Nuclear magnetic resonance imaging (NMRI), Para-sagittal MRI of the head

Radio astronomy (interferometry imaging systems)

The Very Large Array in New Mexico, an example of a radio telescope.

General formulation of inverse problems

- General non linear inverse problems:

$$g(\mathbf{s}) = [\mathcal{H} f(\mathbf{r})](\mathbf{s}) + \epsilon(\mathbf{s}), \quad \mathbf{r} \in \mathcal{R}, \quad \mathbf{s} \in \mathcal{S}$$

- Linear models:

$$g(\mathbf{s}) = \int f(\mathbf{r})\, h(\mathbf{r},\mathbf{s})\, \mathrm{d}\mathbf{r} + \epsilon(\mathbf{s})$$

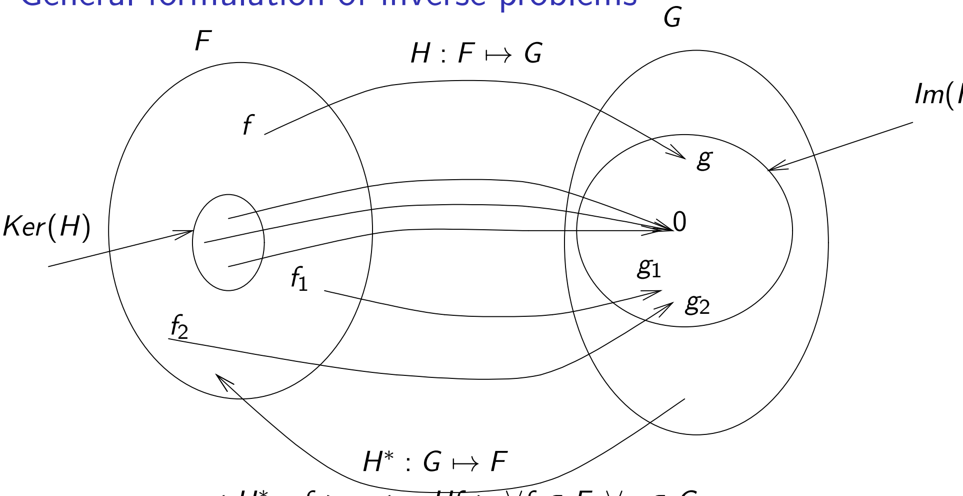
  If $h(\mathbf{r},\mathbf{s}) = h(\mathbf{r}-\mathbf{s}) \longrightarrow$ Convolution.

- Discrete data:

$$g(\mathbf{s}_i) = \int h(\mathbf{s}_i,\mathbf{r})\, f(\mathbf{r})\, \mathrm{d}\mathbf{r} + \epsilon(\mathbf{s}_i), \quad i = 1,\cdots,m$$

- Inversion: Given the forward model $\mathcal{H}$ and the data $\mathbf{g} = \{g(\mathbf{s}_i),\, i = 1,\cdots,m)\}$ estimate $f(\mathbf{r})$
- Well-posed and Ill-posed problems (Hadamard): existance, uniqueness and stability
- Need for prior information

General formulation of inverse problems



$$< H^* g, f > = < g, Hf > \forall f \in F, \forall g \in G$$

Analytical methods (mathematical physics)

$$g(\mathbf{s}_i) = \int h(\mathbf{s}_i, \mathbf{r})\, f(\mathbf{r})\, \mathrm{d}\mathbf{r} + \epsilon(\mathbf{s}_i), \quad i = 1, \cdots, m$$

$$g(\mathbf{s}) = \int h(\mathbf{s}, \mathbf{r})\, f(\mathbf{r})\, \mathrm{d}\mathbf{r}$$

$$\widehat{f}(\mathbf{r}) = \int w(\mathbf{s}, \mathbf{r})\, g(\mathbf{s})\, \mathrm{d}\mathbf{s}$$

$w(\mathbf{s}, \mathbf{r})$ minimizing a criterion:

$$\begin{aligned} Q(w(\mathbf{s}, \mathbf{r})) &= \left\| g(\mathbf{s}) - [\mathcal{H}\,\widehat{f}(\mathbf{r})](\mathbf{s}) \right\|_2^2 = \int \left| g(\mathbf{s}) - [\mathcal{H}\,\widehat{f}(\mathbf{r})](\mathbf{s}) \right|^2 \, \mathrm{d}\mathbf{s} \\ &= \int \left| g(\mathbf{s}) - \int h(\mathbf{s}, \mathbf{r})\, \widehat{f}(\mathbf{r})\, \mathrm{d}\mathbf{r} \right|^2 \, \mathrm{d}\mathbf{s} \\ &= \int \left| g(\mathbf{s}) - \int\int h(\mathbf{s}, \mathbf{r}) w(\mathbf{s}, \mathbf{r})\, g(\mathbf{s})\, \mathrm{d}\mathbf{s}\, \mathrm{d}\mathbf{r} \right|^2 \, \mathrm{d}\mathbf{s} \end{aligned}$$

Trivial solution: $h(\mathbf{s}, \mathbf{r}) w(\mathbf{s}, \mathbf{r}) = \delta(\mathbf{r})\delta(\mathbf{s})$

Analytical methods

- Trivial solution:

$$w(\mathbf{s},\mathbf{r}) = h^{-1}(\mathbf{s},\mathbf{r})$$

  Example: Fourier Transform:

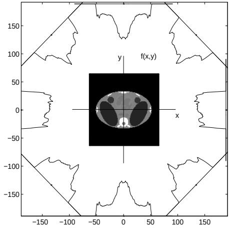
$$g(\mathbf{s}) = \int f(\mathbf{r}) \exp\left\{-j\mathbf{s}.\mathbf{r}\right\} \, \mathrm{d}\mathbf{r}$$

$$h(\mathbf{s},\mathbf{r}) = \exp\left\{-j\mathbf{s}.\mathbf{r}\right\} \longrightarrow w(\mathbf{s},\mathbf{r}) = \exp\left\{+j\mathbf{s}.\mathbf{r}\right\}$$

$$\hat{f}(\mathbf{r}) = \int g(\mathbf{s}) \exp\left\{+j\mathbf{s}.\mathbf{r}\right\} \, \mathrm{d}\mathbf{s}$$

- Known classical solutions for specific expressions of $h(\mathbf{s},\mathbf{r})$:
  - 1D cases: 1D Fourier, Hilbert, Weil, Melin, ...
  - 2D cases: 2D Fourier, Radon, ...

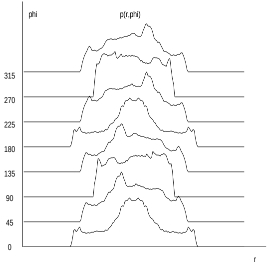
X ray Tomography



$$g(r,\phi) = -\ln\left(\frac{I}{I_0}\right) = \int_{L_{r,\phi}} f(x,y)\,\mathrm{d}l$$

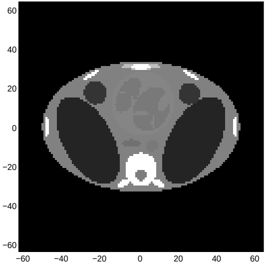
$$g(r,\phi) = \iint_D f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

$$f(x,y) \longrightarrow \boxed{\text{RT}} \longrightarrow g(r,\phi)$$

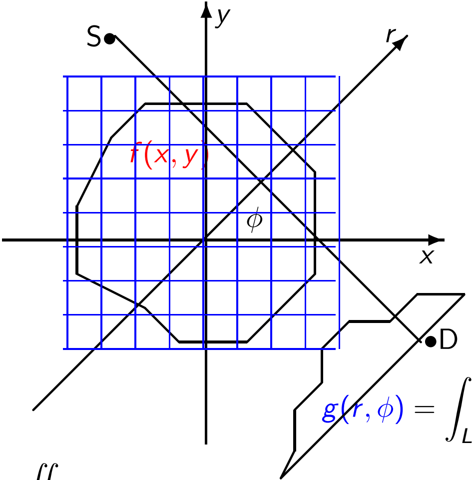


IRT ?

$$\Longrightarrow$$



Analytical Inversion methods



$$g(r,\phi) = \int_L f(x,y)\, \mathrm{d}l$$

Radon:

$$g(r,\phi) = \iint_D f(x,y)\, \delta(r - x\cos\phi - y\sin\phi)\, \mathrm{d}x\, \mathrm{d}y$$

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right) \int_0^{\pi} \int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} g(r,\phi)}{(r - x\cos\phi - y\sin\phi)}\, \mathrm{d}r\, \mathrm{d}\phi$$

Filtered Backprojection method

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right) \int_0^{\pi} \int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} g(r,\phi)}{(r - x\cos\phi - y\sin\phi)} \, \mathrm{d}r \, \mathrm{d}\phi$$

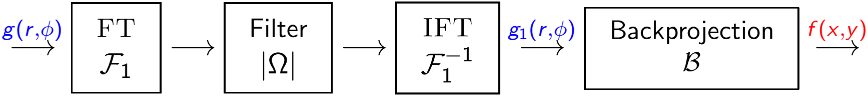
$$\text{Derivation } \mathcal{D} : \qquad \overline{g}(r,\phi) = \frac{\partial g(r,\phi)}{\partial r}$$

$$\text{Hilbert Transform} \mathcal{H} : \qquad g_1(r',\phi) = \frac{1}{\pi} \int_0^{\infty} \frac{\overline{g}(r,\phi)}{(r - r')} \, \mathrm{d}r$$

$$\text{Backprojection } \mathcal{B} : \ f(x,y) = \frac{1}{2\pi} \int_0^{\pi} g_1(r' = x\cos\phi + y\sin\phi, \phi) \, \mathrm{d}\phi$$

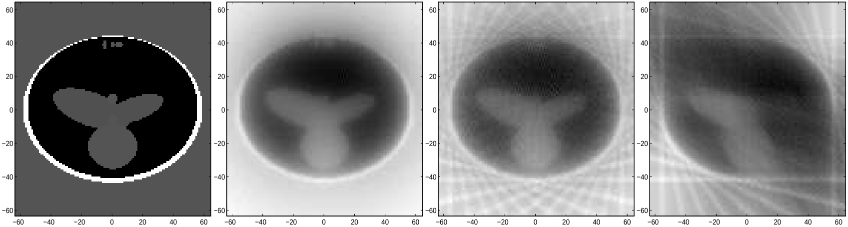
$$f(x,y) = \mathcal{B}\, \mathcal{H}\, \mathcal{D}\, g(r,\phi) = \mathcal{B}\, \mathcal{F}_1^{-1}\, |\Omega|\, \mathcal{F}_1\, g(r,\phi)$$

- Backprojection of filtered projections:



$$\xrightarrow{g(r,\phi)} \boxed{\begin{matrix}\text{FT} \\ \mathcal{F}_1\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{Filter} \\ |\Omega|\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{IFT} \\ \mathcal{F}_1^{-1}\end{matrix}} \xrightarrow{g_1(r,\phi)} \boxed{\begin{matrix}\text{Backprojection} \\ \mathcal{B}\end{matrix}} \xrightarrow{f(x,y)}$$

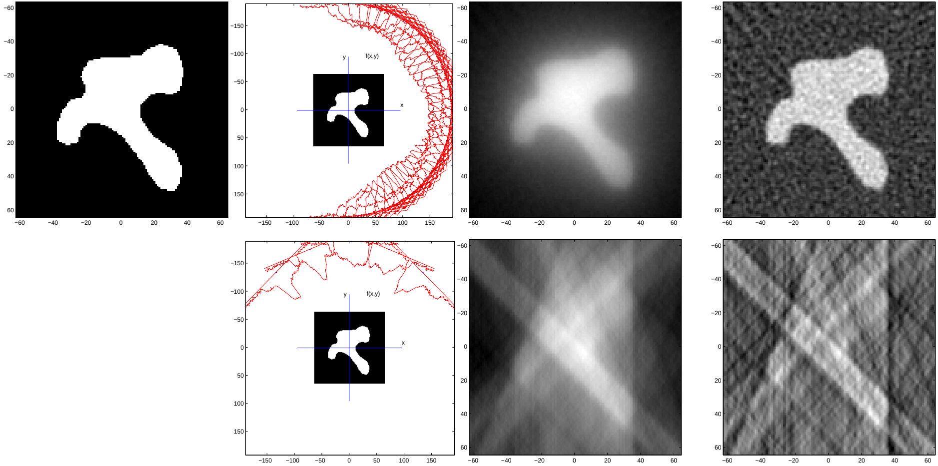
Limitations : Limited angle or noisy data



Original 64 proj. 16 proj. 8 proj. $[0, \pi/2]$

- Limited angle or noisy data
- Accounting for detector size
- Other measurement geometries: fan beam, ...

Limitations : Limited angle or noisy data



Original Data Backprojection Filtered Backprojectio

Parametric methods

- $f(\mathbf{r})$ is described in a parametric form with a very few number of parameters $\boldsymbol{\theta}$ and one searches $\widehat{\boldsymbol{\theta}}$ which minimizes a criterion such as:
- Least Squares (LS): $Q(\boldsymbol{\theta}) = \sum_i |g_i - [\mathcal{H} f(\boldsymbol{\theta})]_i|^2$
- Robust criteria : $Q(\boldsymbol{\theta}) = \sum_i \phi(|g_i - [\mathcal{H} f(\boldsymbol{\theta})]_i|)$
  with different functions ϕ $(L_1$, Hubert, ...).
- Likelihood : $\mathcal{L}(\boldsymbol{\theta}) = -\ln p(\mathbf{g}|\boldsymbol{\theta})$
- Penalized likelihood : $\mathcal{L}(\boldsymbol{\theta}) = -\ln p(\mathbf{g}|\boldsymbol{\theta}) + \lambda\Omega(\boldsymbol{\theta})$

Examples:

- Spectrometry: $f(t)$ modelled as a sum og gaussians
  $f(t) = \sum_{k=1}^{K} a_k \mathcal{N}(t|\mu_k, v_k)$ $\quad \boldsymbol{\theta} = \{a_k, \mu_k, v_k\}$
- Tomography in CND: $f(x, y)$ is modelled as a superposition of circular or elleiptical discs $\quad \boldsymbol{\theta} = \{a_k, \mu_k, r_k\}$

Non parametric methods

$$g(\mathbf{s}_i) = \int h(\mathbf{s}_i, \mathbf{r})\, f(\mathbf{r})\, \mathrm{d}\mathbf{r} + \epsilon(\mathbf{s}_i), \quad i = 1, \cdots, M$$

- $f(\mathbf{r})$ is assumed to be well approximated by

$$f(\mathbf{r}) \simeq \sum_{j=1}^{N} f_j\, b_j(\mathbf{r})$$

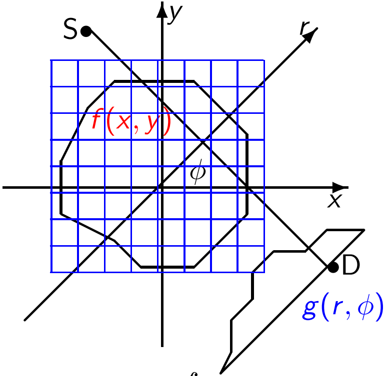
with $\{b_j(\mathbf{r})\}$ a basis or any other set of known functions

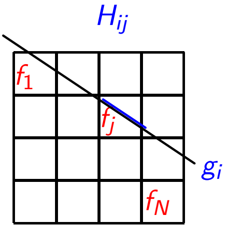
$$g(\mathbf{s}_i) = g_i \simeq \sum_{j=1}^{N} f_j \int h(\mathbf{s}_i, \mathbf{r})\, b_j(\mathbf{r})\, \mathrm{d}\mathbf{r}, \quad i = 1, \cdots, M$$

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon} \ \text{ with } \ H_{ij} = \int h(\mathbf{s}_i, \mathbf{r})\, b_j(\mathbf{r})\, \mathrm{d}\mathbf{r}$$

- $\mathbf{H}$ is huge dimensional
- LS solution : $\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \{Q(\mathbf{f})\}$ with $Q(\mathbf{f}) = \sum_i |g_i - [\mathbf{H}\mathbf{f}]_i|^2 = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2$ does not give satisfactory result.

Algebraic methods: Discretization





$$f(x,y)=\sum_j f_j\, b_j(x,y)$$

$$b_j(x,y)=\begin{cases}1 & \text{if } (x,y)\in \text{ pixel } j\\ 0 & \text{else}\end{cases}$$

$$g(r,\phi)=\int_L f(x,y)\,\mathrm{d}l$$

$$g_i=\sum_{j=1}^{N} H_{ij}\; f_j+\epsilon_i$$

$$\mathbf{g}=\mathbf{H}\mathbf{f}+\epsilon$$

Inversion: Deterministic methods

Data matching

- Observation model
  $g_i = h_i(\mathbf{f}) + \epsilon_i, \quad i = 1, \ldots, M \longrightarrow \mathbf{g} = \mathbf{H}(\mathbf{f}) + \boldsymbol{\epsilon}$
- Misatch between data and output of the model $\Delta(\mathbf{g}, \mathbf{H}(\mathbf{f}))$

$$\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \{\Delta(\mathbf{g}, \mathbf{H}(\mathbf{f}))\}$$

- Examples:

– LS $\quad \Delta(\mathbf{g}, \mathbf{H}(\mathbf{f})) = \|\mathbf{g} - \mathbf{H}(\mathbf{f})\|^2 = \sum_i |g_i - h_i(\mathbf{f})|^2$

– L_p $\quad \Delta(\mathbf{g}, \mathbf{H}(\mathbf{f})) = \|\mathbf{g} - \mathbf{H}(\mathbf{f})\|^p = \sum_i |g_i - h_i(\mathbf{f})|^p, \quad 1 < p < 2$

– KL $\quad \Delta(\mathbf{g}, \mathbf{H}(\mathbf{f})) = \sum_i g_i \ln \frac{g_i}{h_i(\mathbf{f})}$

- In general, does not give satisfactory results for inverse problems.

Regularization theory

Inverse problems = Ill posed problems

$\longrightarrow$ Need for prior information

Functional space (Tikhonov):

$$g = \mathcal{H}(f) + \epsilon \longrightarrow J(f) = ||g - \mathcal{H}(f)||_2^2 + \lambda||\mathcal{D}f||_2^2$$

Finite dimensional space (Philips & Towmey): $\mathbf{g} = \mathbf{H}(\mathbf{f}) + \boldsymbol{\epsilon}$

- Minimum norme LS (MNLS): $J(\mathbf{f}) = ||\mathbf{g} - \mathbf{H}(\mathbf{f})||^2 + \lambda||\mathbf{f}||^2$
- Classical regularization: $J(\mathbf{f}) = ||\mathbf{g} - \mathbf{H}(\mathbf{f})||^2 + \lambda||\mathbf{D}\mathbf{f}||^2$
- More general regularization:

$$J(\mathbf{f}) = \mathcal{Q}(\mathbf{g} - \mathbf{H}(\mathbf{f})) + \lambda\Omega(\mathbf{D}\mathbf{f})$$

or

$$J(\mathbf{f}) = \Delta_1(\mathbf{g}, \mathbf{H}(\mathbf{f})) + \lambda\Delta_2(\mathbf{f}, \mathbf{f}_\infty)$$

Limitations:

- Errors are considered implicitly white and Gaussian
- Limited prior information on the solution
- Lack of tools for the determination of the hyperparameters

Inversion: Probabilistic methods

Taking account of errors and uncertainties ⟶ Probability theory

- Maximum Likelihood (ML)
- Minimum Inaccuracy (MI)
- Probability Distribution Matching (PDM)
- Maximum Entropy (ME) and Information Theory (IT)
- Bayesian Inference (BAYES)

Advantages:

- Explicit account of the errors and noise
- A large class of priors via explicit or implicit modeling
- A coherent approach to combine information content of the data and priors

Limitations:

- Practical implementation and cost of calculation

Bayesian estimation approach

$$\mathcal{M}: \qquad \mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon}$$

- Observation model $\mathcal{M}$ + Hypothesis on the noise $\boldsymbol{\epsilon} \longrightarrow$
  $p(\mathbf{g}|\mathbf{f};\mathcal{M}) = p_{\boldsymbol{\epsilon}}(\mathbf{g} - \mathbf{H}\mathbf{f})$
- A priori information $\quad p(\mathbf{f}|\mathcal{M})$
- Bayes : $\quad p(\mathbf{f}|\mathbf{g};\mathcal{M}) = \dfrac{p(\mathbf{g}|\mathbf{f};\mathcal{M})\, p(\mathbf{f}|\mathcal{M})}{p(\mathbf{g}|\mathcal{M})}$

Link with regularization :

Maximum A Posteriori (MAP) :

$$\begin{aligned} \widehat{\mathbf{f}} &= \arg\max_{\mathbf{f}} \{p(\mathbf{f}|\mathbf{g})\} = \arg\max_{\mathbf{f}} \{p(\mathbf{g}|\mathbf{f})\ p(\mathbf{f})\} \\ &= \arg\min_{\mathbf{f}} \{-\ln p(\mathbf{g}|\mathbf{f}) - \ln p(\mathbf{f})\} \end{aligned}$$

with $\quad Q(\mathbf{g}, \mathbf{H}\mathbf{f}) = -\ln p(\mathbf{g}|\mathbf{f}) \quad$ and $\quad \lambda\Omega(\mathbf{f}) = -\ln p(\mathbf{f})$

Case of linear models and Gaussian priors

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon}$$

- Hypothesis on the noise: $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma_\epsilon^2 \mathbf{I}) \longrightarrow$
  $p(\mathbf{g}|\mathbf{f}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2\right\}$
- Hypothesis on $\mathbf{f}$: $\mathbf{f} \sim \mathcal{N}(0, \sigma_f^2 (\mathbf{D}'\mathbf{D})^{-1}) \longrightarrow$
  $p(\mathbf{f}) \propto \exp\left\{-\frac{1}{2\sigma_f^2}\|\mathbf{D}\mathbf{f}\|^2\right\}$
- A posteriori:
  $$p(\mathbf{f}|\mathbf{g}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 - \frac{1}{2\sigma_f^2}\|\mathbf{D}\mathbf{f}\|^2\right\}$$
- MAP : $\widehat{\mathbf{f}} = \arg\max_{\mathbf{f}} \{p(\mathbf{f}|\mathbf{g})\} = \arg\min_{\mathbf{f}} \{J(\mathbf{f})\}$
  with $J(\mathbf{f}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda \|\mathbf{D}\mathbf{f}\|^2, \qquad \lambda = \frac{\sigma_\epsilon^2}{\sigma_f^2}$
- Advantage : characterization of the solution

$$\mathbf{f}|\mathbf{g} \sim \mathcal{N}(\widehat{\mathbf{f}}, \widehat{\mathbf{P}}) \;\text{ with }\; \widehat{\mathbf{f}} = \widehat{\mathbf{P}}\mathbf{H}'\mathbf{g}, \quad \widehat{\mathbf{P}} = \left(\mathbf{H}'\mathbf{H} + \lambda \mathbf{D}'\mathbf{D}\right)^{-1}$$

MAP estimation with other priors:

$$\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \{J(\mathbf{f})\} \quad \text{with} \quad J(\mathbf{f}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda\Omega(\mathbf{f})$$

Separable priors:

- Gaussian: $p(f_j) \propto \exp\left\{-\alpha|f_j|^2\right\} \longrightarrow \Omega(\mathbf{f}) = \alpha \sum_j |f_j|^2$
- Gamma: $p(f_j) \propto f_j^{\alpha} \exp\left\{-\beta f_j\right\} \longrightarrow \Omega(\mathbf{f}) = \alpha \sum_j \ln f_j + \beta f_j$
- Beta:
  $p(f_j) \propto f_j^{\alpha}(1-f_j)^{\beta} \longrightarrow \Omega(\mathbf{f}) = \alpha \sum_j \ln f_j + \beta \sum_j \ln(1-f_j)$
- Generalized Gaussian:
  $p(f_j) \propto \exp\left\{-\alpha|f_j|^p\right\}, \quad 1 < p < 2 \longrightarrow \quad \Omega(\mathbf{f}) = \alpha \sum_j |f_j|^p,$

Markovian models:

$$p(f_j|\mathbf{f}) \propto \exp\left\{-\alpha \sum_{i \in N_j} \phi(f_j, f_i)\right\} \longrightarrow \quad \Omega(\mathbf{f}) = \alpha \sum_j \sum_{i \in N_j} \phi(f_j, f_i),$$

MAP estimation with markovien priors:

$$\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \{J(\mathbf{f})\} \quad \text{with} \quad J(\mathbf{f}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda\Omega(\mathbf{f})$$

$$\Omega(\mathbf{f}) = \sum_j \phi(f_j - f_{j-1})$$

with $\phi(t)$:

Convex functions:

$$|t|^\alpha, \ \sqrt{1+t^2} - 1, \ \log(\cosh(t)), \ \left\{ \begin{array}{ll} t^2 & |t| \le T \\ 2T|t| - T^2 & |t| > T \end{array} \right.$$

or Non convex functions:

$$\log(1+t^2), \quad \frac{t^2}{1+t^2}, \quad \arctan(t^2), \quad \left\{ \begin{array}{ll} t^2 & |t| \le T \\ T^2 & |t| > T \end{array} \right.$$

Main advantages of the Bayesian approach

- MAP = Regularization
- Posterior mean ? Marginal MAP ?
- More information in the posterior law than only its mode or its mean
- Meaning and tools for estimating hyper parameters
- Meaning and tools for model selection
- More specific and specialized priors, particularly through the hidden variables
- More computational tools:
  - Expectation-Maximization for computing the maximum likelihood parameters
  - MCMC for posterior exploration
  - Variational Bayes for analytical computation of the posterior marginals
  - ...

Content

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State space model: Continuous case

Dynamic systems:

- Single Input Single Output (SISO) system:

$$\left\{\begin{array}{lll} \dot{x}(t) & = A\,x(t) + B\,u(t) & \text{State equation} \\ y(t) & = C\,x(t) + D\,v(t) & \text{Observation equation} \end{array}\right.$$

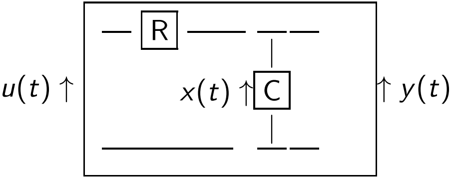
- Multiple Input Multiple Output (MIMO) system:

$$\left\{\begin{array}{lll} \dot{\mathbf{x}}(t) & = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) & \text{State equation} \\ \mathbf{y}(t) & = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{v}(t) & \text{Observation equation} \end{array}\right.$$

$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are the matrices of the system.

State space modeling

- A SISO system
- A simple electric system



- State space model

$$u(t) = R\, i(t) + v_c(t) = RC\, \dot{v}_c(t) + v_c(t) = RC\, \dot{x}(t) + x(t)$$

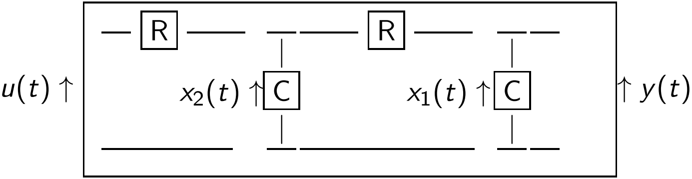
$$\begin{cases} \dot{x}(t) & = \left(\frac{-1}{RC}\right) x(t) + \left(\frac{-1}{RC}\right) u(t) \\ y(t) & = x(t) \end{cases}$$

- $RC = 1$:

$$\begin{cases} \dot{x}(t) = -x(t) - u(t) \\ y(t) = x(t) \end{cases} \rightarrow LT \rightarrow \begin{cases} pX(p) + X(p) = U(p) \rightarrow X(p) = \frac{1}{p+1} U(p) \\ y(t) = e^{-t} * u(t) = h(t) * u(t) \end{cases}$$

State space modeling

- A more complex electric system example



- State space model

$$u(t) = RC\,\dot{x}_2(t) + x_2(t), \quad x_2(t) = RC\,\dot{x}_1(t) + x_1(t)$$

- $RC = 1$:

$$\left\{ \begin{array}{lcl} \left[\begin{array}{c} \dot{x}_1(t) \\ \dot{x}_2(t) \end{array}\right] & = & \left[\begin{array}{cc} -1 & 1 \\ 0 & -1 \end{array}\right] \left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ 1 \end{array}\right] u(t) \\ y(t) & = & \left[\begin{array}{c} 1 \\ 0 \end{array}\right] x_1(t) \end{array} \right.$$

Input-Output model

- Linear Systems
  - Single Input Single Output (SISO) systems

$$y(t) = \int h(t,\tau)\, u(\tau)\, \mathrm{d}\tau$$

  - Multi Input Multi Output (MIMO) systems

$$\mathbf{y}(t) = \int \mathbf{H}(t,\tau)\, \mathbf{u}(\tau)\, \mathrm{d}\tau$$

- Linear Time Invariant System
  - SISO Convolution

$$y(t) = h(t) * u(t) = \int h(t-\tau)\, u(\tau)\, \mathrm{d}\tau$$

  - MIMO Convolution

$$\mathbf{y}(t) = \int \mathbf{H}(t-\tau)\, \mathbf{u}(\tau)\, \mathrm{d}\tau$$

  - Impulse response $h(t)$ or $\mathbf{H}(t) = \left[\begin{array}{ccc} . & . & . \\ . & h_{ij}(t) & . \\ . & . & . \end{array}\right]$

Link between the two modeling

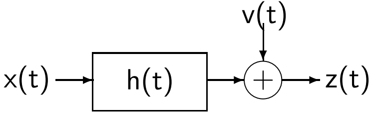
$$\begin{cases} \dot{\mathbf{x}}(t) & = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \quad \text{State equation} \\ \mathbf{y}(t) & = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{v}(t) \quad \text{Observation equation} \end{cases}$$

- $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$
- $\mathbf{x}(t+\Delta t) \simeq \mathbf{x}(t) + \Delta t\,\mathbf{A}\mathbf{x}(t) = (\mathbf{I} + \Delta t\,\mathbf{A})\mathbf{x}(t)$
- $\mathbf{x}(t+n\Delta t) \simeq (\mathbf{I} + \Delta t\,\mathbf{A})^n\,\mathbf{x}(t)$
- $\lim_{\Delta t \mapsto 0}(\mathbf{I} + \Delta t\,\mathbf{A})^{t/\Delta t} = \exp\{t\mathbf{A}\}$
- $\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t)$ with $\mathbf{x}(0) = 0 \longrightarrow \mathbf{x}(t) = \exp\{t\mathbf{A}\}$

State space and input-Output modeling

Transmission of a AR1 signal through a perturbed channel

$$u(t) \longrightarrow \boxed{\quad \text{AR1} \quad} \longrightarrow x(t)$$



$$x(t) = a\,x(t-1) + u(t) \qquad\qquad z(t) = \sum_k h_k\,x(t-k) + v(t)$$

- A FIR channel
  $\mathbf{h} = [h_0, h_1, \cdots, h_{p-1}]' \longrightarrow z(n) = \sum_{k=0}^{p} h_k x(n-k) + v(n)$

$$\left\{\begin{array}{lcl} x_{n+1} & = & a\,x_n + u_n \\ z_n & = & \displaystyle\sum_{k=0}^{p-1} h_n\,x_{n-k} + v_n \end{array}\right. \rightarrow ? \rightarrow \left\{\begin{array}{lcl} \mathbf{x}_{n+1} & = & \mathbf{F}\,\mathbf{x}_n + \mathbf{G}u_n \\ z_n & = & \mathbf{H}\,\mathbf{x}_n + v_n \end{array}\right.$$

State space and input-Output modeling

- A FIR channel $z(n) = \sum_{k=0}^{p} h_k x(n-k) + v(n)$

$$\begin{aligned} \mathbf{x}_n &= [x_n, x_{n-1}, \cdots, x_{n-p+1}]' \\ \mathbf{x}_{n+1} &= [x_{n+1}, x_n, \cdots, x_{n-p+2}]' \end{aligned}$$

$$\begin{bmatrix} x_{n+1} \\ x_n \\ x_{n-1} \\ \vdots \\ \vdots \\ x_{n-p+1} \end{bmatrix} = \begin{bmatrix} a & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \\ & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ \vdots \\ \vdots \\ x_{n-p+1} \\ x_{n-p+2} \end{bmatrix}$$

$$\begin{aligned} \mathbf{G} &= [1, 0, \cdots, 0]' \\ \mathbf{H} &= [h_0, h_1, \cdots, h_n, 0]' \end{aligned} \longrightarrow z_n = \mathbf{H}\,\mathbf{x}_n + v_n$$

State space and input-Output modeling

- A FIR channel $z(n) = \sum_{k=0}^{p} h_k x(n-k) + v(n)$

$$\left\{ \begin{array}{lcl} \mathbf{x}_{n+1} & = & \mathbf{F}\,\mathbf{x}_n + \mathbf{G} u_n \\ z_n & = & \mathbf{H}\,\mathbf{x}_n + v_n \end{array} \right.$$

$$\mathbf{x}_n = [x_n, x_{n-1}, \cdots, x_{n-p+1}]'$$

$$\mathbf{x}_{n+1} = [x_{n+1}, x_n, \cdots, x_{n-p+2}]'$$

$$\mathbf{G} = [1, 0, \cdots, 0]'$$

$$\mathbf{H} = [h_0, h_1, \cdots, h_n]'$$

$$\mathbf{F} = \begin{bmatrix} a & 0 & \ldots & \ldots & 0 \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & 1 & 0 \end{bmatrix}$$

- A perfect but noisy channel $h(t) = h_0\,\delta(t) \longrightarrow p = 1$

$$\left\{ \begin{array}{lcl} x_{n+1} & = & a\,x_n + u_n \\ z_n & = & h_0\,x_n + v_n \end{array} \right.$$

State space modeling: Examples

$$\begin{cases} x_{n+1} & = & a\, x_n + u_n \\ z_n & = & h\, x_n + v_n \end{cases}$$

$$\begin{cases} u_n & \sim & \mathcal{N}(0, q) \\ v_n & \sim & \mathcal{N}(0, r) \\ x_0 & \sim & \mathcal{N}(m_0, p_0) \end{cases}$$

- Try to obtain $\widehat{x}_{n+1|n}$ as a function of $\widehat{x}_{n|n}$ recursively

$$\begin{cases} \widehat{x}_{n+1|n} & = & a\, \widehat{x}_{n|n} \\ \widehat{x}_{n|n} & = & \widehat{x}_{n|n-1} + k_n\, (z_n - h\, \widehat{x}_{n|n-1}) \\ k_n & = & \dfrac{p_{n|n-1}\, h}{h^2 p_{n|n-1} + r} = \dfrac{1}{h}\, \dfrac{p_{n|n-1}}{p_{n|n-1} + r/h^2} \end{cases}$$

$$\begin{cases} p_{n+1|n} & = & a^2\, p_{n|n} + q \\ p_{n|n} & = & \dfrac{1}{h}\, \dfrac{p_{n|n-1}}{\frac{r}{h^2}\, p_{n|n-1+1}} \end{cases}$$

State space modeling of the systems

- Time Varying systems:

$$\left\{\begin{array}{lcll} \mathbf{x}_{n+1} & = & \mathbf{f}_n(\mathbf{x}_n, \mathbf{u}_n) & \text{state equation} \\ \mathbf{z}_n & = & \mathbf{h}_n(\mathbf{x}_n, \mathbf{v}_n) & \text{observation equation} \end{array}\right.$$

- Time Variiying but Linear system

$$\left\{\begin{array}{lcl} \mathbf{x}_{n+1} & = & \mathbf{F}_n \, \mathbf{x}_n + \mathbf{G}_n \, \mathbf{u}_n \\ \mathbf{z}_n & = & \mathbf{H}_n \, \mathbf{x}_n + \mathbf{v}_n \end{array}\right.$$

- Time Invariant and Linear system

$$\left\{\begin{array}{lcl} \mathbf{x}_{n+1} & = & \mathbf{F} \, \mathbf{x}_n + \mathbf{G} \, \mathbf{u}_n \\ \mathbf{z}_n & = & \mathbf{H} \, \mathbf{x}_n + \mathbf{v}_n \end{array}\right.$$

State space modeling examples

- One-dimensional motion
  Track-While-Scan (TWS) Radar

$$\begin{cases} X_t = \frac{\partial P_t}{\partial t} \\ A_t = \frac{\partial V_t}{\partial t} \end{cases} \longrightarrow \begin{cases} X_{n+1} \simeq X_n + T\, V_n \\ V_{n+1} \simeq V_n + T\, A_n \end{cases}$$

$$\begin{cases} \mathbf{x} = \begin{bmatrix} X \\ V \end{bmatrix} \\ u = A \\ z = X + v \end{cases} \longrightarrow \begin{cases} \mathbf{x}_n = \begin{bmatrix} X_n \\ V_n \end{bmatrix} \\ u_n = A_n \\ z_n = X_n + v_n \end{cases}$$

$$\begin{cases} \mathbf{x}_{n+1} & = \mathbf{F}\mathbf{x}_n + \mathbf{G} u_n \\ Z_n & = \mathbf{H}\mathbf{x}_n + v_n \end{cases} \text{ with } \quad \mathbf{F} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 0 \\ T \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\begin{cases} \mathbf{f}_n(\mathbf{x}_n, \mathbf{u}_n) = \mathbf{F}\, \mathbf{x}_n + \mathbf{G}\, \mathbf{u}_n \\ \mathbf{h}_n(\mathbf{x}_n, \mathbf{v}_n) = \mathbf{H}\, \mathbf{x}_n + \mathbf{v}_n \end{cases}$$

State space modeling examples

- 1D motion of heavy targets
  Track-While-Scan (TWS) Radar with dependent acceleration sequences
- heavy target:

$$A_{n+1} = \rho A_n + W_n, \quad n = 0, 1, \ldots$$

- ρ near to 0: low inertia target
- ρ near to 1: high inertia target.

$$\begin{bmatrix} X_{n+1} \\ V_{n+1} \\ A_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} X_n \\ V_n \\ A_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} W_n$$

$$Z_n = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_n \\ V_n \\ A_n \end{bmatrix} + e_n$$

Kalman Filtering: Recursive Linear Filtering

$$\underset{\text{non observable}}{x(n)} \quad \longrightarrow \boxed{\text{Linear System}} \longrightarrow \quad \underset{\text{observable}}{z(n)}$$

- Objective: Estimate $x(n)$ from the observed values of $\{z(n), n = 1, \ldots, k\}$.
- $\widehat{x}(n)$ is then a function of the data $\{z(n), n = 1, \ldots, k\}$

$$\widehat{x}(n \mid z(1), z(2), \ldots, z(k)) \overset{\triangle}{=} \widehat{x}(n \mid k)$$

- $\widehat{x}(n + k|n)$ is called the k-th *order prediction* of $z(n)$ and the estimation procedure is called *prediction*.
- $\widehat{x}(n|n)$ is the *filtered value* of $z(n)$ and the estimation procedure is called *filtering*.
- $\widehat{x}(n|n + l)$ is the *smoothed value* of $z(n)$ and the estimation procedure is called *smoothing*.

State space modeling: General case

$$\begin{cases} \mathbf{x}_{k+1} & = \mathbf{F}_k \, \mathbf{x}_k + \mathbf{G}_k \, \mathbf{u}_k \quad \text{state equation,} \\ \mathbf{z}_k & = \mathbf{H}_k \; \mathbf{x}_k + \mathbf{v}_k \quad \text{observation equation} \end{cases}$$

- $\mathbf{x}_k$ N-dimensional *state vector*
- $\mathbf{z}_k$ P-dimensional *observations vector*
- $\mathbf{v}_k$ P-dimensional *observations error vector*
- $\mathbf{u}_k$ M-dimensional *state representation error*
- $\mathbf{F}_k$, $\mathbf{G}_k$ and $\mathbf{H}_k$ with respective dimensions of (N, N), (N, M) and (P, N) are the *state transition*, the *state input* and the *observation matrices* and are assumed to be known.
- The noise sequences $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$ and the *initial state* $\mathbf{x}_0$ are assumed to be centered, white and jointly Gaussian.

$$\mathrm{E}\left\{ \begin{bmatrix} \mathbf{v}_k \\ \mathbf{x}_0 \\ \mathbf{u}_k \end{bmatrix} \left[\mathbf{v}_l^t, \mathbf{x}_0^t, \mathbf{u}_l^t\right] \right\} = \begin{bmatrix} \mathbf{R}_k & 0 & 0 \\ 0 & \mathbf{P}_0 & 0 \\ 0 & 0 & \mathbf{Q}_k \end{bmatrix} \delta_{kl}$$

Kalman Filtering: Prediction, Filtering and Smoothing

Objective: Find the best estimate $\widehat{\mathbf{x}}_{k|l}$ of $\mathbf{x}_k$ from the observations $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_l$.

$$\widehat{\mathbf{x}}_k(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_l) \overset{\triangle}{=} \widehat{\mathbf{x}}(k \mid l)$$

- If $k > l$ prediction. For example $l = n, k = n + 1$:

$$\widehat{\mathbf{x}}_{n+1}(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n) \overset{\triangle}{=} \widehat{\mathbf{x}}(n + 1 \mid n)$$

- If $k = l$ filtering. For example $l = n, k = n$:

$$\widehat{\mathbf{x}}_n(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n) \overset{\triangle}{=} \widehat{\mathbf{x}}(n \mid n)$$

- If $k < l$ smoothing. For example $l = n + 1, k = n$:

$$\widehat{\mathbf{x}}_n(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n+1}) \overset{\triangle}{=} \widehat{\mathbf{x}}(n \mid n + 1)$$

Kalman Filtering: 3 Interpretations

Three different approaches can be used to obtain the Kalman filtering equations:

- Least Mean Square (LMS) estimation:

  $$\widehat{\mathbf{x}}_{k|l} \overset{\triangle}{=} \mathrm{LMS}(\mathbf{x}_k \mid \mathbf{z}_1, \ldots, \mathbf{z}_l)$$

  which minimizes

  $$\mathrm{E}\left\{[\mathbf{x}_k - \widehat{\mathbf{x}}_{k|l}]^t \, \mathbf{W}_k \, [\mathbf{x}_k - \widehat{\mathbf{x}}_{k|l}]\right\}$$

- Maximum A posteriori (MAP) estimate:

  $$\widehat{\mathbf{x}}_{k|l} = \arg\max_{\mathbf{x}} \{p(\mathbf{x}_k \mid \mathbf{z}_1, \ldots, \mathbf{z}_l)\}$$

- Bayesian MSE estimate:

  $$\widehat{\mathbf{x}}_{k|l} = \mathrm{E}\{\mathbf{x}_k \mid \mathbf{z}_1, \ldots, \mathbf{z}_k\}$$

Kalman Filtering: Bayesian approach

The main procedure is to apply the Bayes rule recursively to find the expression of the posterior law $p(\mathbf{x}_k \mid \mathbf{z}_1, \ldots, \mathbf{z}_l)$.

1. All variables are assumed Gaussian ;
2. All the conditional laws such as $p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})$ and $p(\mathbf{z}_{k+1}|\mathbf{z}_{1:k})$ are Gaussian. So, the posterior law
$$p(\mathbf{x}_{k+1}|\mathbf{z}_{1:k+1}) = p(\mathbf{x}_{k+1}|\mathbf{z}_{1:k}) \frac{p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})}{p(\mathbf{z}_{k+1}|\mathbf{z}_{1:k})}$$
is also Gaussian.

To obtain the equations in the general case we note

- $\widehat{\mathbf{x}}_{k|k}$ the estimate of the state vector at time k from the observations up to time k ;
- $\widehat{\mathbf{x}}_{k+1|k}$ the estimate of the state vector at time $k+1$ from the observations up to the instant k ;
- $\mathbf{e}_{k+1} = \mathbf{z}_{k+1} - H_{k+1}\,\widehat{\mathbf{x}}_{k+1|k}$ the *innovation process* of the observations at the instant $k+1$

Kalman Filtering: Bayesian approach

- The covariance matrix of the innovation by

  $$\mathbf{R}^e_{k+1} = \mathrm{E}\left\{\mathbf{e}_{k+1|k}\,\mathbf{e}^t_{k+1|k}\right\}$$

  which is diagonal;

- The covariance matrix of the prediction error by

  $$\mathbf{P}_{k+1|k} = \mathrm{E}\left\{\left[\mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1|k}\right]\left[\mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1|k}\right]^t\right\},$$

- The posterior covariance matrix of the estimation error by

  $$\mathbf{P}_{k+1|k+1} = \mathrm{E}\left\{\left[\mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1|k+1}\right]\left[\mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1|k+1}\right]^t\right\}$$

  which is also called the covariance matrix of the filtering error.

Kalman Filtering: Bayesian approach

$$
\begin{aligned}
\mathrm{E}\left\{\mathbf{x}_k | \mathbf{z}_{1:k}\right\} &\stackrel{\triangle}{=} \widehat{\mathbf{x}}_{k|k} \\
\operatorname{cov}[\mathbf{x}_k | \mathbf{z}_{1:k}] &\stackrel{\triangle}{=} \mathbf{P}_{k|k} \\
\mathrm{E}\left\{\mathbf{x}_{k+1} | \mathbf{z}_{1:k}\right\} &\stackrel{\triangle}{=} \widehat{\mathbf{x}}_{k+1|k} \\
\operatorname{cov}[\mathbf{x}_{k+1} | \mathbf{z}_{1:k}] &\stackrel{\triangle}{=} \mathbf{P}_{k+1|k}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left\{\mathbf{z}_{k+1} | \mathbf{x}_{k+1}\right\} &= \mathbf{H}_{k+1}\, \widehat{\mathbf{x}}_{k+1} \\
\operatorname{cov}[\mathbf{z}_{k+1} | \mathbf{x}_{k+1}] &= \mathbf{R}_{k+1}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left\{\mathbf{x}_{k+1} | \mathbf{z}_{1:k}\right\} &= \mathbf{F}_k\, \widehat{\mathbf{x}}_{k|k} \\
\operatorname{cov}[\mathbf{x}_{k+1} | \mathbf{z}_{1:k}] &= \mathbf{F}_k\, \mathbf{P}_k \mathbf{F}_k^t + \mathbf{G}_k\, \mathbf{Q}_k\, \mathbf{G}_k^t
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left\{\mathbf{z}_{k+1} | \mathbf{z}_{1:k}\right\} &= \mathbf{H}_{k+1}\, \mathbf{F}_k\, \widehat{\mathbf{x}}_{k|k} \\
\operatorname{cov}[\mathbf{z}_{k+1} | \mathbf{z}_{1:k}] &= \mathbf{H}_{k+1}\, \mathbf{P}_{k+1|k} \mathbf{H}_{k+1}^t + \mathbf{R}_{k+1}
\end{aligned}
$$

Kalman Filtering: Bayesian approach

Replacing the expressions of $p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})$ and $p(\mathbf{z}_{k+1}|\mathbf{z}_{1:k})$:

$$p(\mathbf{x}_{k+1}|\mathbf{z}_{1:k+1}) = A \exp\left\{-\frac{1}{2}[\mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1|k}]^t \mathbf{P}_{k+1|k+1}^{-1} [\mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1|k}]\right\}$$

with

$$A = \frac{1}{(2\pi)^{n/2}} \left\{ \left| \mathbf{H}_{k+1}\, \mathbf{P}_{k+1|k}\, \mathbf{H}_{k+1}^t + \mathbf{R}_k \right|^{1/2} \left| \mathbf{R}_k \right|^{-1/2} \left| \mathbf{P}_{k+1|k} \right| \right\}^{-1/2}$$

$$\begin{aligned}
\widehat{\mathbf{x}}_{k+1|k} &= \mathbf{F}_k\, \widehat{\mathbf{x}}_{k|k} + \mathbf{P}_{k+1|k}\, \mathbf{H}_{k+1}^t \\
&\quad \left[\mathbf{R}_{k+1} + \mathbf{H}_{k+1}\, \mathbf{P}_{k+1|k}\, \mathbf{H}_{k+1}^t\right]^{-1} \left[\mathbf{z}_{k+1} - \mathbf{H}_{k+1}\, \mathbf{F}_k\, \widehat{\mathbf{x}}_{k|k}\right] \\
\mathbf{P}_{k+1|k} &= \mathbf{F}_k\, \mathbf{P}_{k|k}\, \mathbf{F}_k^t + \mathbf{G}_k\, \mathbf{Q}_k\, \mathbf{G}_k^t \\
\mathbf{P}_{k+1|k+1}^{-1} &= \mathbf{P}_{k+1|k}^{-1} + \mathbf{H}_{k+1}^t \mathbf{R}_{k+1}^{-1}\, \mathbf{H}_{k+1}
\end{aligned}$$

Kalman Filtering: Different forms of equation

- **Prediction-Correction form :**
  - **Prediction (Time update) :**

$$\begin{aligned} \widehat{\mathbf{x}}_{k+1|k} &= \mathbf{F}_k \, \widehat{\mathbf{x}}_{k|k} \\ \mathbf{P}_{k+1|k} &= \mathbf{F}_k \, \mathbf{P}_{k|k} \, \mathbf{F}_k^t + \mathbf{G}_k \, \mathbf{Q}_k \, \mathbf{G}_k^t \end{aligned}$$

  - **Correction (measurement update) :**

$$\begin{aligned} \widehat{\mathbf{x}}_{k+1|k+1} &= \widehat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}^g [\mathbf{z}_{k+1} - \mathbf{H}_{k+1} \, \widehat{\mathbf{x}}_{k+1|k}] \\ \mathbf{K}_{k+1}^g &= \mathbf{P}_{k+1|k} \, \mathbf{H}_{k+1}^t (\mathbf{R}_{k+1}^e)^{-1} \\ \mathbf{R}_{k+1}^e &= \mathbf{R}_{k+1} + \mathbf{H}_{k+1} \, \mathbf{P}_{k+1|k} \, \mathbf{H}_{k+1}^t \\ \mathbf{P}_{k+1|k+1} &= [\mathbf{I} - \mathbf{K}_{k+1}^f \mathbf{H}_{k+1}] \mathbf{P}_{k+1|k} \end{aligned}$$

Kalman Filtering: Different forms of equation

- **Compact form for prediction :**

$$
\begin{aligned}
\widehat{\mathbf{x}}_{k+1|k} &= \mathbf{F}_k\,\widehat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{R}^e_k)^{-1}[\mathbf{z}_k - \mathbf{H}_k\,\widehat{\mathbf{x}}_{k|k-1}] \\
\mathbf{R}^e_k &= \mathbf{R}_k + \mathbf{H}_k\,\mathbf{P}_{k|k-1}\,\mathbf{H}^t_k \\
\mathbf{K}_k &= \mathbf{F}_k\,\mathbf{P}_{k|k-1}\,\mathbf{H}^t_k \\
\mathbf{P}_{k+1|k} &= \mathbf{F}_k\,\mathbf{P}_{k|k-1}\,\mathbf{F}^t_k + \mathbf{G}_k\,\mathbf{Q}_k\,\mathbf{G}^t_k - \mathbf{K}_k(\mathbf{R}^e_k)^{-1}\,\mathbf{K}^t_k
\end{aligned}
$$

- In all cases the initialization is :

$$
\widehat{\mathbf{x}}_{0|-1} = 0 \quad \mathbf{P}_{0|-1} = \mathbf{P}_0
$$

Kalman Filtering: Different forms of equation

- **Compact form for filtering :**

$$\begin{array}{rcl} \widehat{\mathbf{x}}_{k|k} & = & \widehat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k^g[\mathbf{z}_k - \mathbf{H}_k\,\widehat{\mathbf{x}}_{k|k-1}] \\ \mathbf{K}_k^g & = & \mathbf{P}_{k|k-1}\,\mathbf{H}_k^t[\mathbf{R}_k + \mathbf{H}_k\,\mathbf{P}_{k|k-1}\,\mathbf{H}_k^t]^{-1} \\ \mathbf{P}_{k|k} & = & \mathbf{P}_{k|k-1} - \mathbf{K}_k^g\mathbf{H}_k\mathbf{P}_{k|k-1} \\ \mathbf{P}_{k+1|k} & = & \mathbf{F}_k\mathbf{P}_{k|k}\,\mathbf{F}_k^t + \mathbf{G}_k\,\mathbf{Q}_k\,\mathbf{G}_k^t \end{array}$$

- **Very compact form for prediction :**

$$\begin{array}{rcl} \widehat{\mathbf{x}}_{k+1|k} & = & \mathbf{F}_k\widehat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k^g[\mathbf{z}_k - \mathbf{H}_k\,\widehat{\mathbf{x}}_{k|k-1}] \\ \mathbf{K}_k^g & = & \mathbf{P}_{k|k-1}\,\mathbf{H}_k^t[\mathbf{R}_k + \mathbf{H}_k\,\mathbf{P}_{k|k-1}\,\mathbf{H}_k^t]^{-1} \\ \mathbf{P}_{k+1|k} & = & \mathbf{F}_k\mathbf{P}_{k|k-1}\,\mathbf{F}_k^t - \mathbf{F}_k\mathbf{K}_k^g\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{F}_k^t + \mathbf{G}_k\,\mathbf{Q}_k\,\mathbf{G}_k^t \end{array}$$

where $\mathbf{K}_k$ is called the *Kalman filter gain* and $\mathbf{K}_k^g = \mathbf{K}_k(\mathbf{R}_k^e)^{-1}$ the generalized *Kalman filter gain*.

Kalman Filtering equations: 1D case

An AR1 signal is observed through a perturbed chanel:

$$\left\{\begin{array}{lcl} x_{n+1} & = & f\, x_n + u_n \\ z_n & = & h\, x_n + v_n \end{array}\right. \qquad \left\{\begin{array}{lcl} u_n & \sim & \mathcal{N}(0, q) \\ v_n & \sim & \mathcal{N}(0, r) \\ x_0 & \sim & \mathcal{N}(m_0, p_0) \end{array}\right.$$

$$\left\{\begin{array}{lcl} \widehat{x}_{n+1|n} & = & f\, \widehat{x}_{n|n} \\ \widehat{x}_{n|n} & = & \widehat{x}_{n|n-1} + k_n\, (z_n - h\, \widehat{x}_{n|n-1}) \\ k_n & = & (h^2 p_{n|n-1} + r)^{-1} p_{n|n-1} h = \frac{p_{n|n-1} h}{h^2 p_{n|n-1} + r} = \frac{1}{h}\, \frac{p_{n|n-1}}{p_{n|n-1} + r/h^2} \end{array}\right.$$

$$\left\{\begin{array}{lcl} p_{n+1|n} & = & f^2\, p_{n|n} + q \\ p_{n|n} & = & \frac{1}{h}\, \frac{p_{n|n-1}}{\frac{r}{h^2}\, p_{n|n-1+1}} \end{array}\right.$$

We can eliminate the coupling between these equations and obtain

$$p_{n+1|n} = \frac{f^2\, p_{n|n-1}}{\frac{h^2}{r}\, p_{n|n-1} + 1} + q$$

As n increases, $p_{n+1|n}$ and so the gain k_n approaches a constant.

Kalman Filtering equations: 1D case

If $p_{n+1|n}$ does approach a constant, say p_∞, then p_∞ must satisfy

$$p_\infty = \frac{f^2\, p_\infty}{\frac{h^2}{r}\, p_\infty + 1} \; + \; q$$

This equation is quadratic and has a unique positive solution

$$p_\infty = \frac{1}{2}\left\{\left[\frac{r}{h^2}(1-f^2) - q\right]^2 + \frac{4rq}{h^2}\right\}^{1/2} - \frac{r}{2h^2}(1-f^2) + q$$

$$\begin{aligned} \left|p_{n+1|n} - p_\infty\right| &= f^2 \left| \frac{p_{n|n-1}}{\frac{h^2}{r}\, p_{n|n-1} + 1} - \frac{p_\infty}{\frac{h^2}{r}\, p_\infty + 1} \right| \\ &\leq f^2 \left|p_{n|n-1} - p_\infty\right| \end{aligned}$$

$$\left|p_{n+1|n} - p_\infty\right| \leq f^{2(n+1)} \left|p_0 - p_\infty\right|$$

$$\leq \quad f^2 \left|p_{n|n-1} - p_\infty\right|$$

- If $|f| < 1$ then $p_{n+1|n}$ converges to p_∞.
- $|f| < 1$ is a sufficient condition for Kalman-Bucy filter to approach a steady state.

Kalman Filtering: Applications

Track-While-Scan (TWS) Radar

$$\begin{aligned}\begin{bmatrix} X_{n+1} \\ V_{n+1} \end{bmatrix} &= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_n \\ V_n \end{bmatrix} + \begin{bmatrix} 0 \\ T \end{bmatrix} A_n \\ Z_n &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_n \\ V_n \end{bmatrix} + e_n\end{aligned}$$

- For a more general case in 3D we have a state vector with 6 components (3 positions and 3 velocities).
- But, if we assume that the measurement noise in 3 dimensions are independent of one another and independent to the components of the acceleration, the problem can be treated as 3 independent one-dimensional moving target.

Kalman Filtering: Applications

Track-While-Scan (TWS) Radar

$$\begin{aligned}
\begin{bmatrix}\widehat{X}_{n+1|n}\\ \widehat{V}_{n+1|n}\end{bmatrix} &= \begin{bmatrix}\widehat{X}_{n|n}+T\widehat{V}_{n|n}\\ \widehat{V}_{n|n}\end{bmatrix}\\
\begin{bmatrix}\widehat{X}_{n|n}\\ \widehat{V}_{n|n}\end{bmatrix} &= \begin{bmatrix}\widehat{X}_{n|n-1}\\ \widehat{V}_{n|n-1}\end{bmatrix}+\begin{bmatrix}K_{n,1}\\ K_{n,2}\end{bmatrix}\begin{bmatrix}Z_n-\widehat{X}_{n|n-1}\end{bmatrix}\\
\begin{bmatrix}K_{n,1}\\ K_{n,2}\end{bmatrix} &= \begin{bmatrix}\frac{P(1,1)}{P(1,1)+r}\\ \frac{P(2,1)}{P(1,1)+r}\end{bmatrix}
\end{aligned}$$

where $P(k,l)$ is the $(k-l)$th component of the matrix $P_{n|n-1}$. To reduce the computation, the time varying elements of the Kalman gain vector can be replaced with some constants (the steady states values) to obtain

$$\begin{bmatrix}\widehat{X}_{n|n}\\ \widehat{V}_{n|n}\end{bmatrix} = \begin{bmatrix}\widehat{X}_{n|n-1}\\ \widehat{V}_{n|n-1}\end{bmatrix}+\begin{bmatrix}\alpha\\ \beta/T\end{bmatrix}\begin{bmatrix}Z_n-\widehat{X}_{n|n-1}\end{bmatrix}$$

with constatnt values for α and β.

Kalman Filtering: Applications

Track-While-Scan (TWS) Radar with dependent acceleration sequences

- heavy target:

$$A_{n+1} = \rho A_n + W_n, \quad n = 0, 1, \ldots$$

- ρ near to 0: low inertia target and ρ near to 1: high inertia target.

$$\begin{bmatrix} X_{n+1} \\ V_{n+1} \\ A_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} X_n \\ V_n \\ A_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} W_n$$

$$Z_n = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_n \\ V_n \\ A_n \end{bmatrix} + e_n$$

Kalman Filtering: Applications

- **Track-While-Scan (TWS) Radar**

$$\begin{bmatrix} \widehat{X}_{n+1|n} \\ \widehat{V}_{n+1|n} \\ \widehat{A}_{n+1|n} \end{bmatrix} = \begin{bmatrix} \widehat{X}_{n|n} + T\widehat{V}_{n|n} \\ \widehat{V}_{n|n} + T\widehat{A}_{n|n} \\ \rho\widehat{A}_{n|n} \end{bmatrix}$$

$$\begin{bmatrix} \widehat{X}_{n|n} \\ \widehat{V}_{n|n} \\ \widehat{A}_{n|n} \end{bmatrix} = \begin{bmatrix} \widehat{X}_{n|n-1} \\ \widehat{V}_{n|n-1} \\ \widehat{A}_{n|n-1} \end{bmatrix} + \begin{bmatrix} K_{n,1} \\ K_{n,2} \\ K_{n,3} \end{bmatrix} \begin{bmatrix} Z_n - \widehat{X}_{n|n-1} \end{bmatrix}$$

$$\begin{bmatrix} K_{n,1} \\ K_{n,2} \\ K_{n,3} \end{bmatrix} = \begin{bmatrix} \frac{P(1,1)}{P(1,1)+r} \\ \frac{P(2,1)}{P(1,1)+r} \; \frac{P(3,1)}{P(1,1)+r} \end{bmatrix}$$

where $P(k,l)$ is the $(k-l)$th component of the matrix $P_{n|n-1}$.

- To reduce the computation:

$$\begin{bmatrix} \widehat{X}_{n|n} \\ \widehat{V}_{n|n} \\ \widehat{A}_{n|n} \end{bmatrix} \begin{bmatrix} \widehat{X}_{n|n-1} \\ \widehat{V}_{n|n-1} \\ \widehat{A}_{n|n-1} \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta/T \\ \gamma/T^2 \end{bmatrix} \begin{bmatrix} Z_n - \widehat{X}_{n|n-1} \end{bmatrix}$$

Kalman Filtering: Fast Kalman Algorithms

$$\begin{aligned}
\widehat{\mathbf{x}}_{k+1|k} &= \mathbf{F}\widehat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{R}^e_k)^{-1}[\mathbf{z}_k - \mathbf{H}\widehat{\mathbf{x}}_{k|k-1}] \\
\mathbf{R}^e_k &= \mathbf{R} + \mathbf{H}\mathbf{P}_{k|k-1}\,\mathbf{H}^t \\
\mathbf{K}_k &= \mathbf{F}\mathbf{P}_{k|k-1}\,\mathbf{H}^t \\
\mathbf{P}_{k+1|k} &= \mathbf{F}\mathbf{P}_{k|k-1}\,\mathbf{F}^t + \mathbf{G}\mathbf{Q}\mathbf{G}^t - \mathbf{K}_k(\mathbf{R}^e_k)^{-1}\,\mathbf{K}^t_k
\end{aligned}$$

$$\begin{aligned}
\delta\mathbf{P}_k &= \mathbf{P}_{k|k-1} - \mathbf{P}_{k-1|k-2} \\
\delta\mathbf{K}^g_k &= \mathbf{K}^g_k - \mathbf{K}^g_{k-1} \\
\delta\mathbf{R}^e_k &= \mathbf{R}^e_k - \mathbf{R}^e_{k-1}
\end{aligned}$$

$$\begin{aligned}
\delta\mathbf{P}_{k+1} &= [\mathbf{F} - \mathbf{K}^g_{k-1}\mathbf{H}][\delta\mathbf{P}_k - \delta\mathbf{P}_k\,\mathbf{H}^t(\mathbf{R}^e_k)^{-1}\,\mathbf{H}\mathbf{P}_k][\mathbf{F} - \mathbf{K}^g_{k-1}\mathbf{H}]^t \\
&= [\mathbf{F} - \mathbf{K}^g_k\mathbf{H}][\delta\mathbf{P}_k + \delta\mathbf{P}_k\,\mathbf{H}^t(\mathbf{R}^e_{k-1})^{-1}\,\mathbf{H}\mathbf{P}_k][\mathbf{F} - \mathbf{K}^g_k\mathbf{H}]^t
\end{aligned}$$

Kalman Filtering: Fast Kalman Algorithms

- If $\delta\mathbf{P}_k$ can be factorized as:

$$\delta\mathbf{P}_1 = \mathbf{z}_0\mathbf{M}_0\mathbf{z}_0^t, \longrightarrow \delta\mathbf{P}_{k+1} = \mathbf{z}_k\,\mathbf{M}_k\,\mathbf{z}_k^t$$

$$\begin{aligned}
\delta\mathbf{P}_{k+1} &= \mathbf{z}_k\,\mathbf{M}_k\,\mathbf{z}_k^t \\
\mathbf{z}_k &= [\mathbf{F} - \mathbf{K}_k^g\mathbf{H}]\mathbf{z}_{k-1} \\
\mathbf{M}_k &= \mathbf{M}_{k-1} + \mathbf{M}_{k-1}\,\mathbf{z}_{k-1}^t\mathbf{H}^t(\mathbf{R}_{k-1}^e)^{-1}\,\mathbf{H}\mathbf{z}_{k-1}\,\mathbf{M}_{k-1} \\
\mathbf{R}_{k-1}^e &= \mathbf{R}_k^e + \mathbf{H}\mathbf{z}_k\,\mathbf{M}_k\,\mathbf{z}_k^t\mathbf{H}^t \\
\mathbf{K}_{k+1} &= \mathbf{K}_{k+1}^g\,\mathbf{R}_{k+1}^e = \mathbf{K}_k + \mathbf{F}\mathbf{z}_k\,\mathbf{M}_k\,\mathbf{z}_k^t\mathbf{H}^t \\
\mathbf{P}_{k|k-1} &= \mathbf{P}_0 + \sum_{j=0}^{k-1}\mathbf{z}_j\mathbf{M}_j\mathbf{z}_j^t
\end{aligned}$$

which are called *Chandrasekhar* equations.

Kalman Filtering: Fast Kalman Algorithms

- Note that if $\alpha = \text{rang}\,\{\delta \mathbf{P}_1\}$ where

  $$\delta \mathbf{P}_1 = \mathbf{F}\mathbf{P}_0\mathbf{F}^t + \mathbf{G}\mathbf{Q}\mathbf{G}^t - \mathbf{K}_0(\mathbf{R}_0^e)^{-1}\mathbf{K}_0^t - \mathbf{P}_0$$

  then $\mathbf{z}_k$ has dimensions (N, α), $\mathbf{M}_k$ has dimensions (α, α).
- So, in place of updating $\mathbf{P}_k$ with dimensions (N, N) we only have to update the matrixes $\mathbf{z}_k$ and $\mathbf{M}_k$ with dimensions (N, α) and (α, α).
- Note also that $\mathbf{M}_0$ is the signature matrix of $\delta \mathbf{P}_1$ and the value of α depends on the choice of the initial covariance matrix $\mathbf{P}_0$.
- It is not unusual to have $\alpha = 1$ which greately reduces the computation cost.

Kalman Filter equations for signal deconvolution

$$z(k) = \sum_{i=0}^{p-1} h(i)x(k-i) + \epsilon(k)$$

$$\begin{bmatrix} z(1) \\ \vdots \\ z(k) \\ \vdots \\ z(M) \end{bmatrix} = \begin{bmatrix} h_{(p-1)} & \cdots & h_{(0)} & \cdots & \cdots \\ \vdots & & & & \vdots \\ 0 & h_{(p-1)} & \cdots & h_{(0)} & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & h_{(p-1)} & \cdots & h_{(0)} \end{bmatrix} \begin{bmatrix} x(-p) \\ \vdots \\ x(0) \\ \vdots \\ x(M) \end{bmatrix} + \begin{bmatrix} \epsilon(1) \\ \vdots \\ \epsilon(k) \\ \vdots \\ \epsilon(M) \end{bmatrix}$$

- **Constant state vector model**

$$\left\{ \begin{array}{lll} \mathbf{x}_{k+1} & = & \mathbf{x}_k = \mathbf{x} = [x_{-p}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n]^t \\ z_k & = & \mathbf{h}_k^t \cdot \mathbf{x}_k + v_k \end{array} \right.$$

$$\mathbf{h}_k = \begin{bmatrix} 0 & 0 & 0 & h_{p-1} & \ldots & h_0 & 0 & 0 \end{bmatrix}^t$$

where coefficient h_0 is in the k-th position.

Kalman Filter equations for signal deconvolution

- **Constant state vector model**

$$\left\{\begin{array}{lcl}\mathbf{u}_k & = & \mathbf{0} \\ \mathbf{F}_k & = & \mathbf{G}_k = \mathbf{I} \\ \mathbf{h}_{k+1}^t & = & \mathbf{D}\mathbf{h}_k^t\end{array}\right. \quad \text{with } \mathbf{D} = \begin{bmatrix} 0 & \dots & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \vdots & \vdots \\ \vdots & & & \vdots & \vdots \\ 0 & & \dots & 1 & 0 \end{bmatrix}$$

$$\mathrm{E}\left\{\mathbf{x}\right\} = \mathbf{x}_0 \qquad \mathrm{E}\left\{[\mathbf{x}-\mathbf{x}_0][\mathbf{x}-\mathbf{x}_0]^t\right\} = \mathbf{P}_0$$

$$\mathrm{E}\left\{v_k\right\} = 0 \qquad \mathrm{E}\left\{v_k v_j\right\} = r\,\delta_{kj}$$

$$\begin{array}{rcl}\widehat{\mathbf{x}}_{k|k} & = & \widehat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k (r_k^e)^{-1} [y_k - \mathbf{h}_k^t \cdot \widehat{\mathbf{x}}_{k|k-1}] \\ r_k^e & = & r + \mathbf{h}_k^t \mathbf{P}_{k|k-1}\,\mathbf{h}_k \\ \mathbf{K}_k & = & \mathbf{P}_{k|k-1}\,\mathbf{h}_k^t \\ \mathbf{P}_{k+1|k} & = & \mathbf{P}_{k|k-1} - \mathbf{K}_k (r_k^e)^{-1}\,\mathbf{K}_k^t\end{array}$$

- y_k and r_k^e are scalar.
- $\mathbf{x}$ is a N-dimensional.
- The covariance matrix $\mathbf{P}$ has the dimensions $(N \times N)$.

Kalman Filter equations for signal deconvolution

- **Non constant state space model**

$$\mathbf{h} = [h_0, \ldots, h_{p-1}]^t, \quad \mathbf{x}_k = [x_k, x_{k-1}, \ldots, x_{k-p+1}]^t$$

$$z(k) = \mathbf{h}^t \mathbf{x}_k + \epsilon(k), \quad \text{dimension of } \mathbf{x}_k = p \leq N$$

$$x(n+1) = \sum_{i=1}^{q} a(i)\, x(n-i+1) + u(n+1)$$

$$\mathrm{E}\left\{u_n\right\} = 0, \quad \mathrm{E}\left\{|u_n|^2\right\} = \beta^2, \quad \mathrm{E}\left\{u_m u_n\right\} = 0, \quad m \neq n$$

$$\begin{bmatrix} x(n+1) \\ x(n) \\ \vdots \\ \vdots \\ x(n-q+2) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \ldots & \ldots & a_q \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ \vdots \\ x(n-q+1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} u(n+1)$$

$$\begin{cases} \mathbf{x}_{k+1} & = & \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{u}_{k+1} \\ y_k & = & \mathbf{h}^t \cdot \mathbf{x}_k + \epsilon_k \end{cases}$$

Kalman Filter equations for signal deconvolution

- **Non constant state space model**

$$\begin{cases} \mathbf{x}_{k+1} & = & \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{u}_{k+1} \\ y_k & = & \mathbf{h}^t \cdot \mathbf{x}_k + \epsilon_k \end{cases}$$

$$\mathbf{F} = \begin{bmatrix} a_1 & a_2 & \ldots & \ldots & a_q \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & 1 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Advantages: **F**, **G** and **H** are constant and we can use fast algorithms.
- Drawbacks: Determination of q and a_k, $k = 1, \ldots, q$.

Kalman Filter equations for signal deconvolution

- If the input signal is also assumed to be causal, we obtain :

$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix} = \begin{bmatrix} h(0) & & & & & \\ h(1) & \ddots & & & & \\ \vdots & & & & & \\ h(p) & \cdots & & h(0) & & \\ 0 & \ddots & & & \ddots & \\ \vdots & & & & & \\ 0 & \cdots & 0 & h(p) & \cdots & h(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f(M) \end{bmatrix}$$

and finally if $p = M$ we have :

$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(M) \end{bmatrix} = \begin{bmatrix} h(0) & & & \\ h(1) & h(0) & & \\ \vdots & & \ddots & \\ h(M) & \cdots & h(1) & h(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(M) \end{bmatrix}$$

Remark that, in all cases matrix **H** is TOEPLITZ.

Kalman Filter equations for signal deconvolution

- Circulante form when system and input signal are both causal.

$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} h(0) & 0 & \cdots & & & 0 & & h(p) & \cdots & h(1) \\ h(1) & \ddots & & & & & & & \ddots & \vdots \\ \vdots & & & & & & & & & h(p) \\ h(p) & \cdots & & h(0) & 0 & & & & & 0 \\ 0 & \ddots & & & \ddots & & & & & \vdots \\ \vdots & & & & & & & & & \\ 0 & \cdots & 0 & h(p) & \cdots & & & h(0) & & 0 \\ \vdots & & & & & & & & & \vdots \\ & & & & \ddots & & & & \ddots & 0 \\ 0 & \cdots & & \cdots & 0 & h(p) & \cdots & & & h(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f(M) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- **f** and **g** have been completed artificially by some zeros. This operation is called zero-filling
- The main advantage to do so is that the matrix **H** is now a circulant matrix.

Kalman Filter equations for signal deconvolution

$$\left\{\begin{array}{lll} \mathbf{z}_k & = \mathbf{H}_k\,\mathbf{x}_k + \mathbf{v}_k & \text{observation equation} \\ \mathbf{x}_{k+1} & = \mathbf{F}_k\,\mathbf{x}_k + \mathbf{G}_k\,\mathbf{u}_k & \text{state equation} \end{array}\right.$$

$$\begin{array}{rcl} \widehat{\mathbf{x}}_{k+1|k} & = & \mathbf{F}_k\,\widehat{\mathbf{x}}_{k|k} \\ \mathbf{P}_{k+1|k} & = & \mathbf{F}_k\,\mathbf{P}_{k|k}\,\mathbf{F}_k^t + \mathbf{G}_k\,\mathbf{Q}_k\,\mathbf{G}_k^t \\ \widehat{\mathbf{x}}_{k+1|k+1} & = & \widehat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}^f[\mathbf{z}_{k+1} - \mathbf{H}_{k+1}\,\widehat{\mathbf{x}}_{k+1|k}] \\ \mathbf{K}_{k+1}^f & = & \mathbf{P}_{k+1|k}\,\mathbf{H}_{k+1}^t(\mathbf{R}_{k+1}^e)^{-1} \\ \mathbf{R}_{k+1}^e & = & \mathbf{R}_{k+1} + \mathbf{H}_{k+1}\,\mathbf{P}_{k+1|k}\,\mathbf{H}_{k+1}^t \\ \mathbf{P}_{k+1|k+1} & = & [\mathbf{I} - \mathbf{K}_{k+1}^f\mathbf{H}_{k+1}]\mathbf{P}_{k+1|k} \end{array}$$

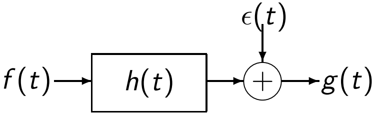
Content

1. Introduction: Signals and Images, Linear transformations (Convolution, Fourier, Laplace, Hilbert, Radon, ..., Discrete convolution, Z transform, DFT, FFT, ...)
2. Modeling: parametric and non-parametric, MA, AR and ARMA models, Linear prediction
3. Deconvolution and Parameter Estimation: Deterministic (LS, Regularization) and Probabilistic methods (Wiener Filtering)
4. Inverse problems and Bayesian estimation
5. Kalman Filtering and smoothing
6. **Case study: Signal deconvolution**
7. Case study: Image restoration
8. Case study: Image reconstruction and Computed Tomography

6. Case study: Signal deconvolution

- Convolution, Identification and Deconvolution
- Forward and Inverse problems: Well-posedness and Ill-posedness
- Naîve methods of Deconvolution
- Classical methods: Wiener filtering
- Bayesian approach to deconvolution
- Simple and Blind Deconvolution
- Deterministic and probabilistic methods
- Joint source and canal estimation

Convolution: Discretization



$$g(t) = \int f(t')\, h(t-t')\, \mathrm{d}t' + \epsilon(t) = \int h(t')\, f(t-t')\, \mathrm{d}t' + \epsilon(t)$$

- The signals $f(t)$, $g(t)$, $h(t)$ are discretized with the same sampling period $\Delta T = 1$,
- The impulse response is finite (FIR) : $h(t) = 0$, for t such that $t < -q\Delta T$ or $\forall t > p\Delta T$.

$$g(m) = \sum_{k=-q}^{p} h(k)\, f(m-k) + \epsilon(m), \quad m = 0, \cdots, M$$

Convolution: Discretized matrix vector forms

$$\begin{bmatrix} g(0)\\ g(1)\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ g(M)\end{bmatrix} = \begin{bmatrix} h(p) & \cdots & h(0) & \cdots & h(-q) & 0 & \cdots & \cdots & \cdots & 0\\ 0 & \ddots & & \ddots & & \ddots & & & & \vdots\\ \vdots & & & & & & & & & \vdots\\ \vdots & & h(p) & \cdots & h(0) & \cdots & h(-q) & & & \vdots\\ \vdots & & & & & & & & & \vdots\\ \vdots & & & \ddots & & & \ddots & & & \vdots\\ \vdots & & & & \ddots & & & & & 0\\ 0 & \cdots & \cdots & & 0 & h(p) & \cdots & h(0) & \cdots & h(-q)\end{bmatrix} \begin{bmatrix} f(-p)\\ \vdots\\ f(0)\\ f(1)\\ \vdots\\ \vdots\\ \vdots\\ f(M)\\ f(M+1)\\ \vdots\\ f(M+q)\end{bmatrix}$$

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \epsilon$$

- $\mathbf{g}$ is a $(M+1)$-dimensional vector,
- $\mathbf{f}$ has dimension $M+p+q+1$,
- $\mathbf{h} = [h(p), \cdots, h(0), \cdots, h(-q)]$ has dimension $(p+q+1)$
- $\mathbf{H}$ has dimensions $(M+1) \times (M+p+q+1)$.

Convolution: Discretized matrix vector form

- If system is causal ($q = 0$) we obtain

$$
\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix}
=
\begin{bmatrix}
h(p) & \cdots & h(0) & 0 & \cdots & \cdots & 0 \\
0 & & & & & & \vdots \\
\vdots & & & & & & \vdots \\
\vdots & & h(p) & \cdots & h(0) & & \vdots \\
\vdots & & & & & & \vdots \\
\vdots & & & & & & 0 \\
0 & \cdots & \cdots & 0 & h(p) & \cdots & h(0)
\end{bmatrix}
\begin{bmatrix} f(-p) \\ \vdots \\ f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f(M) \end{bmatrix}
$$

- $\mathbf{g}$ is a $(M+1)$-dimensional vector,
- $\mathbf{f}$ has dimension $M+p+1$,
- $\mathbf{h} = [h(p), \cdots, h(0)]$ has dimension $(p+1)$
- $\mathbf{H}$ has dimensions $(M+1) \times (M+p+1)$.

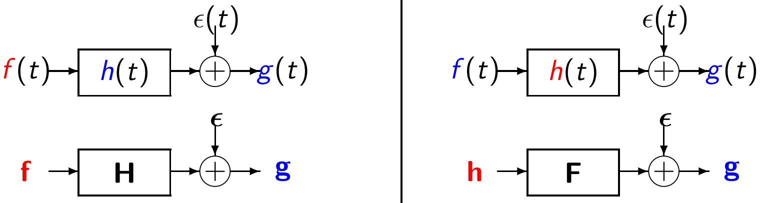
Convolution: Causal systems and causal input

$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix} = \begin{bmatrix} h(0) & & & & & \\ h(1) & \ddots & & & & \\ \vdots & & & & & \\ h(p) & \cdots & & h(0) & & \\ 0 & \ddots & & & \ddots & \\ \vdots & & & & & \\ 0 & \cdots & 0 & h(p) & \cdots & h(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f(M) \end{bmatrix}$$

- $\mathbf{g}$ is a $(M+1)$-dimensional vector,
- $\mathbf{f}$ has dimension $M+1$,
- $\mathbf{h} = [h(p), \cdots, h(0)]$ has dimension $(p+1)$
- $\mathbf{H}$ has dimensions $(M+1) \times (M+1)$.

Convolution, Identification, Simple Deconvolution and Blind deconvolution problems

$$g(t)=\int f(t')\,h(t-t')\,\mathrm{d}t'+\epsilon(t)=\int h(t')\,f(t-t')\,\mathrm{d}t'+\epsilon(t)$$



$$\mathbf{g}=\mathbf{H}\ \mathbf{f}+\boldsymbol{\epsilon} \qquad \mathbf{g}=\mathbf{F}\ \mathbf{h}+\boldsymbol{\epsilon}$$

$$\mathbf{g}=\mathbf{H}\,\mathbf{f}+\boldsymbol{\epsilon}=\mathbf{F}\,\mathbf{h}+\boldsymbol{\epsilon}$$

- Convolution: Given h and f compute g
- Identification: Given f and g estimate h
- Simple Deconvolution: Given h and g estimate f
- Blind Deconvolution: Given g estimate h and f

Convolution: Discretization for Identification

Causal systems and causal input

$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix} = \begin{bmatrix} 0 & & . & 0 & f(0) \\ . & & . & f(0) & f(1) \\ . & & f(0) & f(1) & \vdots \\ . & . & . & . & \vdots \\ f(0) & f(1) & & & f(M-p) \\ f(1) & . & & . & . \\ . & & & . & . \\ . & & . & & . \\ . & . & & & . \\ f(M-p) & . & . & . & f(M) \end{bmatrix} \begin{bmatrix} h(p) \\ h(p-1) \\ \vdots \\ \vdots \\ h(1) \\ h(0) \end{bmatrix}$$

$$\mathbf{g} = \mathbf{F}\,\mathbf{h} + \epsilon$$

- $\mathbf{g}$ is a $(M+1)$-dimensional vector,
- $\mathbf{F}$ has dimension $(M+1) \times (p+1)$,

Identification and Deconvolution

Deconvolution	Identification
$\mathbf{g} = \mathbf{H}\,\mathbf{f} + \epsilon$	$\mathbf{g} = \mathbf{F}\,\mathbf{h} + \epsilon$
$J(\mathbf{f}) = \lVert\mathbf{g} - \mathbf{H}\mathbf{f}\rVert^2 + \lambda_f \lVert\mathbf{D}_f\mathbf{f}\rVert^2$ $\nabla J(\mathbf{f}) = -2\mathbf{H}'(\mathbf{g} - \mathbf{H}\mathbf{f}) + 2\lambda_f \mathbf{D}_f'\mathbf{D}_f\mathbf{f}$ $\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f'\mathbf{D}_f]^{-1}\mathbf{H}'\mathbf{g}$ $\widehat{f}(\omega) = \frac{H^*(\omega)}{\lvert H(\omega)\rvert^2 + \lambda_f \lvert D_f(\omega)\rvert^2}\, g(\omega)$	$J(\mathbf{h}) = \lVert\mathbf{g} - \mathbf{F}\mathbf{h}\rVert^2 + \lambda_h \lVert\mathbf{D}_h\mathbf{h}\rVert^2$ $\nabla J(\mathbf{h}) = -2\mathbf{F}'(\mathbf{g} - \mathbf{F}\mathbf{h}) + 2\lambda_h \mathbf{D}_h'\mathbf{D}_h\mathbf{h}$ $\widehat{\mathbf{h}} = [\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h'\mathbf{D}_h]^{-1}\mathbf{F}'\mathbf{g}$ $\widehat{f}(\omega) = \frac{\lvert F^*(\omega)}{\lvert F(\omega)\rvert^2 + \lambda_h \lvert D_h(\omega)\rvert^2}\, g(\omega)$
$\widehat{f}(\omega) = \frac{H^*(\omega)}{\lvert H(\omega)\rvert^2 + \frac{S_{\epsilon\epsilon}(\omega)}{S_{ff}(\omega)}}\, g(\omega)$	$\widehat{f}(\omega) = \frac{F^*(\omega)}{\lvert F(\omega)\rvert^2 + \frac{S_{\epsilon\epsilon}(\omega)}{S_{hh}(\omega)}}\, g(\omega)$
$p(\mathbf{g}\vert\mathbf{f}) = \mathcal{N}(\mathbf{H}\mathbf{f}, \boldsymbol{\Sigma}_\epsilon)$ $p(\mathbf{f}) = \mathcal{N}(0, \boldsymbol{\Sigma}_f)$	$p(\mathbf{g}\vert\mathbf{h}) = \mathcal{N}(\mathbf{F}\mathbf{h}, \boldsymbol{\Sigma}_\epsilon)$ $p(\mathbf{h}) = \mathcal{N}(0, \boldsymbol{\Sigma}_h)$
$p(\mathbf{f}\vert\mathbf{g}) = \mathcal{N}(\widehat{\mathbf{f}}, \widehat{\boldsymbol{\Sigma}}_f)$ $\widehat{\boldsymbol{\Sigma}}_f = [\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f'\mathbf{D}_f]^{-1}$ $\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f'\mathbf{D}_f]^{-1}\mathbf{H}'\mathbf{g}$	$p(\mathbf{h}\vert\mathbf{g}) = \mathcal{N}(\widehat{\mathbf{h}}, \widehat{\boldsymbol{\Sigma}}_h)$ $\widehat{\boldsymbol{\Sigma}}_h = [\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h'\mathbf{D}_h]^{-1}$ $\widehat{\mathbf{h}} = [\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h'\mathbf{D}_h]^{-1}\mathbf{F}'\mathbf{g}$

Blind Deconvolution: Regularization

Deconvolution	Identification
$\mathbf{g} = \mathbf{H}\,\mathbf{f} + \epsilon$	$\mathbf{g} = \mathbf{F}\,\mathbf{h} + \epsilon$
$J(\mathbf{f}) = \Vert\mathbf{g} - \mathbf{H}\mathbf{f}\Vert^2 + \lambda_f \Vert\mathbf{D}_f\mathbf{f}\Vert^2$	$J(\mathbf{h}) = \Vert\mathbf{g} - \mathbf{F}\mathbf{h}\Vert^2 + \lambda_h \Vert\mathbf{D}_h\mathbf{h}\Vert^2$

- Joint Criterion

$$J(\mathbf{f},\mathbf{h}) = \Vert\mathbf{g} - \mathbf{H}\mathbf{f}\Vert^2 + \lambda_f \Vert\mathbf{D}_f\mathbf{f}\Vert^2 + \lambda_h \Vert\mathbf{D}_h\mathbf{h}\Vert^2$$

- iterative algorithm

Deconvolution	Identification
$\nabla_f J(\mathbf{f},\mathbf{h}) = -2\mathbf{H}'(\mathbf{g} - \mathbf{H}\mathbf{f}) + 2\lambda_f \mathbf{D}_f'\mathbf{D}_f\mathbf{f}$	$\nabla_h J(\mathbf{f},\mathbf{h}) = -2\mathbf{F}'(\mathbf{g} - \mathbf{F}\mathbf{h}) + 2\lambda_h \mathbf{D}_h'\mathbf{D}_h\mathbf{h}$
$\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f'\mathbf{D}_f]^{-1}\mathbf{H}'\mathbf{g}$	$\widehat{\mathbf{h}} = [\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h'\mathbf{D}_h]^{-1}\mathbf{F}'\mathbf{g}$
$\widehat{f}(\omega) = \frac{1}{H(\omega)} \frac{\vert H(\omega)\vert^2}{\vert H(\omega)\vert^2 + \lambda_f \vert D_f(\omega)\vert^2}\, g(\omega)$	$\widehat{f}(\omega) = \frac{1}{F(\omega)} \frac{\vert F(\omega)\vert^2}{\vert F(\omega)\vert^2 + \lambda_h \vert D_h(\omega)\vert^2}\, g(\omega)$

Blind Deconvolution: Bayesian approach

Deconvolution	Identification
$\mathbf{g} = \mathbf{H}\,\mathbf{f} + \boldsymbol{\epsilon}$	$\mathbf{g} = \mathbf{F}\,\mathbf{h} + \boldsymbol{\epsilon}$
$p(\mathbf{g} \mid \mathbf{f}) = \mathcal{N}(\mathbf{H}\mathbf{f}, \boldsymbol{\Sigma}_\epsilon)$	$p(\mathbf{g} \mid \mathbf{h}) = \mathcal{N}(\mathbf{F}\mathbf{h}, \boldsymbol{\Sigma}_\epsilon)$
$p(\mathbf{f}) = \mathcal{N}(0, \boldsymbol{\Sigma}_f)$	$p(\mathbf{h}) = \mathcal{N}(0, \boldsymbol{\Sigma}_h)$
$p(\mathbf{f} \mid \mathbf{g}) = \mathcal{N}(\widehat{\mathbf{f}}, \widehat{\boldsymbol{\Sigma}}_f)$	$p(\mathbf{h} \mid \mathbf{g}) = \mathcal{N}(\widehat{\mathbf{h}}, \widehat{\boldsymbol{\Sigma}}_h)$
$\widehat{\boldsymbol{\Sigma}}_f = [\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f'\mathbf{D}_f]^{-1}$	$\widehat{\boldsymbol{\Sigma}}_h = [\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h'\mathbf{D}_h]^{-1}$
$\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f'\mathbf{D}_f]^{-1}\mathbf{H}'\mathbf{g}$	$\widehat{\mathbf{h}} = [\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h'\mathbf{D}_h]^{-1}\mathbf{F}'\mathbf{g}$

- Joint posterior law:

$$p(\mathbf{f}, \mathbf{h} \mid \mathbf{g}) \propto p(\mathbf{g} \mid \mathbf{f}, \mathbf{h})\, p(\mathbf{f})\, p(h\mathbf{h})$$

$$p(\mathbf{f}, \mathbf{h} \mid \mathbf{g}) \propto \exp\left\{-J(\mathbf{f}, \mathbf{h})\right\}$$

with

$$J(\mathbf{f}, \mathbf{h}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda_f \|\mathbf{D}_f \mathbf{f}\|^2 + \lambda_h \|\mathbf{D}_h \mathbf{h}\|^2$$

- iterative algorithm

Blind Deconvolution: Bayesian Joint MAP criterion

- Joint posterior law:

$$p(\mathbf{f},\mathbf{h}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f},\mathbf{h})\, p(\mathbf{f})\, p(h\mathbf{h})$$

$$p(\mathbf{f},\mathbf{h}|\mathbf{g}) \propto \exp\left\{-J(\mathbf{f},\mathbf{h})\right\}$$

with

$$J(\mathbf{f},\mathbf{h}) = \|\mathbf{g}-\mathbf{H}\mathbf{f}\|^2 + \lambda_f\|\mathbf{D}_f\mathbf{f}\|^2 + \lambda_h\|\mathbf{D}_h\mathbf{h}\|^2$$

- iterative algorithm

Deconvolution	Identification
$p(\mathbf{g}\|\mathbf{f},\mathbf{H}) = \mathcal{N}(\mathbf{H}\mathbf{f},\boldsymbol{\Sigma}_\epsilon)$	$p(\mathbf{g}\|\mathbf{h},\mathbf{F}) = \mathcal{N}(\mathbf{F}\mathbf{h},\boldsymbol{\Sigma}_\epsilon)$
$p(\mathbf{f}) = \mathcal{N}(0,\boldsymbol{\Sigma}_f)$	$p(\mathbf{h}) = \mathcal{N}(0,\boldsymbol{\Sigma}_h)$
$p(\mathbf{f}\|\mathbf{g},\mathbf{H}) = \mathcal{N}(\widehat{\mathbf{f}},\widehat{\boldsymbol{\Sigma}}_f)$	$p(\mathbf{h}\|\mathbf{g},\mathbf{F}) = \mathcal{N}(\widehat{\mathbf{h}},\widehat{\boldsymbol{\Sigma}}_h)$
$\widehat{\boldsymbol{\Sigma}}_f = [\mathbf{H}'\mathbf{H}+\lambda_f\mathbf{D}_f'\mathbf{D}_f]^{-1}$	$\widehat{\boldsymbol{\Sigma}}_h = [\mathbf{F}'\mathbf{F}+\lambda_h\mathbf{D}_h'\mathbf{D}_h]^{-1}$
$\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H}+\lambda_f\mathbf{D}_f'\mathbf{D}_f]^{-1}\mathbf{H}'\mathbf{g}$	$\widehat{\mathbf{h}} = [\mathbf{F}'\mathbf{F}+\lambda_h\mathbf{D}_h'\mathbf{D}_h]^{-1}\mathbf{F}'\mathbf{g}$

Blind Deconvolution: Marginalization and EM algorithm

- Joint posterior law:
- Marginalization

$$p(\mathbf{f},\mathbf{h}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f},\mathbf{h})\, p(\mathbf{f})\, p(h\mathbf{h})$$

$$p(\mathbf{h}|\mathbf{g}) = \int p(\mathbf{f},\mathbf{h}|\mathbf{g})\, \mathrm{d}\mathbf{f}$$

$$\widehat{\mathbf{h}} = \arg\max_{\mathbf{h}} \{p(\mathbf{h}|\mathbf{g})\} \longrightarrow \widehat{\mathbf{f}} = \arg\max_{\mathbf{f}} \left\{p(\mathbf{f}|\mathbf{g},\widehat{\mathbf{h}})\right\}$$

- Expression of $p(\mathbf{h}|\mathbf{g})$ and its maximization are complexes
- Expectation-Maximization Algorithm

$$\ln p(\mathbf{f},\mathbf{h}|\mathbf{g}) \propto J(\mathbf{f},\mathbf{h}) = \|\mathbf{g}-\mathbf{H}\mathbf{f}\|^2 + \lambda_f\|\mathbf{D}_f\mathbf{f}\|^2 + \lambda_h\|\mathbf{D}_h\mathbf{h}\|^2$$

  - Iterative algorithm
  - Expectation: Compute

$$Q(\mathbf{h},\mathbf{h}^{k-1}) = \mathrm{E}_{p(\mathbf{f},\mathbf{h}^{k-1}|\mathbf{g})}\{J(\mathbf{f},\mathbf{h})\} = \langle \ln p(\mathbf{f},\mathbf{h}|\mathbf{g})\rangle_{p(\mathbf{f},\mathbf{h}^{k-1}|\mathbf{g})}$$

  - Maximization:

$$\mathbf{h}^k = \arg\max_{\mathbf{h}} \left\{Q(\mathbf{h},\mathbf{h}^{k-1})\right\}$$

Blind Deconvolution: Variational Bayesian Approximation algorithm

- Joint posterior law:

$$p(\mathbf{f},\mathbf{h}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f},\mathbf{h})\, p(\mathbf{f})\, p(h\mathbf{h})$$

- Approximation: $p(\mathbf{f},\mathbf{h}|\mathbf{g})$ by $q(\mathbf{f},\mathbf{h}|\mathbf{g}) = q_1(\mathbf{f})\, q_2(\mathbf{h})$
- Criterion of approximation: Kullback-Leiler

$$\mathrm{KL}(q|p) = \int q \ln \frac{q}{p} = \int q_1\, q_2 \ln \frac{q_1\, q_2}{p}$$

$$\begin{aligned} \mathrm{KL}(q_1\, q_2|p) &= \int q_1 \ln q_1 + \int q_2 \ln q_2 - \int q \ln p \\ &= -\mathcal{H}(q_1) - \mathcal{H}(q_2) + \langle -\ln p((\mathbf{f},\mathbf{h}|\mathbf{g})\rangle_q \end{aligned}$$

- When the expression of q_1 and q_2 are obtained, use them.

Variational Bayesian Approximation algorithm

- Kullback-Leibler criterion

$$\begin{aligned} \mathrm{KL}(q_1\, q_2|p) &= \int q_1 \,\ln q_1 + \int q_2 \,\ln q_2 + \int q \ln p \\ &= -\mathcal{H}(q_1) - \mathcal{H}(q_2) + \langle -\ln p((\mathbf{f},\mathbf{h}|\mathbf{g})\rangle_q \end{aligned}$$

- Free energy

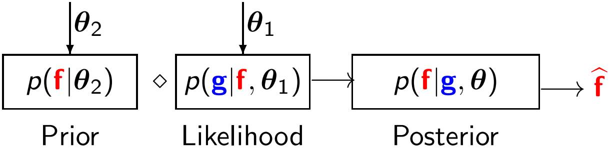
$$\mathcal{F}(q_1\, q_2) = -\left\langle \ln p((\mathbf{f},\mathbf{h}|\mathbf{g})\right\rangle_{q_1 q_2}$$

- Equivalence between optimization of $\mathrm{KL}(q_1\, q_2|p)$ and $\mathcal{F}(q_1\, q_2)$
- Alternate optimization:

$$\begin{aligned} \widehat{q}_1 &= \arg\min_{q_1}\left\{\mathrm{KL}(q_1\, q_2|p)\right\} = \arg\min_{q_1}\left\{\mathcal{F}(q_1\, q_2)\right\} \\ \widehat{q}_2 &= \arg\min_{q_2}\left\{\mathrm{KL}(q_1\, q_2|p)\right\} = \arg\min_{q_2}\left\{\mathcal{F}(q_1\, q_2)\right\} \end{aligned}$$

Summary of Bayesian estimation for Deconvolution

- Simple Bayesian Model and Estimation for Deconvolution

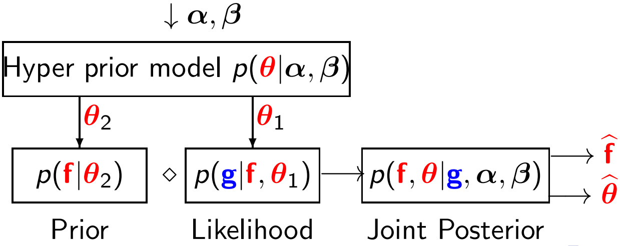


θ_2 θ_1

$p(\mathbf{f}|\theta_2)$ $\diamond$ $p(\mathbf{g}|\mathbf{f},\theta_1)$ $\longrightarrow$ $p(\mathbf{f}|\mathbf{g},\theta)$ $\longrightarrow \widehat{\mathbf{f}}$

Prior Likelihood Posterior

- Full Bayesian Model and Hyperparameter Estimation for Deconvolution



$\downarrow \alpha,\beta$

Hyper prior model $p(\theta|\alpha,\beta)$

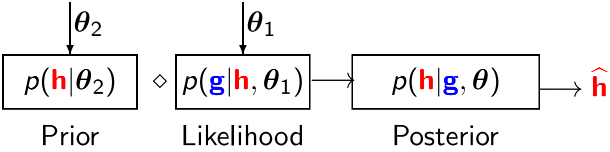
θ_2 θ_1

$p(\mathbf{f}|\theta_2)$ $\diamond$ $p(\mathbf{g}|\mathbf{f},\theta_1)$ $\longrightarrow$ $p(\mathbf{f},\theta|\mathbf{g},\alpha,\beta)$ $\longrightarrow \widehat{\mathbf{f}}$ $\longrightarrow \widehat{\theta}$

Prior Likelihood Joint Posterior

Summary of Bayesian estimation for Identification

- Simple Bayesian Model and Estimation for Identification

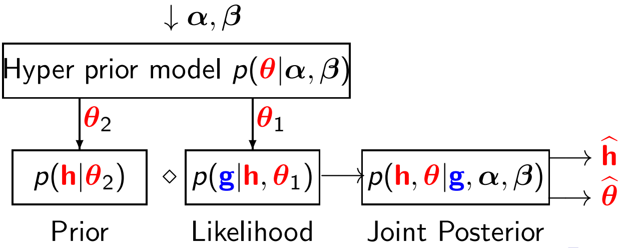


$\boldsymbol{\theta}_2$ $\boldsymbol{\theta}_1$

$p(\mathbf{h}|\boldsymbol{\theta}_2) \diamond p(\mathbf{g}|\mathbf{h},\boldsymbol{\theta}_1) \longrightarrow p(\mathbf{h}|\mathbf{g},\boldsymbol{\theta}) \longrightarrow \widehat{\mathbf{h}}$

Prior Likelihood Posterior

- Full Bayesian Model and Hyperparameter Estimation for Identification



$\downarrow \boldsymbol{\alpha},\boldsymbol{\beta}$

Hyper prior model $p(\boldsymbol{\theta}|\boldsymbol{\alpha},\boldsymbol{\beta})$

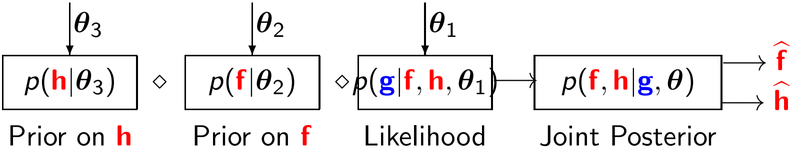
$\boldsymbol{\theta}_2$ $\boldsymbol{\theta}_1$

$p(\mathbf{h}|\boldsymbol{\theta}_2) \diamond p(\mathbf{g}|\mathbf{h},\boldsymbol{\theta}_1) \longrightarrow p(\mathbf{h},\boldsymbol{\theta}|\mathbf{g},\boldsymbol{\alpha},\boldsymbol{\beta})$ $\longrightarrow \widehat{\mathbf{h}}$ $\longrightarrow \widehat{\boldsymbol{\theta}}$

Prior Likelihood Joint Posterior

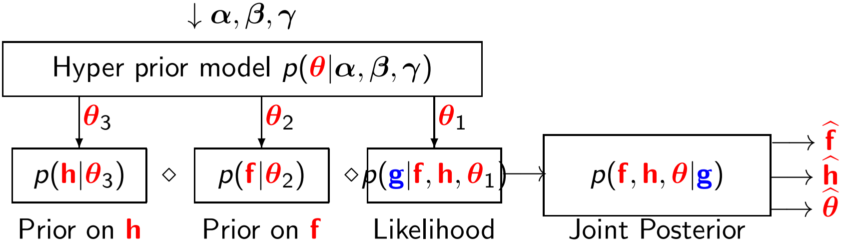
Summary of Bayesian estimation for Blind Deconvolution

Known hyperparameters $\boldsymbol{\theta}$



$$\begin{tikzcd} \boxed{p(\mathbf{h}|\boldsymbol{\theta}_3)} \arrow[r, phantom, "\diamond"] & \boxed{p(\mathbf{f}|\boldsymbol{\theta}_2)} \arrow[r, phantom, "\diamond"] & \boxed{p(\mathbf{g}|\mathbf{f},\mathbf{h},\boldsymbol{\theta}_1)} \arrow[r] & \boxed{p(\mathbf{f},\mathbf{h}|\mathbf{g},\boldsymbol{\theta})} \arrow[r] \arrow[r] & \widehat{\mathbf{f}},\ \widehat{\mathbf{h}} \end{tikzcd}$$

Unknown hyperparameters $\boldsymbol{\theta}$



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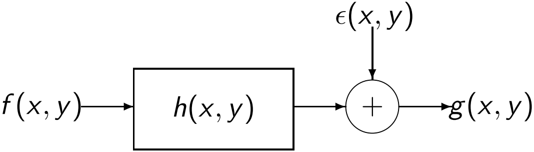
8. Case study: Image reconstruction and Computed Tomography

7. Case study: Image restoration

- Image restoration
- Classical methods: Wiener filtering in 2D
- Bayesian approach to deconvolution
- Simple and Blind Deconvolution
- Practical issues and computational costs

Convolution in 2D: Discretization



$$g(x,y) = \int f(x',y')\, h(x-x')\, \mathrm{d}x\, \mathrm{d}y + \epsilon(x,y)$$

- Images $f(x,y)$, $g(x,y)$, $h(x,y)$ are discretized with the same sampling period $\Delta x = \Delta y = 1$,
- The impulse response is finite (FIR) : $h(x,y) = 0$, for $|x - p\Delta x| < 0$ and $|y - p\Delta y| < 0$.

$$g(m,n) = \sum_{k=-p}^{p} \sum_{l=-p}^{p} h(k,l)\, f(m-k, n-l) + \epsilon(m,n)$$

Convolution: Discretized matrix vector forms

$$g(m,n)=\sum_{k=-p}^{p}\sum_{l=-p}^{p} h(k,l)\, f(m-k,n-l)+\epsilon(m,n)$$

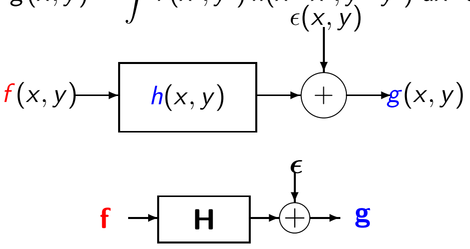
- Matlab notation `g=conv2(h,f);`

$$\mathbf{g}=\mathbf{H}\mathbf{f}+\epsilon$$

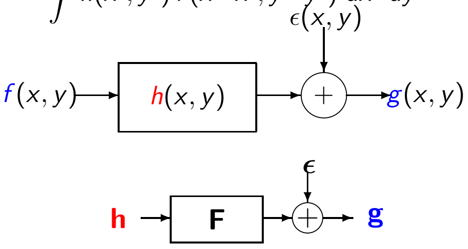
- $\mathbf{g}$ is a vector of dimension $(M \times M)$.
- $\mathbf{f}$ is a vector of dimension $(M+2p+1)\times(M+2p+1)$.
- $\mathbf{H}$ is a Toeplitz-Bloc-Toeplitz matrix of dimensions $(M \times M)\times((M+2p+1)\times(M+2p+1))$.

Convolution, Identification, Simple Deconvolution and Blind deconvolution problems

$$g(x,y) = \int f(x',y')\, h(x-x',y-y')\, \mathrm{d}x'\, \mathrm{d}y' = \int h(x',y')\, f(x-x',y-y')\, \mathrm{d}x'\, \mathrm{d}y'$$



$\mathbf{g} = \mathbf{H}\ \mathbf{f} + \epsilon$



$\mathbf{g} = \mathbf{F}\ \mathbf{h} + \epsilon$

$$\mathbf{g} = \mathbf{H}\,\mathbf{f} + \epsilon = \mathbf{F}\,\mathbf{h} + \epsilon$$

- Convolution: Given h and f compute g
- Identification: Given f and g estimate h
- Simple Deconvolution: Given h and g estimate f
- Blind Deconvolution: Given g estimate h and f

Identification and Deconvolution

Deconvolution	Identification
$\mathbf{g} = \mathbf{H}\,\mathbf{f} + \epsilon$	$\mathbf{g} = \mathbf{F}\,\mathbf{h} + \epsilon$
$J(\mathbf{f}) = \Vert\mathbf{g} - \mathbf{H}\mathbf{f}\Vert^2 + \lambda_f \Vert\mathbf{D}_f\mathbf{f}\Vert^2$ $\nabla J(\mathbf{f}) = -2\mathbf{H}'(\mathbf{g} - \mathbf{H}\mathbf{f}) + 2\lambda_f \mathbf{D}_f'\mathbf{D}_f\mathbf{f}$ $\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f'\mathbf{D}_f]^{-1}\mathbf{H}'\mathbf{g}$	$J(\mathbf{h}) = \Vert\mathbf{g} - \mathbf{F}\mathbf{h}\Vert^2 + \lambda_h \Vert\mathbf{D}_h\mathbf{h}\Vert^2$ $\nabla J(\mathbf{h}) = -2\mathbf{F}'(\mathbf{g} - \mathbf{F}\mathbf{h}) + 2\lambda_h \mathbf{D}_h'\mathbf{D}_h\mathbf{h}$ $\widehat{\mathbf{h}} = [\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h'\mathbf{D}_h]^{-1}\mathbf{F}'\mathbf{g}$
$\widehat{f}(u,v) = \frac{H^*(u,v)}{\lvert H(u,v)\rvert^2 + \lambda_f \lvert D_f(u,v)\rvert^2}\, g(u,v)$ $\widehat{f}(u,v) = \frac{H^*(u,v)}{\lvert H(u,v)\rvert^2 + \frac{S_{\epsilon\epsilon}(u,v)}{S_{ff}(u,v)}}\, g(u,v)$	$\widehat{f}(u,v) = \frac{F^*(u,v)}{\lvert F(u,v)\rvert^2 + \lambda_h \lvert D_h(u,v)\rvert^2}\, g(u,v)$ $\widehat{f}(u,v) = \frac{F^*(u,v)}{\lvert F(u,v)\rvert^2 + \frac{S_{\epsilon\epsilon}(u,v)}{S_{ff}(u,v)}}\, g(u,v)$
$p(\mathbf{g}\mid\mathbf{f}) = \mathcal{N}(\mathbf{H}\mathbf{f}, \mathbf{\Sigma}_\epsilon)$ $p(\mathbf{f}) = \mathcal{N}(0, \mathbf{\Sigma}_f)$	$p(\mathbf{g}\mid\mathbf{h}) = \mathcal{N}(\mathbf{F}\mathbf{h}, \mathbf{\Sigma}_\epsilon)$ $p(\mathbf{h}) = \mathcal{N}(0, \mathbf{\Sigma}_h)$
$p(\mathbf{f}\mid\mathbf{g}) = \mathcal{N}(\widehat{\mathbf{f}}, \widehat{\mathbf{\Sigma}}_f)$ $\widehat{\mathbf{\Sigma}}_f = [\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f'\mathbf{D}_f]^{-1}$ $\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f'\mathbf{D}_f]^{-1}\mathbf{H}'\mathbf{g}$	$p(\mathbf{h}\mid\mathbf{g}) = \mathcal{N}(\widehat{\mathbf{h}}, \widehat{\mathbf{\Sigma}}_h)$ $\widehat{\mathbf{\Sigma}}_h = [\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h'\mathbf{D}_h]^{-1}$ $\widehat{\mathbf{h}} = [\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h'\mathbf{D}_h]^{-1}\mathbf{F}'\mathbf{g}$

Blind Deconvolution: Regularization

Deconvolution	Identification
$\mathbf{g} = \mathbf{H}\,\mathbf{f} + \epsilon$	$\mathbf{g} = \mathbf{F}\,\mathbf{h} + \epsilon$
$J(\mathbf{f}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda_f \|\mathbf{D}_f \mathbf{f}\|^2$	$J(\mathbf{h}) = \|\mathbf{g} - \mathbf{F}\mathbf{h}\|^2 + \lambda_h \|\mathbf{D}_h \mathbf{h}\|^2$

- Joint Criterion

$$J(\mathbf{f}, \mathbf{h}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda_f \|\mathbf{D}_f \mathbf{f}\|^2 + \lambda_h \|\mathbf{D}_h \mathbf{h}\|^2$$

- iterative algorithm

Deconvolution	Identification
$\nabla_f J(\mathbf{f}, \mathbf{h}) = -2\mathbf{H}'(\mathbf{g} - \mathbf{H}\mathbf{f}) + 2\lambda_f \mathbf{D}_f' \mathbf{D}_f \mathbf{f}$	$\nabla_h J(\mathbf{f}, \mathbf{h}) = -2\mathbf{F}'(\mathbf{g} - \mathbf{F}\mathbf{h}) + 2\lambda_h \mathbf{D}_h' \mathbf{D}_h \mathbf{h}$
$\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f' \mathbf{D}_f]^{-1} \mathbf{H}' \mathbf{g}$	$\widehat{\mathbf{h}} = [\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h' \mathbf{D}_h]^{-1} \mathbf{F}' \mathbf{g}$
$\widehat{f}(u, v) = \frac{H^*(u,v)}{\|H(u,v)\|^2 + \lambda_f \|D_f(u,v)\|^2}\, g(u, v)$	$\widehat{f}(u, v) = \frac{F^*(u,v)}{\|F(u,v)\|^2 + \lambda_h \|D_h(u,v)\|^2}\, g(u, v)$

Blind Deconvolution: Bayesian approach

Deconvolution	Identification
$\mathbf{g} = \mathbf{H}\,\mathbf{f} + \boldsymbol{\epsilon}$	$\mathbf{g} = \mathbf{F}\,\mathbf{h} + \boldsymbol{\epsilon}$
$p(\mathbf{g}\vert\mathbf{f}) = \mathcal{N}(\mathbf{H}\mathbf{f}, \boldsymbol{\Sigma}_\epsilon)$	$p(\mathbf{g}\vert\mathbf{h}) = \mathcal{N}(\mathbf{F}\mathbf{h}, \boldsymbol{\Sigma}_\epsilon)$
$p(\mathbf{f}) = \mathcal{N}(0, \boldsymbol{\Sigma}_f)$	$p(\mathbf{h}) = \mathcal{N}(0, \boldsymbol{\Sigma}_h)$
$p(\mathbf{f}\vert\mathbf{g}) = \mathcal{N}(\widehat{\mathbf{f}}, \widehat{\boldsymbol{\Sigma}}_f)$	$p(\mathbf{h}\vert\mathbf{g}) = \mathcal{N}(\widehat{\mathbf{h}}, \widehat{\boldsymbol{\Sigma}}_h)$
$\widehat{\boldsymbol{\Sigma}}_f = [\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f'\mathbf{D}_f]^{-1}$	$\widehat{\boldsymbol{\Sigma}}_h = [\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h'\mathbf{D}_h]^{-1}$
$\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f'\mathbf{D}_f]^{-1}\mathbf{H}'\mathbf{g}$	$\widehat{\mathbf{h}} = [\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h'\mathbf{D}_h]^{-1}\mathbf{F}'\mathbf{g}$

- Joint posterior law:

$$p(\mathbf{f},\mathbf{h}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f},\mathbf{h})\,p(\mathbf{f})\,p(h\mathbf{h})$$

$$p(\mathbf{f},\mathbf{h}|\mathbf{g}) \propto \exp\{-J(\mathbf{f},\mathbf{h})\}$$

with

$$J(\mathbf{f},\mathbf{h}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda_f\|\mathbf{D}_f\mathbf{f}\|^2 + \lambda_h\|\mathbf{D}_h\mathbf{h}\|^2$$

- iterative algorithm

Blind Deconvolution: Bayesian Joint MAP criterion

- Joint posterior law:

$$p(\mathbf{f},\mathbf{h}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f},\mathbf{h})\, p(\mathbf{f})\, p(h\mathbf{h})$$

$$p(\mathbf{f},\mathbf{h}|\mathbf{g}) \propto \exp\{-J(\mathbf{f},\mathbf{h})\}$$

with

$$J(\mathbf{f},\mathbf{h}) = \|\mathbf{g}-\mathbf{H}\mathbf{f}\|^2 + \lambda_f \|\mathbf{D}_f\mathbf{f}\|^2 + \lambda_h \|\mathbf{D}_h\mathbf{h}\|^2$$

- iterative algorithm

Deconvolution	Identification
$p(\mathbf{g}\mid\mathbf{f},\mathbf{H}) = \mathcal{N}(\mathbf{H}\mathbf{f},\mathbf{\Sigma}_\epsilon)$	$p(\mathbf{g}\mid\mathbf{h},\mathbf{F}) = \mathcal{N}(\mathbf{F}\mathbf{h},\mathbf{\Sigma}_\epsilon)$
$p(\mathbf{f}) = \mathcal{N}(0,\mathbf{\Sigma}_f)$	$p(\mathbf{h}) = \mathcal{N}(0,\mathbf{\Sigma}_h)$
$p(\mathbf{f}\mid\mathbf{g},\mathbf{H}) = \mathcal{N}(\widehat{\mathbf{f}},\widehat{\mathbf{\Sigma}}_f)$	$p(\mathbf{h}\mid\mathbf{g},\mathbf{F}) = \mathcal{N}(\widehat{\mathbf{h}},\widehat{\mathbf{\Sigma}}_h)$
$\widehat{\mathbf{\Sigma}}_f = [\mathbf{H}'\mathbf{H}+\lambda_f\mathbf{D}_f'\mathbf{D}_f]^{-1}$	$\widehat{\mathbf{\Sigma}}_h = [\mathbf{F}'\mathbf{F}+\lambda_h\mathbf{D}_h'\mathbf{D}_h]^{-1}$
$\widehat{\mathbf{f}} = [\mathbf{H}'\mathbf{H}+\lambda_f\mathbf{D}_f'\mathbf{D}_f]^{-1}\mathbf{H}'\mathbf{g}$	$\widehat{\mathbf{h}} = [\mathbf{F}'\mathbf{F}+\lambda_h\mathbf{D}_h'\mathbf{D}_h]^{-1}\mathbf{F}'\mathbf{g}$

Blind Deconvolution: Marginalization and EM algorithm

- Joint posterior law:
- Marginalization

$$p(\mathbf{f},\mathbf{h}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f},\mathbf{h})\, p(\mathbf{f})\, p(h\mathbf{h})$$

$$p(\mathbf{h}|\mathbf{g}) = \int p(\mathbf{f},\mathbf{h}|\mathbf{g})\, \mathrm{d}\mathbf{f}$$

$$\widehat{\mathbf{h}} = \arg\max_{\mathbf{h}} \{p(\mathbf{h}|\mathbf{g})\} \longrightarrow \widehat{\mathbf{f}} = \arg\max_{\mathbf{f}} \left\{p(\mathbf{f}|\mathbf{g},\widehat{\mathbf{h}})\right\}$$

- Expression of $p(\mathbf{h}|\mathbf{g})$ and its maximization are complexes
- Expectation-Maximization Algorithm

$$\ln p(\mathbf{f},\mathbf{h}|\mathbf{g}) \propto J(\mathbf{f},\mathbf{h}) = \|\mathbf{g}-\mathbf{H}\mathbf{f}\|^2 + \lambda_f\|\mathbf{D}_f\mathbf{f}\|^2 + \lambda_h\|\mathbf{D}_h\mathbf{h}\|^2$$

  - Iterative algorithm
  - Expectation: Compute

$$Q(\mathbf{h},\mathbf{h}^{k-1}) = \mathrm{E}_{p(\mathbf{f},\mathbf{h}^{k-1}|\mathbf{g})}\{J(\mathbf{f},\mathbf{h})\} = \langle \ln p(\mathbf{f},\mathbf{h}|\mathbf{g})\rangle_{p(\mathbf{f},\mathbf{h}^{k-1}|\mathbf{g})}$$

  - Maximization:

$$\mathbf{h}^k = \arg\max_{\mathbf{h}}\left\{Q(\mathbf{h},\mathbf{h}^{k-1})\right\}$$

Blind Deconvolution: Variational Bayesian Approximation algorithm

- Joint posterior law:

$$p(\mathbf{f},\mathbf{h}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f},\mathbf{h})\, p(\mathbf{f})\, p(h\mathbf{h})$$

- Approximation: $p(\mathbf{f},\mathbf{h}|\mathbf{g})$ by $q(\mathbf{f},\mathbf{h}|\mathbf{g}) = q_1(\mathbf{f})\, q_2(\mathbf{h})$
- Criterion of approximation: Kullback-Leiler

$$\mathrm{KL}(q|p) = \int q \ln \frac{q}{p} = \int q_1\, q_2 \ln \frac{q_1\, q_2}{p}$$

$$\begin{aligned}\mathrm{KL}(q_1\, q_2|p) &= \int q_1 \ln q_1 + \int q_2 \ln q_2 - \int q \ln p \\ &= -\mathcal{H}(q_1) - \mathcal{H}(q_2) + \langle - \ln p((\mathbf{f},\mathbf{h}|\mathbf{g})\rangle_q\end{aligned}$$

- When the expression of q_1 and q_2 are obtained, use them.

Variational Bayesian Approximation algorithm

- Kullback-Leibler criterion

$$\begin{aligned} \mathrm{KL}(q_1\, q_2|p) &= \int q_1 \ln q_1 + \int q_2 \ln q_2 + \int q \ln p \\ &= -\mathcal{H}(q_1) - \mathcal{H}(q_2) + \langle -\ln p((\mathbf{f},\mathbf{h}|\mathbf{g})\rangle_q \end{aligned}$$

- Free energy

$$\mathcal{F}(q_1\, q_2) = -\langle \ln p((\mathbf{f},\mathbf{h}|\mathbf{g})\rangle_{q_1 q_2}$$

- Equivalence between optimization of $\mathrm{KL}(q_1\, q_2|p)$ and $\mathcal{F}(q_1\, q_2)$
- Alternate optimization:

$$\begin{aligned} \widehat{q}_1 &= \arg\min_{q_1} \{\mathrm{KL}(q_1\, q_2|p)\} = \arg\min_{q_1} \{\mathcal{F}(q_1\, q_2)\} \\ \widehat{q}_2 &= \arg\min_{q_2} \{\mathrm{KL}(q_1\, q_2|p)\} = \arg\min_{q_2} \{\mathcal{F}(q_1\, q_2)\} \end{aligned}$$

Blind Deconvolution algorithm

$$\begin{tikzcd}
\mathbf{h}^{(0)} \arrow[r] & \mathbf{H} \arrow[r] & \boxed{\left(\mathbf{H}'\mathbf{H} + \lambda_f \mathbf{D}_f' \mathbf{D}_f\right)^{-1} \mathbf{H}'\mathbf{g}} \arrow[r] & \widehat{\mathbf{f}}(t) \arrow[d] \\
 & \widehat{\mathbf{h}} \arrow[u] & \boxed{\left(\mathbf{F}'\mathbf{F} + \lambda_h \mathbf{D}_h' \mathbf{D}_h\right)^{-1} \mathbf{F}'\mathbf{g}} \arrow[l] & \mathbf{F} \arrow[l]
\end{tikzcd}$$

Joint Estimation of **h** and **f** with a Gaussian prior model..

VBA: $p(\mathbf{f},\mathbf{h}|\mathbf{g}) \longrightarrow q_1(\mathbf{f}|\mathbf{h},\mathbf{g})\; q_2(\mathbf{h}|\mathbf{f},\mathbf{g})$

$$q_1(\mathbf{f}) = \mathcal{N}(\widehat{\mathbf{f}}, \widehat{\boldsymbol{\Sigma}}_f)$$
$$\left\{ \begin{array}{l} \widehat{\boldsymbol{\Sigma}}_f = (\mathbf{F}'\mathbf{F} + \lambda_f \boldsymbol{\Sigma}_f)^{-1} \\ \widehat{\mathbf{f}}(t) = (\mathbf{F}'\mathbf{F} + \lambda_f \boldsymbol{\Sigma}_f)^{-1}\mathbf{F}'\mathbf{g}(t), \quad \lambda_f = v_\epsilon / v_f \end{array} \right.$$

$$q_2(\mathbf{h}) = \mathcal{N}(\widehat{\mathbf{h}}, \widehat{\boldsymbol{\Sigma}}_h)$$
$$\left\{ \begin{array}{l} \widehat{\boldsymbol{\Sigma}}_h = (\mathbf{H}'\mathbf{H} + \lambda_h \widehat{\boldsymbol{\Sigma}}_h)^{-1} \\ \widehat{\mathbf{h}} = (\mathbf{H}'\mathbf{H} + \lambda_h \widehat{\boldsymbol{\Sigma}}_h)^{-1} \qquad \lambda_a = v_\epsilon / v_a \end{array} \right.$$

Joint Estimation of h and f with a Gaussian prior model..

- Joint MAP:

$$\begin{tikzcd}
\mathbf{h}^{(0)} \arrow[r] & \mathbf{H} \arrow[r] & \boxed{\left(\mathbf{H}'\mathbf{H} + \lambda_f \boldsymbol{\Sigma}_f\right)^{-1} \mathbf{H}'\mathbf{g}} \arrow[r] & \widehat{\mathbf{f}}(t) \arrow[d] \\
 & \widehat{\mathbf{h}} \arrow[u] & \boxed{\left(\mathbf{F}'\mathbf{F} + \lambda_h \boldsymbol{\Sigma}_h\right)^{-1} \mathbf{F}'\mathbf{g}} \arrow[l] & \mathbf{F} \arrow[l]
\end{tikzcd}$$

- VBA:

$$\begin{tikzcd}
\begin{array}{l} \mathbf{h}^{(0)} \longrightarrow \mathbf{H} \longrightarrow \\ \boldsymbol{\Sigma}_f^{(0)} \longrightarrow \boldsymbol{\Sigma}_f \longrightarrow \end{array} & \boxed{\begin{array}{c} \widehat{\mathbf{f}} = \left(\mathbf{H}'\mathbf{H} + \lambda_f \boldsymbol{\Sigma}_f\right)^{-1} \mathbf{H}'\mathbf{g} \\ \widehat{\boldsymbol{\Sigma}}_f = (\mathbf{H}'\mathbf{H} + \lambda_f \boldsymbol{\Sigma}_f)^{-1} \end{array}} & \begin{array}{l} \longrightarrow \widehat{\mathbf{f}} \\ \longrightarrow \widehat{\boldsymbol{\Sigma}} \end{array} \arrow[d, Rightarrow] \\
\begin{array}{l} \widehat{\mathbf{h}} \longleftarrow \\ \widehat{\mathbf{V}} \longleftarrow \end{array} \arrow[u, Rightarrow] & \boxed{\begin{array}{c} \widehat{\mathbf{h}} = \left(\mathbf{F}'\mathbf{F} + \lambda_h \boldsymbol{\Sigma}_h\right)^{-1} \mathbf{F}'\mathbf{g} \\ \widehat{\mathbf{V}} = (\mathbf{F}'\mathbf{F} + \lambda_h \boldsymbol{\Sigma}_h)^{-1} \end{array}} & \begin{array}{l} \longleftarrow \mathbf{F} \\ \longleftarrow \boldsymbol{\Sigma}_h \end{array}
\end{tikzcd}$$

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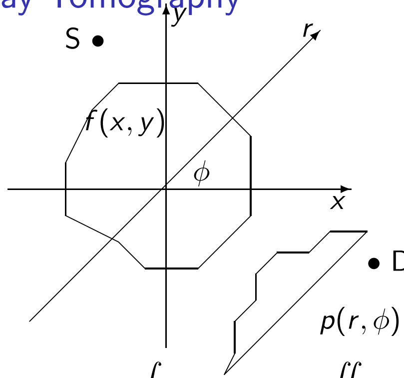
8. **Case study: Image reconstruction and Computed Tomography**

8. Case study: Image reconstruction and Computed Tomography

- X ray Computed Tomography, Radon transform
- Analytical inversion methods
- Discretization and algebraic methods
- Bayesian approach to CT
- Gauss-Markov-Potts model for images
- 3D CT
- Practical issues and computational costs

X ray Tomography



$$I = I_0 \exp\left\{-\mu x\right\}$$

$$I = I_0 \exp\left\{-\sum_j \mu_j x_j\right\}$$

$$I = I_0 \exp\left\{-\int_L f(x)\,\mathrm{d}x\right\}$$

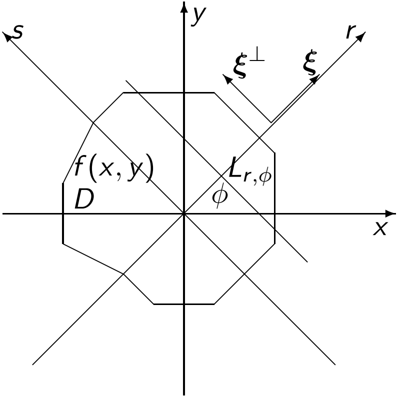
$$-\ln\frac{I}{I_0} = \int_L f(x,y)\,\mathrm{d}l$$

$$p(r,\phi) = \int_L f(x,y)\,\mathrm{d}l = \iint f(x,y)\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

$$f(x,y) \longrightarrow \boxed{\text{Radon Transform}} \longrightarrow p(r,\phi)$$

$$p(r,\phi) \longrightarrow \boxed{\begin{array}{c}\text{Reconstruction}\\ = \\ \text{Inversion de la TR}\end{array}} \longrightarrow f(x,y)$$

Radon Transform: 2D case



$$\boldsymbol{\xi} = [\cos\phi, \sin\phi]', \quad \boldsymbol{\xi}^{\perp} = [-\sin\phi, \cos\phi]'$$

$$\mathbf{x} = [x, y]' = [\rho\cos\theta, \rho\sin\theta], \qquad L_{r,\phi} : r = x\cos\phi + y\sin\phi = \boldsymbol{\xi}' \cdot \mathbf{x}$$

Radon Transform: Different formulations

- 2D case

$$g(r,\phi)=\int_L f(x,y)\,\mathrm{d}l=\iint_D f(x,y)\delta(r-x\cos\phi-y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

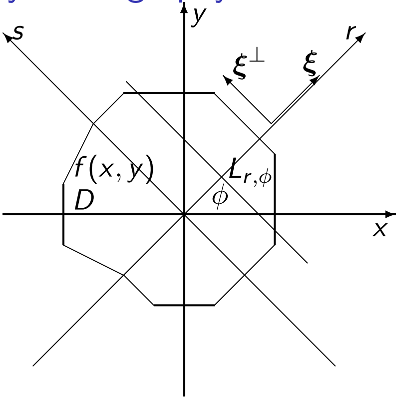
- 2D polar coordinates:

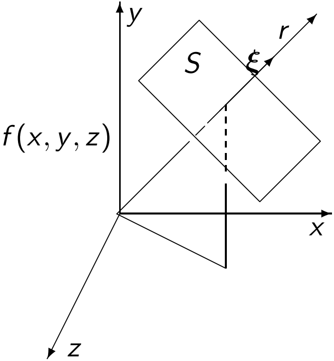
$$g(r,\phi)=\int_0^\infty\int_0^{2\pi} f(\rho,\theta)\delta\big(r-\rho\cos(\phi-\theta)\big)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

- nD case

$$g(r,\boldsymbol{\xi})=\iint f(x,y)\delta(r-\boldsymbol{\xi}'\cdot\mathbf{x})\,\mathrm{d}x\,\mathrm{d}y$$

X ray Tomography: 3D case





$$\mathbf{x}=[x,y,z]', \qquad d\mathbf{x}=\,\mathrm{d}x\,\mathrm{d}y dz, \quad r=\boldsymbol{\xi}'\cdot\mathbf{x}=\xi_1 x+\xi_2 y+\xi_3 z$$

$$g(r,\boldsymbol{\xi})=\int f(\mathbf{x})\delta(r-\boldsymbol{\xi}'\cdot\mathbf{x})\,\mathrm{d}\mathbf{x}$$

$$g(r,\boldsymbol{\xi})=\iiint f(x,y,z)\delta(r-x\sin\theta\cos\phi-y\sin\theta\sin\phi-z\cos\theta)\,\mathrm{d}x\,\mathrm{d}y dz$$

X ray Tomography: *n*-D case

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]', \quad \boldsymbol{\xi} = [\xi_1, \xi_2, \ldots, \xi_n]' \quad \text{avec} \quad |\boldsymbol{\xi}| = 1$$

$$\mathrm{d}\mathbf{x} = \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_n, \quad r = \boldsymbol{\xi}' \cdot \mathbf{x} = \xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n x_n$$

$$\boxed{g(r, \boldsymbol{\xi}) = \int_{R^n} f(\mathbf{x}) \delta(r - \boldsymbol{\xi}' \cdot \mathbf{x}) \, \mathrm{d}\mathbf{x}}$$

Radon Transform: some properties

- Definition:

$$f(x,y) \xrightarrow{\mathcal{R}} g(r,\phi) = \iint f(x,y)\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

- Definition in Polar Coordinates

$$f(\rho,\theta) \xrightarrow{\mathcal{R}} g(r,\phi) = \int_0^{\infty}\int_0^{2\pi} f(\rho,\theta)\delta(r-\rho\cos(\phi-\theta)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

	function	Radon Transform
Linearity:	$af_1(x,y) + bf_2(x,y)$	$ag_1(r,\phi) + bg_2(r,\phi)$
Symetry:	$f(x,y)$	$g(r,\phi) = g(-r,\phi \pm \pi)$
Periodicity:	$f(x,y)$	$g(r,\phi) = g(r,\phi \pm 2k\pi)$
Translation:	$f(x-x_0, y-y_0)$	$g(r - x_0\cos\phi - y_0\sin\phi, \phi)$
Rotation :	$f(\rho,\theta+\theta_0)$	$g(r,\phi+\theta_0)$
Scale change:	$f(ax,ay)$	$\frac{1}{\|a\|}g(ar,\phi), \quad a \neq 0$
Mass Conservation:	$M = \iint_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$	$M = \int_{-\infty}^{+\infty} g(r,\phi)\,\mathrm{d}r$

Radon Transform: *n*–D case

- Definition

$$\mathcal{R}\{f(\mathbf{x})\} = g(r, \boldsymbol{\xi}) = \int f(\mathbf{x}) \delta(r - \boldsymbol{\xi}' \cdot \mathbf{x}) \, \mathrm{d}\mathbf{x}$$

- Linearity

$$\mathcal{R}\{a\, f(\mathbf{x}) + b\, g(\mathbf{x})\} = a\, \mathcal{R}\{f(x)\} + b\, \mathcal{R}\{g(\mathbf{x})\}$$

- Symetry

$$g(-r, -\boldsymbol{\xi}) = g(r, \boldsymbol{\xi})$$

- Homogeneity

$$g(sr, s\boldsymbol{\xi}) = \frac{1}{|s|} \int f(\mathbf{x}) \delta(r - \boldsymbol{\xi}' \cdot \mathbf{x}) \, \mathrm{d}\mathbf{x}$$

$$g(sr, s\boldsymbol{\xi}) = \frac{1}{|s|} g(r, \boldsymbol{\xi})$$

$$g(r, s\boldsymbol{\xi}) = \frac{1}{s} g(\frac{r}{s}, \boldsymbol{\xi}), \quad s > 0$$

$$g(sr, \boldsymbol{\xi}) = s g(r, \boldsymbol{\xi})$$

- Axis transformation

Radon Transform: *n*–D case

- RT of the derivative of a function
- Derivative:

$$\mathcal{R}\{\frac{\partial f}{\partial x_k}\} = \xi_k \frac{\partial g(r,\boldsymbol{\xi})}{\partial r}$$

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]' \quad \mathbf{a}.\nabla f = \left[a_1 \frac{\partial f}{\partial x_1}, \ldots, a_n \frac{\partial f}{\partial x_n}\right]'$$

$$\mathcal{R}\{\mathbf{a}.\nabla f\} = \mathbf{a}.\boldsymbol{\xi} \frac{\partial g(r,\boldsymbol{\xi})}{\partial r}$$

- Second Derivative:

$$\mathcal{R}\{\frac{\partial^2 f}{\partial x_l \partial x_k}\} = \xi_l \, \xi_k \frac{\partial g(r,\boldsymbol{\xi})}{\partial r}$$

$$\Delta f(\mathbf{x}) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}$$

$$\mathcal{R}\{\Delta f\} = |\boldsymbol{\xi}|^2 \frac{\partial g(r,\boldsymbol{\xi})}{\partial r} = \frac{\partial g(r,\boldsymbol{\xi})}{\partial r}$$

Radon Transform: *n*–D case

- Derivative of the RT

$$\frac{\partial g(r,\boldsymbol{\xi})}{\partial \xi_k} = \int f(\mathbf{x}) \frac{\partial}{\partial \xi_k} \delta(r - \boldsymbol{\xi}'.\mathbf{x}) d\mathbf{x}$$

$$\frac{\partial}{\partial \xi_k} \mathcal{R}\{f(\mathbf{x})\} = \frac{\partial g(r,\boldsymbol{\xi})}{\partial \xi_k} = -\frac{\partial}{\partial r} \mathcal{R}\{x_k\, f(\mathbf{x})\}$$

$$\frac{\partial^2}{\partial \xi_l \partial \xi_k} \mathcal{R}\{f(\mathbf{x})\} = \frac{\partial^2 g(r,\boldsymbol{\xi})}{\partial \xi_l \partial \xi_k} = -\frac{\partial}{\partial r} \mathcal{R}\{x_l\, x_k\, f(\mathbf{x})\}$$

- Convolution

$$f(\mathbf{x}) = g * h = \int g(\mathbf{y}) h(\mathbf{x} - \mathbf{y})\, \mathrm{d}\mathbf{y}$$

$$g(r,\boldsymbol{\xi}) = \widetilde{g} * \widetilde{h}$$

Summary

$$\begin{array}{rcl}
f(\mathbf{x}) & \xrightarrow{TR} & g(r,\boldsymbol{\xi}) \\
c_1 f_1(\mathbf{x}) + c_2 f_2(\mathbf{x}) & \xrightarrow{TR} & c_1 g_1(r,\boldsymbol{\xi}) + c_2 g_2(r,\boldsymbol{\xi}) \\
f(\mathbf{A}\mathbf{x}) & \xrightarrow{TR} & |\mathbf{A}^{-1}| g(r, \mathbf{A}^{-t}\mathbf{x}) \\
f(\mathbf{A}^{-1}\mathbf{x}) & \xrightarrow{TR} & |\mathbf{A}| g(r, \mathbf{A}^{t}\mathbf{x}) \\
f(c\,\mathbf{x}) & \xrightarrow{TR} & \frac{1}{c^n}\, g(r, \boldsymbol{\xi}/c) = \frac{1}{c^{n-1}}\, g(cr, \boldsymbol{\xi}) \\
f(\mathbf{x} \pm \mathbf{a}) & \xrightarrow{TR} & g(r \pm \boldsymbol{\xi}'.\mathbf{a}, \boldsymbol{\xi})
\end{array}$$

$$\begin{array}{rcl}
g(sr, s\boldsymbol{\xi}) & = & \frac{1}{|s|}\, g(r,\boldsymbol{\xi}) \\
g(-r, -\boldsymbol{\xi}) & = & g(r,\boldsymbol{\xi}) \\
g(sr, \boldsymbol{\xi}) & = & s\, g(r,\boldsymbol{\xi})
\end{array}$$

Summary

Derivative:	$\frac{\partial f}{\partial x_k}$	$\xi_k \frac{\partial g(r,\boldsymbol{\xi})}{\partial r}$
Gradient :	$\nabla f = \left[\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right]'$	$\frac{\partial g(r,\boldsymbol{\xi})}{\partial r}\left[\xi_1,\ldots,\xi_n\right]'$
	$\mathbf{a}.\nabla f = a_1\frac{\partial f}{\partial x_1}+\cdots+a_1\frac{\partial f}{\partial x_n}$	$\mathbf{a}.\boldsymbol{\xi}\frac{\partial g(r,\boldsymbol{\xi})}{\partial r}$
Second Derivative:	$\frac{\partial^2 f}{\partial x_l \partial x_k}$	$\xi_l\xi_k\frac{\partial g(r,\boldsymbol{\xi})}{\partial r}$
Laplacian:	$\Delta f(\mathbf{x}) = \sum_{k=1}^{n}\frac{\partial f}{\partial x_k}$	$\frac{\partial g(r,\boldsymbol{\xi})}{\partial r}$

Link between RT and FT

$$\mathbf{x} = [x_1, \ldots, x_n]', \quad \boldsymbol{\omega} = [\omega_1, \ldots, \omega_n]'$$

$$\left\{ \begin{array}{rcl} \widehat{f}(\boldsymbol{\omega}) & = & \mathcal{F}_n \left\{ f(\mathbf{x}) \right\} = \int f(\mathbf{x}) \exp\left\{-j\boldsymbol{\omega}' \cdot \mathbf{x}\right\} \, \mathrm{d}\mathbf{x} \\ f(\mathbf{x}) & = & \mathcal{F}_n^{-1} \left\{ \widehat{f}(\boldsymbol{\omega}) \right\} = \int \widehat{f}(\omega) \exp\left\{+j\boldsymbol{\omega}' \cdot \mathbf{x}\right\} \, \mathrm{d}\boldsymbol{\omega} \end{array} \right.$$

$$\left\{ \begin{array}{rcl} \widehat{g}(\Omega, \boldsymbol{\xi}) & = & \mathcal{F}_1 \left\{ g(r, \boldsymbol{\xi}) \right\} = \int g(r, \boldsymbol{\xi}) \exp\left\{-j\Omega r\right\} \, \mathrm{d}r \\ g(r, \boldsymbol{\xi}) & = & \mathcal{F}_1^{-1} \left\{ \widehat{g}(\Omega, \boldsymbol{\xi}) \right\} = \int \widehat{g}(\Omega, \boldsymbol{\xi}) \exp\left\{+j\Omega r\right\} \, \mathrm{d}\Omega \end{array} \right.$$

$$\widehat{f}(\boldsymbol{\omega}) = \int g(r, \boldsymbol{\xi}) \exp\left\{-j\Omega r\right\} \, \mathrm{d}r = \widehat{g}(\Omega, \boldsymbol{\xi}) \quad \forall \boldsymbol{\omega} = \Omega \boldsymbol{\xi}$$

$$\left\{ \begin{array}{rcl} \widehat{f} & = & \mathcal{F}_n \left\{ f \right\} = \mathcal{F}_1 \left\{ g \right\} = \mathcal{F}_1 \left\{ \mathcal{R}\{f\} \right\} \\ \widetilde{f} & = & \mathcal{R}\{f\} = \mathcal{F}_1^{-1} \left\{ \widehat{f} \right\} = \mathcal{F}_1^{-1} \left\{ \mathcal{F}_n \left\{ f \right\} \right\} \end{array} \right.$$

Direct inversion of the RT: Odd dimensions

$$f(\mathbf{x}) = C_n \int_{|\boldsymbol{\xi}|=1} \left[(\frac{\partial}{\partial r})^{n-1} g(r, \boldsymbol{\xi}) \right]_{r=\boldsymbol{\xi}'.\mathbf{x}} \mathrm{d}\boldsymbol{\xi}$$

$$C_n = \frac{1}{2} \frac{(-1)^{(n-1)/2}}{(2\pi)^{(n-1)}} = \frac{1}{2} \frac{1}{(2j\pi)^{(n-1)}}$$

Example :
$n = 3,\ C_3 = \frac{-1}{8\pi^2}$, $\mathbf{x} = [x, y, z]'$,
$\boldsymbol{\xi} = [\xi_1, \xi_2, \xi_3]' = [\sin\theta \cos\phi,\ \sin\theta \sin\phi,\ \cos\theta]'$ and
$d\boldsymbol{\xi} = \sin\theta\, d\theta\, d\phi$.

$$f(\mathbf{x}) = \frac{-1}{8\pi^2} \int_{|\boldsymbol{\xi}|=1} \left[\frac{\partial^2}{\partial r^2} g(r, \boldsymbol{\xi}) \right]_{r=\boldsymbol{\xi}'.\mathbf{x}} d\boldsymbol{\xi}$$

Direct inversion of the RT: Odd dimensions

Spherical coordinates : (ρ, θ, ϕ):

$$g(r, \boldsymbol{\xi}) = \iiint f(x, y, z)\delta(r - x \sin\theta \cos\phi + y \sin\theta \sin\phi + z \cos\theta) \, \mathrm{d}x \; \mathrm{d}ydz$$

$$\overline{g}(r, \boldsymbol{\xi}) = \frac{-1}{8\pi^2} \frac{\partial^2 g(r, \boldsymbol{\xi})}{\partial r^2}$$

$$f(x, y, z) = \mathcal{B}\left\{\overline{g}\right\} \overset{\triangle}{=} \int_0^{\pi} \int_0^{2\pi} \overline{g}(r, \boldsymbol{\xi}) \sin\theta \; \mathrm{d}\phi \; \mathrm{d}\theta$$

$$f(x, y, z) \longrightarrow \boxed{\;\mathcal{R}\;} \longrightarrow g(r, \boldsymbol{\xi})$$

$$g(r, \boldsymbol{\xi}) \longrightarrow \boxed{\frac{-1}{8\pi^2} \frac{\partial^2}{\partial r^2}} \longrightarrow \overline{g}(r, \boldsymbol{\xi}) \longrightarrow \boxed{\;\mathcal{B}\;} \longrightarrow f(x, y, z)$$

Even dimensions

$$f(\mathbf{x}) = \frac{C_n}{j\pi} \int_{|\boldsymbol{\xi}|=1} \mathrm{d}\boldsymbol{\xi} \int \mathrm{d}r \, \frac{(\frac{\partial}{\partial r})^{n-1} g(r, \boldsymbol{\xi})}{(r - \boldsymbol{\xi}'.\mathbf{x})}$$

Example : $n = 2,\ C_2 = \frac{-1}{2\pi}$,
$\mathbf{x} = [x, y]'$, $\boldsymbol{\xi} = [\xi_1,\ \xi_2]' = [\cos\phi,\ \sin\phi]',\ d\boldsymbol{\xi} = d\phi.$

$$f(x, y) = \left(-\frac{1}{2\pi^2}\right) \int_0^{\pi} d\phi \int_{-\infty}^{+\infty} \mathrm{d}r \, \frac{\frac{\partial}{\partial r} g(r, \phi)}{(r - x\cos\phi - y\sin\phi)}$$

$$g(r, \phi) = \iint f(x, y) \delta(r - x\cos\phi + y\sin\phi) \, \mathrm{d}x \, \mathrm{d}y$$

Even dimensions

$$\overline{g}(r,\phi) = \frac{-1}{2\pi^2}\frac{\partial g(r,\phi)}{\partial r}$$

$$g(r,\phi) = \mathcal{H}\left\{\overline{g}(r,\phi)\right\} = \int \frac{\overline{g}(r,\phi)}{(r - x\cos\phi - y\sin\phi)}\,\mathrm{d}r$$

$$f(x,y) = \mathcal{B}\left\{g(r,\phi)\right\} \overset{\triangle}{=} \int_0^{\pi} g(r,\phi)\,\mathrm{d}\phi$$

$$f(x,y) \longrightarrow \boxed{\mathcal{R}} \longrightarrow g(r,\phi)$$

$$g(r,\phi) \rightarrow \boxed{\frac{-1}{2\pi^2}\frac{\partial}{\partial r}} \rightarrow \overline{g}(r,\phi) \rightarrow \boxed{\mathcal{H}} \rightarrow g(x,y) \rightarrow \boxed{\mathcal{B}} \rightarrow f(x,y)$$

Cylindrical coordinates: $(\rho,\theta) \longrightarrow x = \rho\cos\theta$ and $y = \rho\sin\theta$,

$$f(\rho,\theta) = \left(-\frac{1}{2\pi^2}\right)\int_0^{\pi} d\phi \int_{-\infty}^{+\infty} \mathrm{d}r\,\frac{\frac{\partial}{\partial r}g(r,\xi)}{(r - \rho cos(\phi-\theta))}$$

Inversion using the FT

$$\begin{tikzcd}
f(x,y) \arrow[rr, "\mathcal{R}"] \arrow[dr, shift right, "\mathcal{F}_2"'] & & p(r,\phi) \arrow[dl, shift left, "\mathcal{F}_1"] \\
& \widehat{f}(\omega_x,\omega_y) \arrow[ul, shift right, "\mathcal{F}_2^{-1}"'] \arrow[ur, shift left, "\mathcal{F}_1^{-1}"] &
\end{tikzcd}$$

$$\mathcal{F}_2\left\{f\right\} = \mathcal{F}_1\left\{\mathcal{R}\{f\}\right\} = \mathcal{F}_1\left\{g\right\}$$

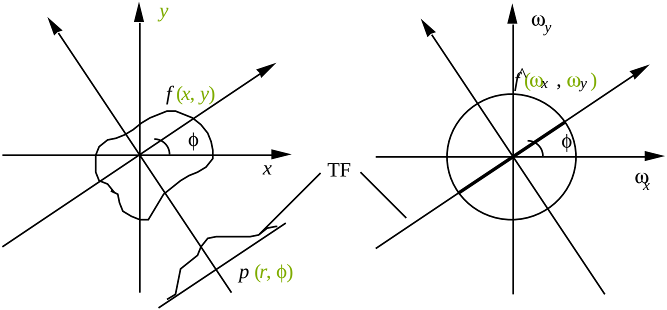
$$G(\Omega,\boldsymbol{\xi}) = G(\Omega,\phi) = \int g(r,\phi)\exp\left\{-j\Omega r\right\}\,\mathrm{d}r$$

$$\widehat{f}(\omega_x,\omega_y) = \iint f(x,y)\exp\left\{-j(\omega_x\,x+\omega_y\,y)\right\}\,\mathrm{d}x\,\mathrm{d}y$$

$$f(x,y) = \iint \widehat{f}(\omega_x,\omega_y)\exp\left\{+j(\omega_x\,x+\omega_y\,y)\right\}\,\mathrm{d}\omega_x\,\mathrm{d}\omega_y$$

$$\widehat{f}(\omega_x,\omega_y) = G(\Omega,\phi) \quad \text{pour} \quad \omega_x = \Omega\sin\phi \quad \text{et} \quad \omega_y = \Omega\cos\phi$$

Inversion using the FT



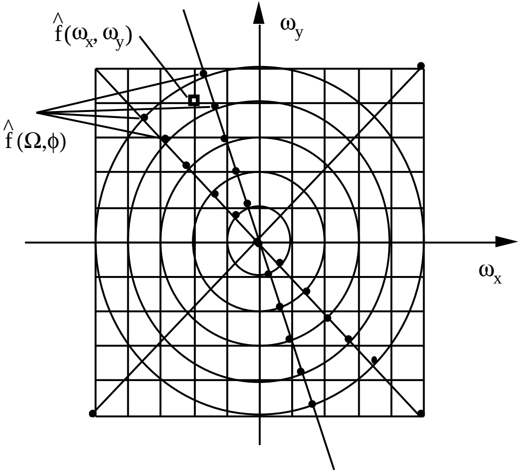
Inversion using FT

$$\begin{tikzcd}
f(x,y) \arrow[r, "\mathcal{R}"] & g(r,\phi) \arrow[d, "\mathcal{F}_1"] \\
\widehat{f}(\omega_x,\omega_y) \arrow[u, "\mathcal{F}_2^{-1}"] & G(\Omega,\phi) \arrow[l, "{\omega_x = \Omega\cos\phi\,,\ \omega_y = \Omega\sin\phi}"']
\end{tikzcd}$$

Algorithm:

- $p(r,\phi_m) \longrightarrow$ 1D FT $\longrightarrow \hat{p}(\Omega,\phi_m), \quad m = 1,\cdots,M$
- Interpolation in the Fourier domaine $\longrightarrow \widehat{f}(\omega_x,\omega_y)$
- $\widehat{f}(\omega_x,\omega_y) \longrightarrow$ 2D FT $f(x,y)$

Inversion using FT



Inversion using Backprojection

For a function $p(r, \phi)$ of $r \in R$ and $\phi \in [0, \pi]$, and
$r = x \cos \phi + y \sin \phi$, $\forall (x, y) \in R^2$

$$\mathcal{B}\left\{p\right\}(x, y) = \int_0^\pi p(x \cos \phi + y \sin \phi, \phi) d\phi$$

This operator is the adjoint operator of the Radon transform $\mathcal{R}$
Polar coordinates:

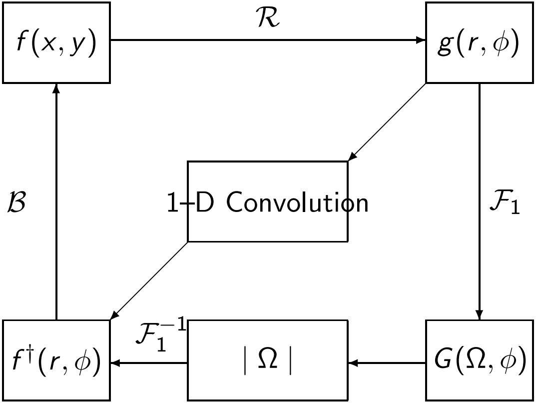
$$\mathcal{B}\left\{p\right\}(x, y) = \int_0^\pi p(\rho \cos(\theta - \phi), \phi) d\phi$$

$$p(r, \phi) = \mathcal{R}\{f(x, y)\}, \quad f_b(x, y) = \mathcal{B}\left\{p(r, \phi)\right\} = \mathcal{B}\left\{\mathcal{R}\{f\}\right\}$$

$$f_b(x, y) = f(x, y) * \frac{1}{|r|} = f(x, y) * \frac{1}{\sqrt{x^2 + y^2}}$$

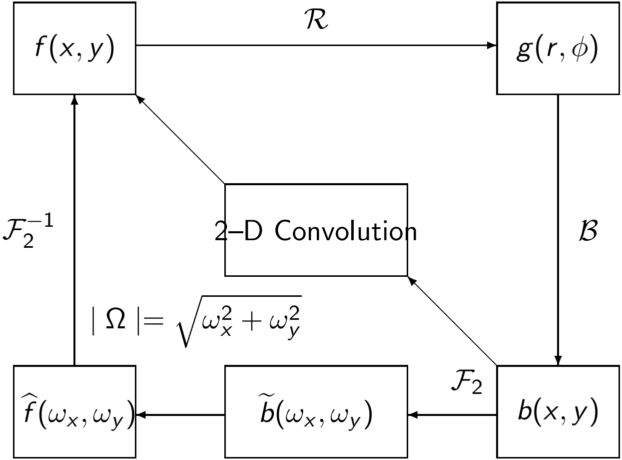
Inversion by Backprojection algorithms

$$f(x,y) = \mathcal{B}\left\{\mathcal{F}_1^{-1}\left\{|\Omega|\mathcal{F}_1\left\{g(r,\phi)\right\}\right\}\right\}$$



Inversion by Backprojection and filtering in 2D Fourier domaine

$$f(x,y) = \mathcal{F}_2^{-1}\left\{|\Omega|\mathcal{F}_2\left\{\mathcal{B}\left\{g(r,\phi)\right\}\right\}\right\}$$



$f(x,y)$

$\mathcal{R}$

$g(r,\phi)$

$\mathcal{F}_2^{-1}$

2–D Convolution

$\mathcal{B}$

$| \Omega |= \sqrt{\omega_x^2+\omega_y^2}$

$\mathcal{F}_2$

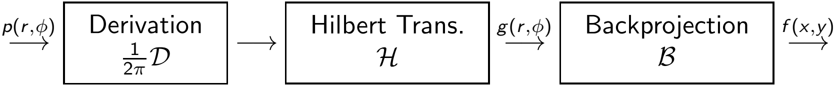
$\widehat{f}(\omega_x,\omega_y)$

$\widetilde{b}(\omega_x,\omega_y)$

$b(x,y)$

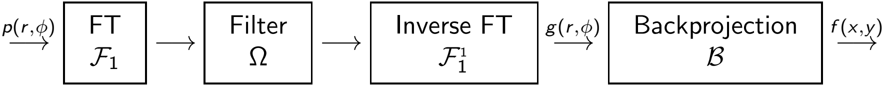
Inversion by Backprojection

- Direct Inversion of Radon Transform



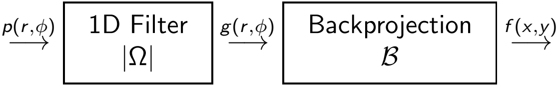
$\xrightarrow{p(r,\phi)}$ Derivation $\frac{1}{2\pi}\mathcal{D}$ $\longrightarrow$ Hilbert Trans. $\mathcal{H}$ $\xrightarrow{g(r,\phi)}$ Backprojection $\mathcal{B}$ $\xrightarrow{f(x,y)}$

- FBP:



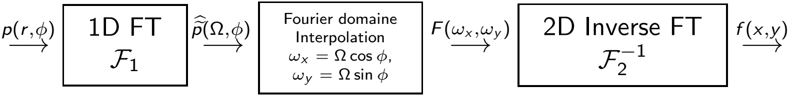
$\xrightarrow{p(r,\phi)}$ FT $\mathcal{F}_1$ $\longrightarrow$ Filter Ω $\longrightarrow$ Inverse FT $\mathcal{F}_1^1$ $\xrightarrow{g(r,\phi)}$ Backprojection $\mathcal{B}$ $\xrightarrow{f(x,y)}$

- Convolution–BackoProjection (CBP)



$\xrightarrow{p(r,\phi)}$ 1D Filter $|\Omega|$ $\xrightarrow{g(r,\phi)}$ Backprojection $\mathcal{B}$ $\xrightarrow{f(x,y)}$

- Fourier Domaine Inversion



$\xrightarrow{p(r,\phi)}$ 1D FT $\mathcal{F}_1$ $\xrightarrow{\widehat{p}(\Omega,\phi)}$ Fourier domaine Interpolation $\omega_x = \Omega\cos\phi$, $\omega_y = \Omega\sin\phi$ $\xrightarrow{F(\omega_x,\omega_y)}$ 2D Inverse FT $\mathcal{F}_2^{-1}$ $\xrightarrow{f(x,y)}$

Inversion by Backprojection

- Backprojection 2D Filtering

$$\xrightarrow{p(r,\phi)} \boxed{\begin{array}{c}\text{Backprojection}\\ \mathcal{B}\end{array}} \xrightarrow{b(x,y)} \boxed{\begin{array}{c}\text{2D FT}\\ \mathcal{F}_2\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{2D Filter}\\ |\Omega|\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{2D Inverse FT}\\ \mathcal{F}_2^{-1}\end{array}} \xrightarrow{f(x,y)}$$

- Backprojection and 2D Convolution

$$\xrightarrow{p(r,\phi)} \boxed{\begin{array}{c}\text{Backprojection}\\ \mathcal{B}\end{array}} \xrightarrow{b(x,y)} \longrightarrow \boxed{\begin{array}{c}\text{2D Filter}\\ |\Omega|\end{array}} \xrightarrow{f(x,y)}$$

Choice of the filter

$$H(\Omega) = |\Omega|$$

In practice:

$$H(\Omega) = |\Omega| W(\Omega)$$

- RAM–LAK
- Shepp–Logan
- Low pass Filter
- Hamming

Inversion par Backprojection: Implementation

convolution–Backprojection

$$\xrightarrow{p(r,\phi)} \boxed{\begin{array}{c}\text{Convolution}\\ h(r)\end{array}} \longrightarrow \xrightarrow{\overline{p}(r,\phi)} \boxed{\begin{array}{c}\text{Backprojection}\\ \mathcal{B}\end{array}} \xrightarrow{f(x,y)}$$

$$r = md, \quad -M/2 \leq m \leq M/2-1, \quad \phi = n\Delta$$

Discrete Convolution

$$\overline{p}(md, n\Delta) \simeq \overline{p}_n(m) \overset{\triangle}{=} \sum_{k=-M/2}^{M/2-1} p_n(k)h(m-k), \quad -M/2 \leq m \leq M/2-1$$

Linear Interpolation

$$\overline{p}(r, n\Delta) \simeq \overline{p}_n(m)+(r/d-m)[\overline{p}_n(m+1)-\overline{p}_n(m)], \; md \leq r \leq (m+1)d$$

Discrete Backprojection

$$f(x,y) \simeq \mathcal{B}_N\overline{p} \overset{\triangle}{=} \Delta \sum_{n=0}^{N-1} \overline{p}(r = x\cos n\Delta + y\sin n\Delta, n\Delta)$$

$$f(i\Delta x, j\Delta y) \simeq \Delta \sum_{n=0}^{N-1} \overline{p}(i\Delta x\cos n\Delta + j\Delta y\sin n\Delta, n\Delta)$$

Algebraic methods

- Discretization of the image

$$f(x,y) \simeq \sum_{i=1}^{I}\sum_{j=1}^{J} a_{i,j}\varphi_{i,j}(x,y)$$

- Discretization of the projections

$$p(r,\phi) \simeq \mathcal{R}f = \sum_{i=1}^{I}\sum_{j=1}^{J} a_{i,j}[\mathcal{R}\varphi_{i,j}] \overset{\triangle}{=} \sum_{i=1}^{I}\sum_{j=1}^{J} a_{i,j}h_{i,j}(r,\phi)$$

$$p(r_m,\phi_n) \simeq \sum_{i=1}^{I}\sum_{j=1}^{J} a_{i,j}h_{i,j}(r_m,\phi_n)$$

- Choice of the basis functions

$$\varphi_{i,j}(x,y) = \begin{cases} 1 & \text{inside the pixel} i,j \\ 0 & \text{elsewhere} \end{cases}$$

$$p(r_m,\phi_n) \simeq \sum_{i=1}^{I}\sum_{j=1}^{J} f_{i,j}h_{i,j}(r_m,\phi_n)$$

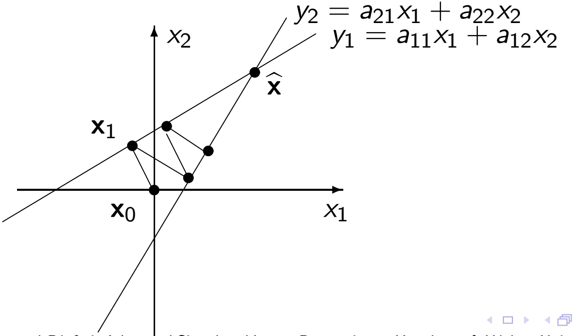
ART : Algebraic Reconstruction Techniques

Kaczmarz (Generalized inversion)

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \mathbf{b}$$

$$\begin{cases} \mathbf{f}^0 = \mathbf{0} \\ \mathbf{f}_{(k+1)} = \mathbf{f}^{(k)} + \frac{g_i - \left\langle \mathbf{h}_i, \mathbf{f}^{(k)} \right\rangle}{\left\langle \mathbf{h}_i, \mathbf{h}_i \right\rangle} \mathbf{h}_i \quad i = 1, 2, \ldots, M \end{cases}$$

where $\mathbf{h}_i$ is the i-th ligne of the matrix $\mathbf{H}$.



ART : Algebraic Reconstruction Techniques

- At each iteration only one of equations (constraints) is used.
- One can also add other constraints. For example:

- without any constraint :

$$f_i(x) = x$$

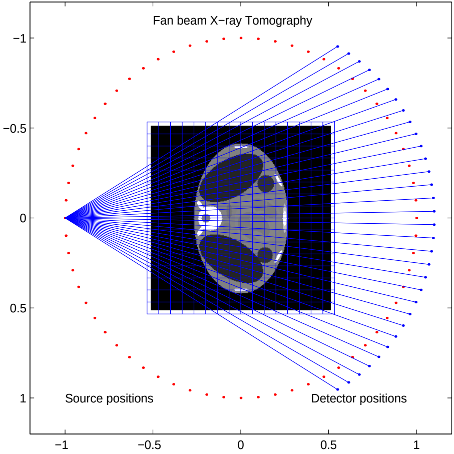
- Positivity :

$$f_i(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

- Positivity and maximum value constraint:

$$f_i(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$

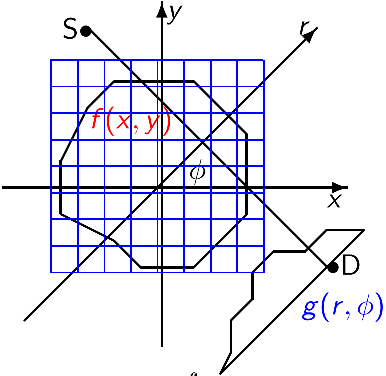
CT as a linear inverse problem

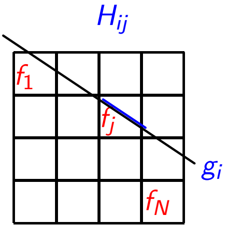


$$g(s_i) = \int_{L_i} f(\mathbf{r})\; \mathrm{d}l_i + \epsilon(s_i) \longrightarrow \text{Discretization} \longrightarrow \mathbf{g} = \mathbf{H}\mathbf{f} + \epsilon$$

- **g**, **f** and **H** are huge dimensional

Algebraic methods: Discretization





$$f(x,y) = \sum_j f_j \, b_j(x,y)$$

$$b_j(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \text{ pixel } j \\ 0 & \text{else} \end{cases}$$

$$g(r,\phi) = \int_L f(x,y) \, \mathrm{d}l$$

$$g_i = \sum_{j=1}^{N} H_{ij} \, f_j + \epsilon_i$$

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \epsilon$$

Inversion: Deterministic methods

Data matching

- Observation model
  $g_i = h_i(\mathbf{f}) + \epsilon_i, \quad i = 1, \ldots, M \longrightarrow \mathbf{g} = \mathbf{H}(\mathbf{f}) + \boldsymbol{\epsilon}$
- Misatch between data and output of the model $\Delta(\mathbf{g}, \mathbf{H}(\mathbf{f}))$

$$\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \{\Delta(\mathbf{g}, \mathbf{H}(\mathbf{f}))\}$$

- Examples:

– LS $$\Delta(\mathbf{g}, \mathbf{H}(\mathbf{f})) = \|\mathbf{g} - \mathbf{H}(\mathbf{f})\|^2 = \sum_i |g_i - h_i(\mathbf{f})|^2$$

– L_p $$\Delta(\mathbf{g}, \mathbf{H}(\mathbf{f})) = \|\mathbf{g} - \mathbf{H}(\mathbf{f})\|^p = \sum_i |g_i - h_i(\mathbf{f})|^p, \; 1 < p < 2$$

– KL $$\Delta(\mathbf{g}, \mathbf{H}(\mathbf{f})) = \sum_i g_i \ln \frac{g_i}{h_i(\mathbf{f})}$$

- In general, does not give satisfactory results for inverse problems.

Deterministic Inversion Algorithms

Least Squares Based Methods

$$\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \{J(\mathbf{f})\} \quad \text{with} \quad J(\mathbf{f}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2$$

$$\nabla J(\mathbf{f}) = -2\mathbf{H}'(\mathbf{g} - \mathbf{H}\mathbf{f})$$

Gradient based algorithms:

- Initialize: $\mathbf{f}^{(0)}$
- Iterate: $\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} - \alpha \nabla J(\mathbf{f}^{(k)})$

At each iteration: $\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \alpha \mathbf{H}' \left(\mathbf{g} - \mathbf{H}\mathbf{f}^{(k)}\right)$

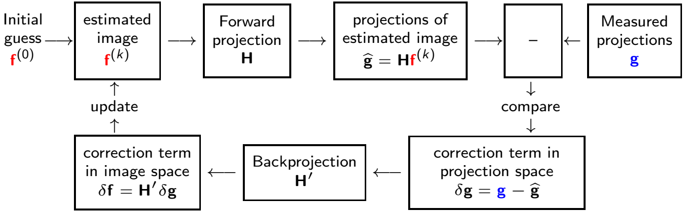
we have to do the following operations:

- Compute $\widehat{\mathbf{g}} = \mathbf{H}\mathbf{f}$ (Forward projection)
- Compute $\delta\mathbf{g} = \mathbf{g} - \widehat{\mathbf{g}}$ (Error or residual)
- Distribute $\delta\mathbf{f} = \mathbf{H}'\delta\mathbf{g}$ (Backprojection of error)
- Update $\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \delta\mathbf{f}$

Gradient based algorithms

Operations at each iteration: $\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \alpha \mathbf{H}' \left(\mathbf{g} - \mathbf{H}\mathbf{f}^{(k)} \right)$

- Compute $\widehat{\mathbf{g}} = \mathbf{H}\mathbf{f}$ (Forward projection)
- Compute $\delta\mathbf{g} = \mathbf{g} - \widehat{\mathbf{g}}$ (Error or residual)
- Distribute $\delta\mathbf{f} = \mathbf{H}'\delta\mathbf{g}$ (Backprojection of error)
- Update $\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \delta\mathbf{f}$



Initial guess $\mathbf{f}^{(0)}$ ⟶ estimated image $\mathbf{f}^{(k)}$ ⟶ Forward projection $\mathbf{H}$ ⟶ projections of estimated image $\widehat{\mathbf{g}} = \mathbf{H}\mathbf{f}^{(k)}$ ⟶ – ⟵ Measured projections $\mathbf{g}$

compare

correction term in image space $\delta\mathbf{f} = \mathbf{H}'\delta\mathbf{g}$ ⟵ Backprojection $\mathbf{H}'$ ⟵ correction term in projection space $\delta\mathbf{g} = \mathbf{g} - \widehat{\mathbf{g}}$

update

Gradient based algorithms

- Fixed step gradient:

$$\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \alpha \mathbf{H}' \left(\mathbf{g} - \mathbf{H}\mathbf{f}^{(k)} \right)$$

- Steepest descent gradient:

$$\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \alpha^{(k)} \mathbf{H}' \left(\mathbf{g} - \mathbf{H}\mathbf{f}^{(k)} \right)$$

  with $\alpha^{(k)} = \arg\min_{\alpha} \left\{ J(\mathbf{f} + \alpha \delta \mathbf{f}) \right\}$

- Conjugate Gradient

$$\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \alpha^{(k)} \mathbf{d}^{(k)}$$

  The successive directions $\mathbf{d}^{(k)}$ have to be conjugate to each other.

Algebraic Reconstruction Techniques

- Main idea: Use the data as they arrive

$$\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \alpha^{(k)} [\mathbf{H}']_{i*} \left(g_i - [\mathbf{H}\mathbf{f}^{(k)}]_i \right)$$

which can also be written as:

$$\begin{aligned} \mathbf{f}^{(k+1)} &= \mathbf{f}^{(k)} + \frac{\left(g_i - [\mathbf{H}\mathbf{f}^{(k)}]_i \right)}{\mathbf{h}'_{i*}\mathbf{h}_{i*}} \mathbf{h}'_{i*} \\ &= \mathbf{f}^{(k)} + \sum_i \frac{\left(g_i - \sum_j H_{ij} f_j^{(k)} \right)}{\sum_i Hij^2} H_{ij} \end{aligned}$$

Algebraic Reconstruction Techniques

- Use the data as they arrive

$$\begin{aligned} \mathbf{f}^{(k+1)} &= \mathbf{f}^{(k)} + \frac{\left(g_i - [\mathbf{H}\mathbf{f}^{(k)}]_i\right)}{\mathbf{h}'_{i*}\mathbf{h}_{i*}}\mathbf{h}'_{i*} \\ &= \mathbf{f}^{(k)} + \sum_i \frac{\left(g_i - \sum_j H_{ij} f_j^{(k)}\right)}{\sum_i Hij^2} H_{ij} \end{aligned}$$

- Update each pixel at each time

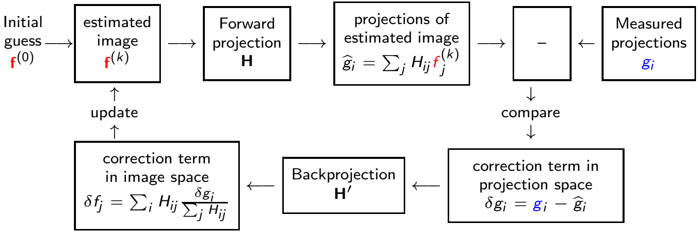
$$f_j^{(k+1)} = f_j^{(k)} + \frac{\left(g_i - \sum_j H_{ij} f_j^{(k)}\right)}{\sum_i Hij^2} H_{ij}$$

Algebraic Reconstruction Techniques (ART)

$$\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \sum_i \frac{\left(g_i - \sum_j H_{ij} f_j^{(k)}\right)}{\sum_i Hij^2} H_{ij}$$

or

$$f_j^{(k+1)} = f_j^{(k)} + \frac{\left(g_i - \sum_j H_{ij} f_j^{(k)}\right)}{\sum_i Hij^2} H_{ij}$$



Initial guess $\mathbf{f}^{(0)}$ ⟶

estimated image $\mathbf{f}^{(k)}$ ⟶

Forward projection $\mathbf{H}$ ⟶

projections of estimated image $\widehat{g}_i = \sum_j H_{ij} f_j^{(k)}$ ⟶

$-$ ⟵ Measured projections g_i

↓ compare ↓

correction term in projection space $\delta g_i = g_i - \widehat{g}_i$ ⟵

Backprojection $\mathbf{H}'$ ⟵

correction term in image space $\delta f_j = \sum_i H_{ij} \frac{\delta g_i}{\sum_j H_{ij}}$

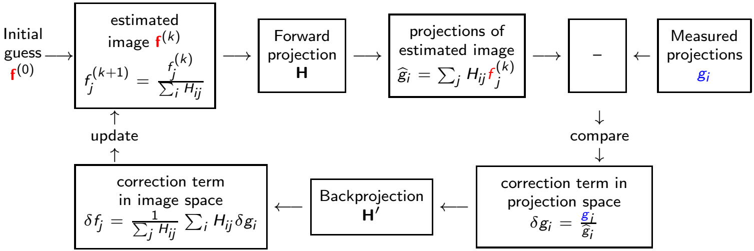
↑ update ↑

Algebraic Reconstruction using KL distance

- $\widehat{\mathbf{f}} = \arg\min_{\mathbf{f}} \{J(\mathbf{f})\}$ with $J(\mathbf{f}) = \sum_i g_i \ln \frac{g_i}{\sum_j H_{ij} f_j}$

$$f_j^{(k+1)} = \frac{f_j^{(k)}}{\sum_i H_{ij}} \sum_i H_{ij} \frac{g_i}{\sum_j H_{ij} f_j^{(k)}}$$

Interestingly, this is the OSEM (Ordered subset Expectation-Maximization) algorithm which is based on Maximum Likelihood and proposed first by Shepp & Vardi.



Initial guess $\mathbf{f}^{(0)}$

estimated image $\mathbf{f}^{(k)}$

$f_j^{(k+1)} = \frac{f_j^{(k)}}{\sum_i H_{ij}}$

Forward projection **H**

projections of estimated image

$\widehat{g}_i = \sum_j H_{ij} f_j^{(k)}$

–

Measured projections g_i

compare

correction term in projection space

$\delta g_i = \frac{g_i}{\widehat{g}_i}$

Backprojection $\mathbf{H}'$

correction term in image space

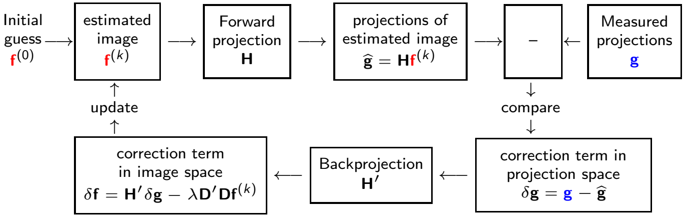
$\delta f_j = \frac{1}{\sum_j H_{ij}} \sum_i H_{ij} \delta g_i$

update

Gradient based algorithms

$$\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \alpha \left[\mathbf{H}' \left(\mathbf{g} - \mathbf{H}\mathbf{f}^{(k)} \right) - \lambda \mathbf{D}'\mathbf{D}\mathbf{f}^{(k)} \right]$$

- Compute $\widehat{\mathbf{g}} = \mathbf{H}\mathbf{f}$ (Forward projection)
- Compute $\delta\mathbf{g} = \mathbf{g} - \widehat{\mathbf{g}}$ (Error or residual)
- Compute $\delta\mathbf{f}_1 = \mathbf{H}'\delta\mathbf{g}$ (Backprojection of error)
- Compute $\delta\mathbf{f}_2 = -\mathbf{D}'\mathbf{D}\mathbf{f}$ (Correction due to regularization)
- Update $\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + [\delta\mathbf{f}_1 + \delta\mathbf{f}_2]$



Initial guess $\mathbf{f}^{(0)}$ ⟶

estimated image $\mathbf{f}^{(k)}$ ⟶

Forward projection $\mathbf{H}$ ⟶

projections of estimated image $\widehat{\mathbf{g}} = \mathbf{H}\mathbf{f}^{(k)}$ ⟶

$-$ ⟵

Measured projections $\mathbf{g}$

↓ compare ↓

correction term in projection space $\delta\mathbf{g} = \mathbf{g} - \widehat{\mathbf{g}}$ ⟵

Backprojection $\mathbf{H}'$ ⟵

correction term in image space $\delta\mathbf{f} = \mathbf{H}'\delta\mathbf{g} - \lambda\mathbf{D}'\mathbf{D}\mathbf{f}^{(k)}$

↑ update ↑

Content

1. Introduction: Signals and Images, Linear transformations (Convolution, Fourier, Laplace, Hilbert, Radon, ..., Discrete convolution, Z transform, DFT, FFT, ...)
2. Modeling: parametric and non-parametric, MA, AR and ARMA models, Linear prediction
3. Deconvolution and Parameter Estimation: Deterministic (LS, Regularization) and Probabilistic methods (Wiener Filtering)
4. Inverse problems and Bayesian estimation
5. Kalman Filtering and smoothing
6. Case study: Signal deconvolution
7. Case study: Image restoration
8. Case study: Image reconstruction and Computed Tomography

7. Case study: Image restoration

- Image restoration
- Classical methods: Wiener filtering in 2D
- Bayesian approach to deconvolution
- Simple and Blind Deconvolution
- Practical issues and computational costs

Full Bayesian approach

$$\mathcal{M}: \qquad \mathbf{g} = \mathbf{H}\mathbf{f} + \epsilon$$

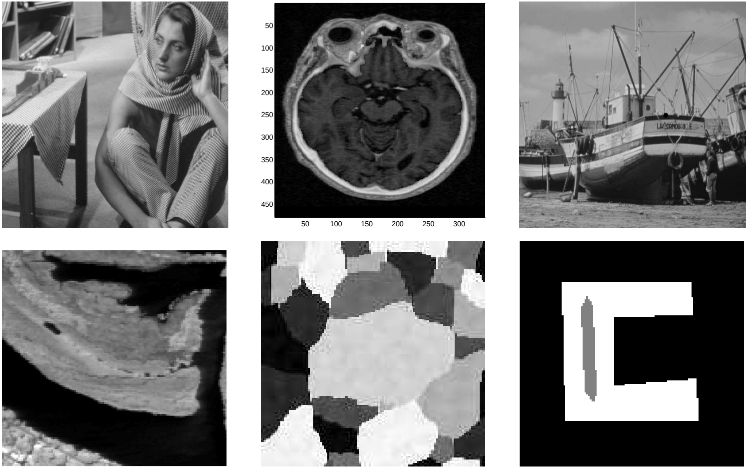
- Forward & errors model: $\longrightarrow p(\mathbf{g}|\mathbf{f}, \boldsymbol{\theta}_1; \mathcal{M})$
- Prior models $\longrightarrow p(\mathbf{f}|\boldsymbol{\theta}_2; \mathcal{M})$
- Hyperparameters $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \longrightarrow p(\boldsymbol{\theta}|\mathcal{M})$
- Bayes: $\longrightarrow p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g}; \mathcal{M}) = \frac{p(\mathbf{g}|\mathbf{f},\boldsymbol{\theta};\mathcal{M})\; p(\mathbf{f}|\boldsymbol{\theta};\mathcal{M})\; p(\boldsymbol{\theta}|\mathcal{M})}{p(\mathbf{g}|\mathcal{M})}$
- Joint MAP: $\qquad (\widehat{\mathbf{f}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\mathbf{f},\boldsymbol{\theta})} \{p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g}; \mathcal{M})\}$
- Marginalization: $\left\{ \begin{array}{ll} p(\mathbf{f}|\mathbf{g}; \mathcal{M}) & = \int p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g}; \mathcal{M})\; \mathrm{d}\mathbf{f} \\ p(\boldsymbol{\theta}|\mathbf{g}; \mathcal{M}) & = \int p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g}; \mathcal{M})\; \mathrm{d}\boldsymbol{\theta} \end{array} \right.$
- Posterior means: $\left\{ \begin{array}{ll} \widehat{\mathbf{f}} & = \int \mathbf{f}\; p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g}; \mathcal{M})\; \mathrm{d}\mathbf{f}\; \mathrm{d}\boldsymbol{\theta} \\ \widehat{\boldsymbol{\theta}} & = \int \boldsymbol{\theta}\; p(\mathbf{f}, \boldsymbol{\theta}|\mathbf{g}; \mathcal{M})\; \mathrm{d}\mathbf{f}\; \mathrm{d}\boldsymbol{\theta} \end{array} \right.$
- Evidence of the model:

$$p(\mathbf{g}|\mathcal{M}) = \iint p(\mathbf{g}|\mathbf{f}, \boldsymbol{\theta}; \mathcal{M}) p(\mathbf{f}|\boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{\theta}|\mathcal{M})\; \mathrm{d}\mathbf{f}\; \mathrm{d}\boldsymbol{\theta}$$

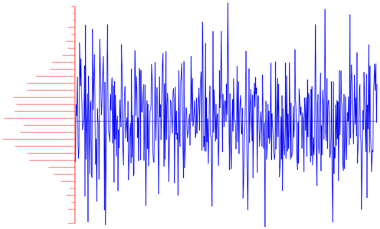
Two main steps in the Bayesian approach

- Prior modeling
  - Separable:
    Gaussian, Generalized Gaussian, Gamma,
    mixture of Gaussians, mixture of Gammas, ...
  - Markovian: Gauss-Markov, GGM, ...
  - Separable or Markovian with hidden variables (contours, region labels)
- Choice of the estimator and computational aspects
  - MAP, Posterior mean, Marginal MAP
  - MAP needs optimization algorithms
  - Posterior mean needs integration methods
  - Marginal MAP needs integration and optimization
  - Approximations:
    - Gaussian approximation (Laplace)
    - Numerical exploration MCMC
    - Variational Bayes (Separable approximation)

Which images I am looking for?

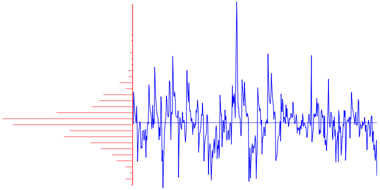


Which signals I am looking for?



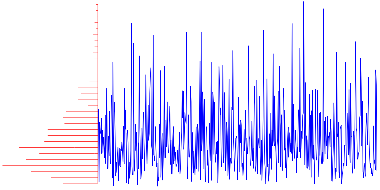
Gaussian

$$p(f_j) \propto \exp\left\{-\alpha|f_j|^2\right\}$$



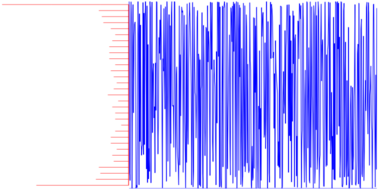
Generalized Gaussian

$$p(f_j) \propto \exp\left\{-\alpha|f_j|^p\right\}, \quad 1 \leq p \leq 2$$



Gamma

$$p(f_j) \propto f_j^\alpha \exp\left\{-\beta f_j\right\}$$



Beta

$$p(f_j) \propto f_j^\alpha (1-f_j)^\beta$$

Different prior models for signals and images

- Separable $p(\mathbf{f}) = \prod_j p_j(f_j) \propto \exp\left\{-\beta \sum_j \phi(f_j)\right\}$

$$p(\mathbf{f}) \propto \exp\left\{-\beta \sum_{\mathbf{r}\in\mathcal{R}} \phi(f(\mathbf{r}))\right\}$$

- Markoviens (simple) $p(f_j|f_{j-1}) \propto \exp\left\{-\beta\phi(f_j - f_{j-1})\right\}$

$$p(\mathbf{f}) \propto \exp\left\{-\beta \sum_{\mathbf{r}\in\mathcal{R}} \sum_{\mathbf{r}'\in\mathcal{V}(\mathbf{r})} \phi(f(\mathbf{r}), f(\mathbf{r}'))\right\}$$

- Markovien with hidden variables
  $z(\mathbf{r})$ (lines, contours, regions)

$$p(\mathbf{f}|\mathbf{z}) \propto \exp\left\{-\beta \sum_{\mathbf{r}\in\mathcal{R}} \sum_{\mathbf{r}'\in\mathcal{V}(\mathbf{r})} \phi(f(\mathbf{r}), f(\mathbf{r}'), z(\mathbf{r}), z(\mathbf{r}'))\right\}$$

Different prior models for images: Separable

- Gaussian:

$$p(f_j) \propto \exp\left\{-\alpha|f_j|^2\right\} \longrightarrow \quad \Omega(\mathbf{f}) = \alpha \sum_j |f_j|^2$$

- Generalized Gaussian (GG):

$$p(f_j) \propto \exp\left\{-\alpha|f_j|^p\right\}, \quad 1 \leq p \leq 2 \longrightarrow \quad \Phi(\mathbf{f}) = \alpha \sum_j |f_j|^p,$$

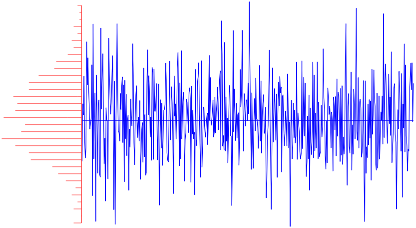
- Gamma: $f_j > 0$

$$p(f_j) \propto f_j^\alpha \exp\left\{-\beta f_j\right\} \longrightarrow \quad \Omega(\mathbf{f}) = \alpha \sum_j \ln f_j + \beta \sum_j f_j,$$

- Beta: $1 > f_j > 0$

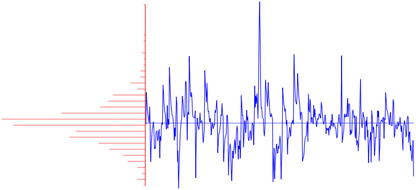
$$p(f_j) \propto f_j^\alpha (1-f_j)^\beta \longrightarrow \quad \Omega(\mathbf{f}) = \alpha \sum_j \ln f_j + \beta \sum_j \ln(1-f_j),$$

Different prior models for images: Separable



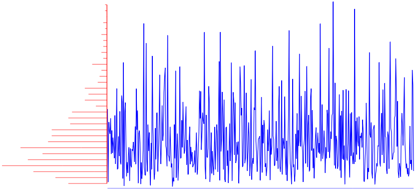
Gaussian

$$p(f_j) \propto \exp\left\{-\alpha|f_j|^2\right\}$$



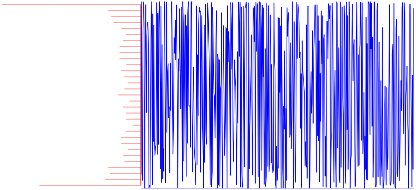
Generalized Gaussian

$$p(f_j) \propto \exp\left\{-\alpha|f_j|^p\right\}, \quad 1 \leq p \leq 2$$



Gamma

$$p(f_j) \propto f_j^\alpha \exp\left\{-\beta f_j\right\}$$



Beta

$$p(f_j) \propto f_j^\alpha (1-f_j)^\beta$$

Different prior models: Simple Markovian

$$p(f_j|\mathbf{f}) \propto \exp\left\{-\alpha \sum_{i\in v_j} \phi(f_j, f_i)\right\} \longrightarrow \quad \Phi(\mathbf{f}) = \alpha \sum_j \sum_{i\in V_j} \phi(f_j, f_i)$$

- 1D case and one neigbor $V_j = j-1$:

$$\Phi(\mathbf{f}) = \alpha \sum_j \phi(f_j - f_{j-1})$$

- 1D Case and two neighbors $V_j = \{j-1, j+1\}$:

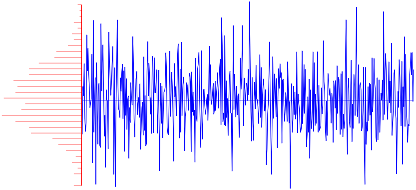
$$\Phi(\mathbf{f}) = \alpha \sum_j \phi\left(f_j - \beta(f_{j-1} + f_{j-1})\right)$$

- 2D case with 4 neighbors:

$$\Phi(\mathbf{f}) = \alpha \sum_{\mathbf{r}\in\mathcal{R}} \phi\left(f(\mathbf{r}) - \beta \sum_{\mathbf{r}'\in\mathcal{V}(\mathbf{r})} f(\mathbf{r}')\right)$$

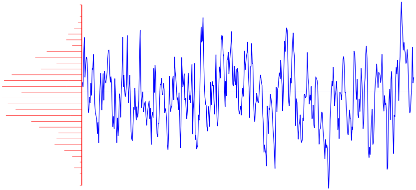
- $\phi(t) = |t|^\gamma$: Generalized Gaussian

Different prior models: Simple Markovian



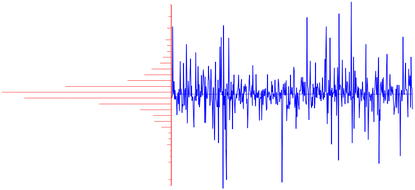
IID Gaussian

$p(f_j) \propto \exp\left\{-\alpha|f_j|^2\right\}$



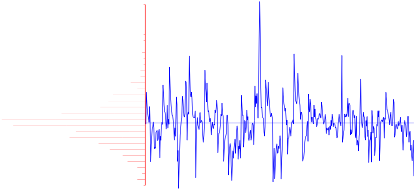
Gauss-Markov

$p(f_j|f_{j-1}) \propto \exp\left\{-\alpha|f_j - f_{j-1}|^2\right\}$



IID GG

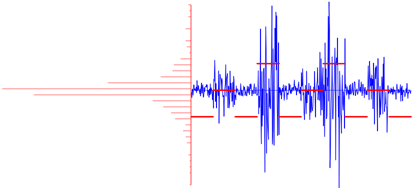
$p(f_j) \propto \exp\left\{-\alpha|f_j|^p\right\}$



Markovian GG

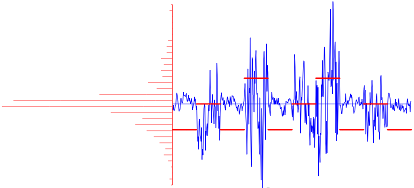
$p(f_j|f_{j-1}) \propto \exp\left\{-\alpha|f_j - f_{j-1}|^p\right\}$

Different prior models: Non-stationnary signals



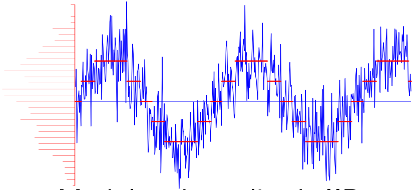
Modulated Variances IID

$p(f_j|z_j) = \mathcal{N}(0, v(z_j))$



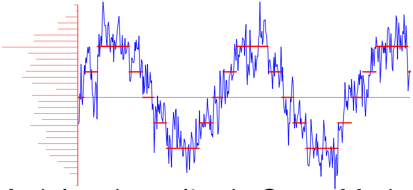
Modulated Variances Gauss-Markov

$p(f_j|f_{j-1}, z_j) = \mathcal{N}(f_{j-1}, v(z_j))$



Modulated amplituds IID

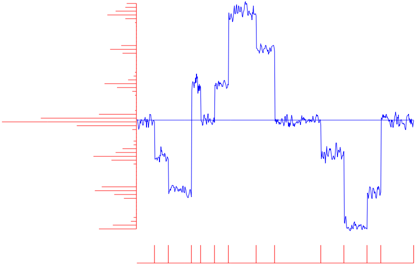
$p(f_j|z_j) = \mathcal{N}(a(z_j), 1)$



Modulated amplituds Gauss-Markov

$p(f_j|f_{j-1}, z_j) = \mathcal{N}(a(f_{j-1}, z_j), 1)$

Different prior models: Markovian with hidden variables

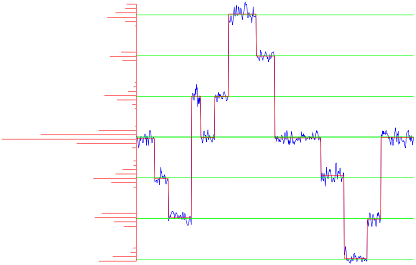


Piecewise Gaussians

(contours hidden variables)

$$p(f_j|q_j,f_{j-1}) = \mathcal{N}\left((1-q_j)f_{j-1},\sigma_f^2\right)$$

$$p(\mathbf{f}|\mathbf{q}) \propto \exp\left\{-\alpha\sum_j \left|f_j-(1-q_j)f_{j-1}\right|^2\right\}$$



Mixture of Gaussians (MoG)

(regions labels hidden variables)

$$p(f_j|z_j=k) = \mathcal{N}\left(m_k,\sigma_k^2\right) \text{ \& } z_j \text{ markovian}$$

$$p(\mathbf{f}|\mathbf{z}) \propto \exp\left\{-\alpha\sum_k\sum_{j\in\mathcal{R}_k}\left(\frac{f_j-m_k}{\sigma_k}\right)^2\right\}$$

Particular case of Gauss-Markov models

$$\left\{\begin{array}{l} \mathbf{g}=\mathbf{H}\mathbf{f}+\boldsymbol{\epsilon} \quad \text{with} \\ \mathbf{f}\sim\mathcal{N}\left(0,\sigma_f^2(\mathbf{D}'\mathbf{D})^{-1}\right) \end{array}\right. = \left\{\begin{array}{l} \mathbf{g}=\mathbf{H}\mathbf{f}+\boldsymbol{\epsilon} \\ \mathbf{f}=\mathbf{C}\mathbf{f}+\mathbf{z} \text{ with } \mathbf{z}\sim\mathcal{N}(0,\sigma_f^2\mathbf{I}) \\ \text{and} \quad \mathbf{D}=(\mathbf{I}-\mathbf{C}) \end{array}\right.$$

$$\mathbf{f}|\mathbf{g}\sim\mathcal{N}(\widehat{\mathbf{f}},\widehat{\mathbf{P}}) \quad \text{with} \quad \widehat{\mathbf{f}}=\widehat{\mathbf{P}}\mathbf{H}'\mathbf{g}, \quad \widehat{\mathbf{P}}=\left(\mathbf{H}'\mathbf{H}+\lambda\mathbf{D}'\mathbf{D}\right)^{-1}$$

$$\widehat{\mathbf{f}}=\arg\min_{\mathbf{f}}\left\{J(\mathbf{f})=\|\mathbf{g}-\mathbf{H}\mathbf{f}\|^2+\lambda\|\mathbf{D}\mathbf{f}\|^2\right\}$$

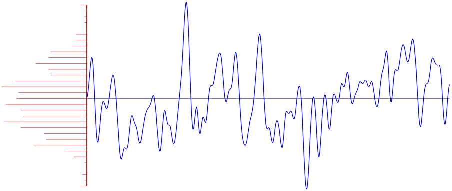
$$\left\{\begin{array}{l} \mathbf{g}=\mathbf{H}\mathbf{f}+\boldsymbol{\epsilon} \\ \text{with} \quad \mathbf{f}\sim\mathcal{N}\left(0,\sigma_f^2(\mathbf{D}\mathbf{D}')\right) \end{array}\right. = \left\{\begin{array}{l} \mathbf{g}=\mathbf{H}\mathbf{f}+\boldsymbol{\epsilon} \\ \mathbf{f}=\mathbf{D}\mathbf{z} \quad \text{with} \quad \mathbf{z}\sim\mathcal{N}(0,\sigma_f^2\mathbf{I}) \end{array}\right.$$

$$\mathbf{z}|\mathbf{g}\sim\mathcal{N}(\widehat{\mathbf{z}},\widehat{\mathbf{P}}) \quad \text{with} \quad \widehat{\mathbf{z}}=\widehat{\mathbf{P}}\mathbf{D}'\mathbf{H}'\mathbf{g}, \quad \widehat{\mathbf{P}}=\left(\mathbf{D}'\mathbf{H}'\mathbf{H}\mathbf{D}+\lambda\mathbf{I}\right)^{-1}$$

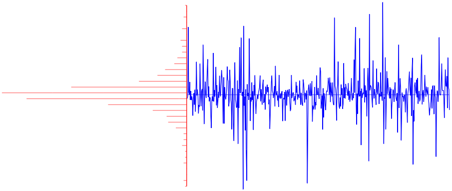
$$\widehat{\mathbf{z}}=\arg\min_{\mathbf{z}}\left\{J(\mathbf{z})=\|\mathbf{g}-\mathbf{H}\mathbf{D}\mathbf{z}\|^2+\lambda\|\mathbf{z}\|^2\right\}\longrightarrow\widehat{\mathbf{f}}=\mathbf{D}\widehat{\mathbf{z}}$$

$\mathbf{z}$ Decomposition coeff on a basis (column of $\mathbf{D}$)

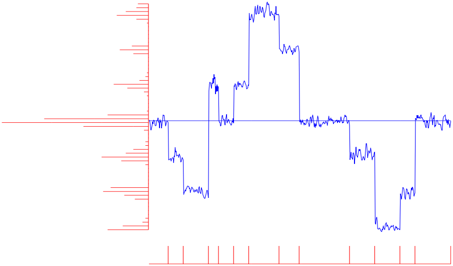
Which images I am looking for?



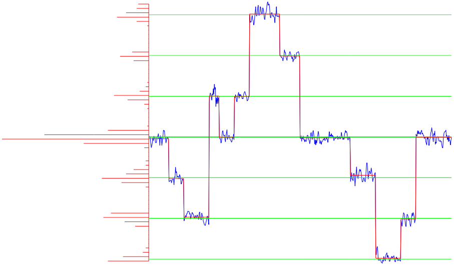
Gauss-Markov



Generalized GM



Piecewize Gaussian



Mixture of GM

Markovien prior models for images

$$\Omega(\mathbf{f}) = \sum_j \phi(f_j - f_{j-1})$$

- Gauss-Markov : $\phi(t) = |t|^2$
- Generalized Gauss-Markov : $\phi(t) = |t|^\alpha$
- Picewize Gauss-Markov or GGM : $\phi(t) = \begin{cases} t^2 & |t| \leq T \\ T^2 & |t| > T \end{cases}$

  or equivalently :

$$\Omega(\mathbf{f}|\mathbf{q}) = \sum_j (1 - q_j)\phi(f_j - f_{j-1})$$

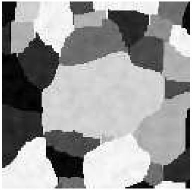
  $\mathbf{q}$ line process (contours)

- Mixture of Gaussians :

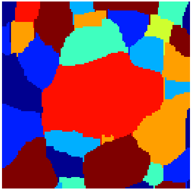
$$\Omega(\mathbf{f}|\mathbf{z}) = \sum_k \sum_{\{j:z_j=k\}} \left(\frac{f_j - m_k}{v_k}\right)^2$$

  $\mathbf{z}$ region labels process.

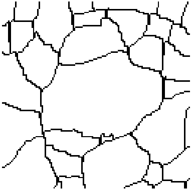
Gauss-Markov-Potts prior models for images



$f(\mathbf{r})$



$z(\mathbf{r})$



$c(\mathbf{r}) = 1 - \delta(z(\mathbf{r}) - z(\mathbf{r}'))$

$$p(f(\mathbf{r})|z(\mathbf{r}) = k, m_k, v_k) = \mathcal{N}(m_k, v_k)$$

$$p(f(\mathbf{r})) = \sum_k P(z(\mathbf{r}) = k)\,\mathcal{N}(m_k, v_k) \quad \text{Mixture of Gaussians}$$

- Separable iid hidden variables: $p(\mathbf{z}) = \prod_{\mathbf{r}} p(z(\mathbf{r}))$
- Markovian hidden variables: $p(\mathbf{z})$ Potts-Markov:

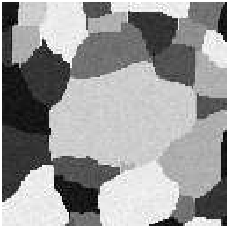
$$p(z(\mathbf{r})|z(\mathbf{r}'), \mathbf{r}' \in \mathcal{V}(\mathbf{r})) \propto \exp\left\{\gamma \sum_{\mathbf{r}'\in\mathcal{V}(\mathbf{r})} \delta(z(\mathbf{r}) - z(\mathbf{r}'))\right\}$$

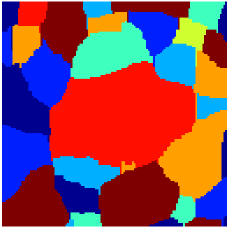
$$p(\mathbf{z}) \propto \exp\left\{\gamma \sum_{\mathbf{r}\in\mathcal{R}} \sum_{\mathbf{r}'\in\mathcal{V}(\mathbf{r})} \delta(z(\mathbf{r}) - z(\mathbf{r}'))\right\}$$

Four different cases

To each pixel of the image is associated 2 variables $f(\mathbf{r})$ and $z(\mathbf{r})$

- $\mathbf{f}|\mathbf{z}$ Gaussian iid, $\mathbf{z}$ iid :
  Mixture of Gaussians
- $\mathbf{f}|\mathbf{z}$ Gauss-Markov, $\mathbf{z}$ iid :
  Mixture of Gauss-Markov
- $\mathbf{f}|\mathbf{z}$ Gaussian iid, $\mathbf{z}$ Potts-Markov :
  Mixture of Independent Gaussians
  (MIG with Hidden Potts)
- $\mathbf{f}|\mathbf{z}$ Markov, $\mathbf{z}$ Potts-Markov :
  Mixture of Gauss-Markov
  (MGM with hidden Potts)


$f(\mathbf{r})$


$z(\mathbf{r})$

Case 1: $\mathbf{f}|\mathbf{z}$ Gaussian iid, $\mathbf{z}$ iid

Independent Mixture of Independent Gaussiens (IMIG):

$$p(f(\mathbf{r})|z(\mathbf{r})=k)=\mathcal{N}(m_k,v_k),\quad \forall\mathbf{r}\in\mathcal{R}$$
$$p(f(\mathbf{r}))=\sum_{k=1}^K \alpha_k\mathcal{N}(m_k,v_k), \text{with } \sum_k \alpha_k=1.$$

$$p(\mathbf{z})=\prod_{\mathbf{r}} p(z(\mathbf{r})=k)=\prod_{\mathbf{r}}\alpha_k=\prod_k \alpha_k^{n_k}$$

Noting

$$m_z(\mathbf{r})=m_k, v_z(\mathbf{r})=v_k, \alpha_z(\mathbf{r})=\alpha_k, \forall\mathbf{r}\in\mathcal{R}_k$$

we have:

$$\begin{aligned} p(\mathbf{f}|\mathbf{z}) &= \prod_{\mathbf{r}\in\mathcal{R}} \mathcal{N}(m_z(\mathbf{r}), v_z(\mathbf{r})) \\ p(\mathbf{z}) &= \prod_{\mathbf{r}} \alpha_z(\mathbf{r}) = \prod_k \alpha_k^{\sum_{\mathbf{r}\in\mathcal{R}}\delta(z(\mathbf{r})-k)} = \prod_k \alpha_k^{n_k} \end{aligned}$$

Case 2: $\mathbf{f}|\mathbf{z}$ Gauss-Markov, $\mathbf{z}$ iid

Independent Mixture of Gauss-Markov (IMGM):

$$p(f(\mathbf{r})|z(\mathbf{r}), z(\mathbf{r}'), f(\mathbf{r}'), \mathbf{r}' \in \mathcal{V}(\mathbf{r})) = \mathcal{N}(\mu_z(\mathbf{r}), v_z(\mathbf{r})), \forall \mathbf{r} \in \mathcal{R}$$

$$\begin{array}{ll} \mu_z(\mathbf{r}) & = \frac{1}{|\mathcal{V}(\mathbf{r})|} \sum_{\mathbf{r}' \in \mathcal{V}(\mathbf{r})} \mu_z^*(\mathbf{r}') \\ \mu_z^*(\mathbf{r}') & = \delta(z(\mathbf{r}') - z(\mathbf{r}))\ f(\mathbf{r}') + (1 - \delta(z(\mathbf{r}') - z(\mathbf{r}))\ m_z(\mathbf{r}') \\ & = \quad (1 - c(\mathbf{r}')) \quad f(\mathbf{r}')\ + \qquad\quad c(\mathbf{r}') \qquad\qquad m_z(\mathbf{r}') \end{array}$$

$$\begin{array}{lll} p(\mathbf{f}|\mathbf{z}) & \propto \prod_{\mathbf{r}} \mathcal{N}(\mu_z(\mathbf{r}), v_z(\mathbf{r})) & \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \mathbf{\Sigma}_k) \\ p(\mathbf{z}) & = \prod_{\mathbf{r}} v_z(\mathbf{r}) & = \prod_k \alpha_k^{n_k} \end{array}$$

with $1_k = 1, \forall \mathbf{r} \in \mathcal{R}_k$ and $\mathbf{\Sigma}_k$ a covariance matrix $(n_k \times n_k)$.

Case 3: $\mathbf{f}|\mathbf{z}$ Gauss iid, $\mathbf{z}$ Potts

Gauss iid as in Case 1:

$$p(\mathbf{f}|\mathbf{z}) = \prod_{\mathbf{r}\in\mathcal{R}} \mathcal{N}(m_z(\mathbf{r}), v_z(\mathbf{r})) = \prod_k \prod_{\mathbf{r}\in\mathcal{R}_k} \mathcal{N}(m_k, v_k)$$

Potts-Markov

$$p(z(\mathbf{r})|z(\mathbf{r}'), \mathbf{r}' \in \mathcal{V}(\mathbf{r})) \propto \exp\left\{\gamma \sum_{\mathbf{r}'\in\mathcal{V}(\mathbf{r})} \delta(z(\mathbf{r}) - z(\mathbf{r}'))\right\}$$

$$p(\mathbf{z}) \propto \exp\left\{\gamma \sum_{\mathbf{r}\in\mathcal{R}} \sum_{\mathbf{r}'\in\mathcal{V}(\mathbf{r})} \delta(z(\mathbf{r}) - z(\mathbf{r}'))\right\}$$

Case 4: $\mathbf{f}|\mathbf{z}$ Gauss-Markov, $\mathbf{z}$ Potts

Gauss-Markov as in Case 2:

$$p(f(\mathbf{r})|z(\mathbf{r}), z(\mathbf{r}'), f(\mathbf{r}'), \mathbf{r}' \in \mathcal{V}(\mathbf{r})) = \mathcal{N}(\mu_z(\mathbf{r}), v_z(\mathbf{r})), \forall \mathbf{r} \in \mathcal{R}$$

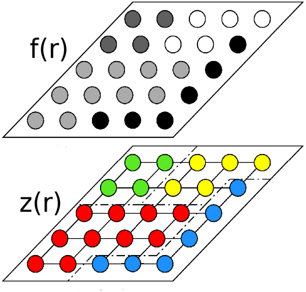
$$\begin{array}{ll} \mu_z(\mathbf{r}) & = \frac{1}{|\mathcal{V}(\mathbf{r})|} \sum_{\mathbf{r}' \in \mathcal{V}(\mathbf{r})} \mu_z^*(\mathbf{r}') \\ \mu_z^*(\mathbf{r}') & = \delta(z(\mathbf{r}') - z(\mathbf{r}))\; f(\mathbf{r}') + (1 - \delta(z(\mathbf{r}') - z(\mathbf{r}))\; m_z(\mathbf{r}') \end{array}$$

$$p(\mathbf{f}|\mathbf{z}) \quad \propto \prod_{\mathbf{r}} \mathcal{N}(\mu_z(\mathbf{r}), v_z(\mathbf{r})) \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \mathbf{\Sigma}_k)$$

Potts-Markov as in Case 3:

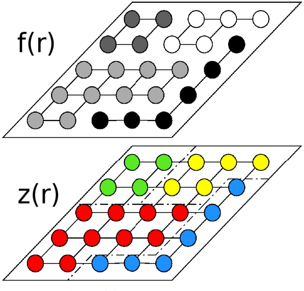
$$p(\mathbf{z}) \propto \exp\left\{ \gamma \sum_{\mathbf{r} \in \mathcal{R}} \sum_{\mathbf{r}' \in \mathcal{V}(\mathbf{r})} \delta(z(\mathbf{r}) - z(\mathbf{r}')) \right\}$$

Summary of the two proposed models



f|**z** Gaussian iid
z Potts-Markov

(MIG with Hidden Potts)



f|**z** Markov
z Potts-Markov

(MGM with hidden Potts)

Bayesian Computation

$$p(\mathbf{f},\mathbf{z},\boldsymbol{\theta}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f},\mathbf{z},v_\epsilon)\, p(\mathbf{f}|\mathbf{z},\mathbf{m},\mathbf{v})\, p(\mathbf{z}|\gamma,\boldsymbol{\alpha})\, p(\boldsymbol{\theta})$$

$\boldsymbol{\theta} = \{v_\epsilon, (\alpha_k, m_k, v_k), k = 1, \cdot, K\}$ $\quad p(\boldsymbol{\theta})$ Conjugate priors

- Direct computation and use of $p(\mathbf{f},\mathbf{z},\boldsymbol{\theta}|\mathbf{g};\mathcal{M})$ is too complex
- Possible approximations :
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)
- Main idea in Variational Bayesian methods:
  Approximate
  $$p(\mathbf{f},\mathbf{z},\boldsymbol{\theta}|\mathbf{g};\mathcal{M}) \quad \text{by} \quad q(\mathbf{f},\mathbf{z},\boldsymbol{\theta}) = q_1(\mathbf{f})\; q_2(\mathbf{z})\; q_3(\boldsymbol{\theta})$$
  - Choice of approximation criterion : $KL(q : p)$
  - Choice of appropriate families of probability laws for $q_1(\mathbf{f})$, $q_2(\mathbf{z})$ and $q_3(\boldsymbol{\theta})$

MCMC based algorithm

$$p(\mathbf{f},\mathbf{z},\boldsymbol{\theta}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f},\mathbf{z},\boldsymbol{\theta})\, p(\mathbf{f}|\mathbf{z},\boldsymbol{\theta})\, p(\mathbf{z})\, p(\boldsymbol{\theta})$$

General scheme:

$$\widehat{\mathbf{f}} \sim p(\mathbf{f}|\widehat{\mathbf{z}},\widehat{\boldsymbol{\theta}},\mathbf{g}) \longrightarrow \widehat{\mathbf{z}} \sim p(\mathbf{z}|\widehat{\mathbf{f}},\widehat{\boldsymbol{\theta}},\mathbf{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta}|\widehat{\mathbf{f}},\widehat{\mathbf{z}},\mathbf{g})$$

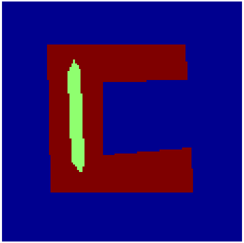
- Estimate $\mathbf{f}$ using $p(\mathbf{f}|\widehat{\mathbf{z}},\widehat{\boldsymbol{\theta}},\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f},\boldsymbol{\theta})\, p(\mathbf{f}|\widehat{\mathbf{z}},\widehat{\boldsymbol{\theta}})$
  Needs optimisation of a quadratic criterion.

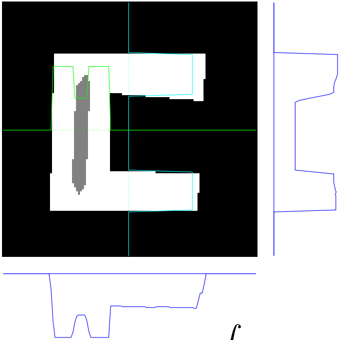
- Estimate $\mathbf{z}$ using $p(\mathbf{z}|\widehat{\mathbf{f}},\widehat{\boldsymbol{\theta}},\mathbf{g}) \propto p(\mathbf{g}|\widehat{\mathbf{f}},\widehat{\mathbf{z}},\widehat{\boldsymbol{\theta}})\, p(\mathbf{z})$
  Needs sampling of a Potts Markov field.

- Estimate $\boldsymbol{\theta}$ using
  $p(\boldsymbol{\theta}|\widehat{\mathbf{f}},\widehat{\mathbf{z}},\mathbf{g}) \propto p(\mathbf{g}|\widehat{\mathbf{f}},\sigma_\epsilon^2\mathbf{I})\, p(\widehat{\mathbf{f}}|\widehat{\mathbf{z}},(m_k,v_k))\, p(\boldsymbol{\theta})$
  Conjugate priors ⟶ analytical expressions.

Application of CT in NDT

Reconstruction from only 2 projections



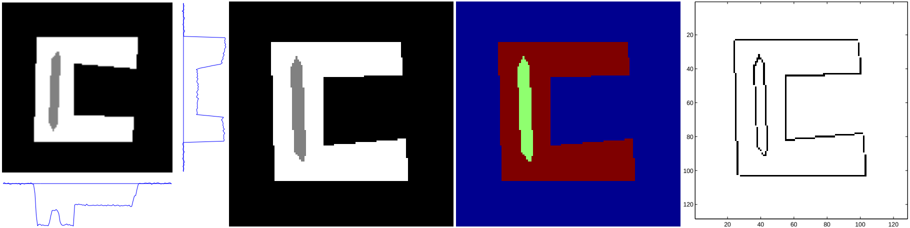


$$g_1(x) = \int f(x,y)\,\mathrm{d}y, \qquad g_2(y) = \int f(x,y)\,\mathrm{d}x$$

- Given the marginals $g_1(x)$ and $g_2(y)$ find the joint distribution $f(x,y)$.
- Infinite number of solutions : $f(x,y) = g_1(x)\,g_2(y)\,\Omega(x,y)$
  $\Omega(x,y)$ is a Copula:

$$\int \Omega(x,y)\,\mathrm{d}x = 1 \quad \text{and} \quad \int \Omega(x,y)\,\mathrm{d}y = 1$$

Application in CT



$\mathbf{g}\mid\mathbf{f}$	$\mathbf{f}\mid\mathbf{z}$	$\mathbf{z}$	$\mathbf{c}$
$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon}$	iid Gaussian	iid	$c(\mathbf{r}) \in \{0,1\}$
$\mathbf{g}\mid\mathbf{f} \sim \mathcal{N}(\mathbf{H}\mathbf{f}, \sigma_\epsilon^2\mathbf{I})$	or	or	$1 - \delta(z(\mathbf{r}) - z(\mathbf{r}'))$
Gaussian	Gauss-Markov	Potts	binary

Proposed algorithm

$$p(\mathbf{f}, \mathbf{z}, \boldsymbol{\theta}|\mathbf{g}) \propto p(\mathbf{g}|\mathbf{f}, \mathbf{z}, \boldsymbol{\theta})\, p(\mathbf{f}|\mathbf{z}, \boldsymbol{\theta})\, p(\boldsymbol{\theta})$$

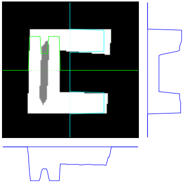
General scheme:

$$\widehat{\mathbf{f}} \sim p(\mathbf{f}|\widehat{\mathbf{z}}, \widehat{\boldsymbol{\theta}}, \mathbf{g}) \longrightarrow \widehat{\mathbf{z}} \sim p(\mathbf{z}|\widehat{\mathbf{f}}, \widehat{\boldsymbol{\theta}}, \mathbf{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta}|\widehat{\mathbf{f}}, \widehat{\mathbf{z}}, \mathbf{g})$$

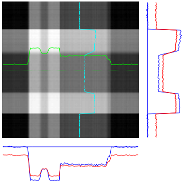
Iterative algorithme:

- Estimate $\mathbf{f}$ using $p(\mathbf{f}|\widehat{\mathbf{z}}, \widehat{\boldsymbol{\theta}}, \mathbf{g}) \propto p(\mathbf{g}|\mathbf{f}, \boldsymbol{\theta})\, p(\mathbf{f}|\widehat{\mathbf{z}}, \widehat{\boldsymbol{\theta}})$
  Needs optimisation of a quadratic criterion.
- Estimate $\mathbf{z}$ using $p(\mathbf{z}|\widehat{\mathbf{f}}, \widehat{\boldsymbol{\theta}}, \mathbf{g}) \propto p(\mathbf{g}|\widehat{\mathbf{f}}, \widehat{\mathbf{z}}, \widehat{\boldsymbol{\theta}})\, p(\mathbf{z})$
  Needs sampling of a Potts Markov field.
- Estimate $\boldsymbol{\theta}$ using
  $p(\boldsymbol{\theta}|\widehat{\mathbf{f}}, \widehat{\mathbf{z}}, \mathbf{g}) \propto p(\mathbf{g}|\widehat{\mathbf{f}}, \sigma_\epsilon^2 \mathbf{I})\, p(\widehat{\mathbf{f}}|\widehat{\mathbf{z}}, (m_k, v_k))\, p(\boldsymbol{\theta})$
  Conjugate priors ⟶ analytical expressions.

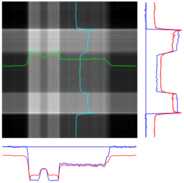
Results



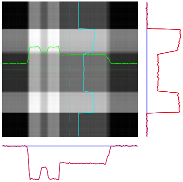
Original



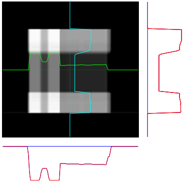
Backprojection



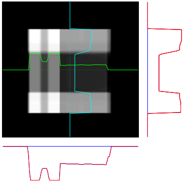
Filtered BP



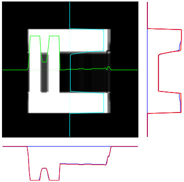
LS



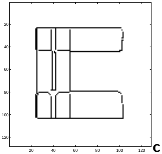
Gauss-Markov+pos

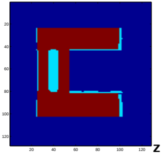


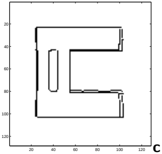
GM+Line process



GM+Label process





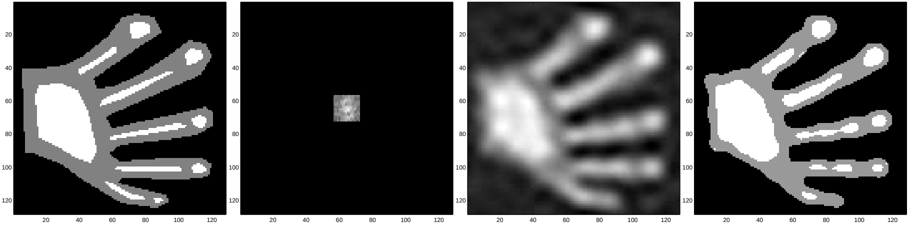


Application in Microwave imaging

$$g(\omega) = \int f(\mathbf{r}) \exp\{-j(\omega.\mathbf{r})\} \, \mathrm{d}\mathbf{r} + \epsilon(\omega)$$

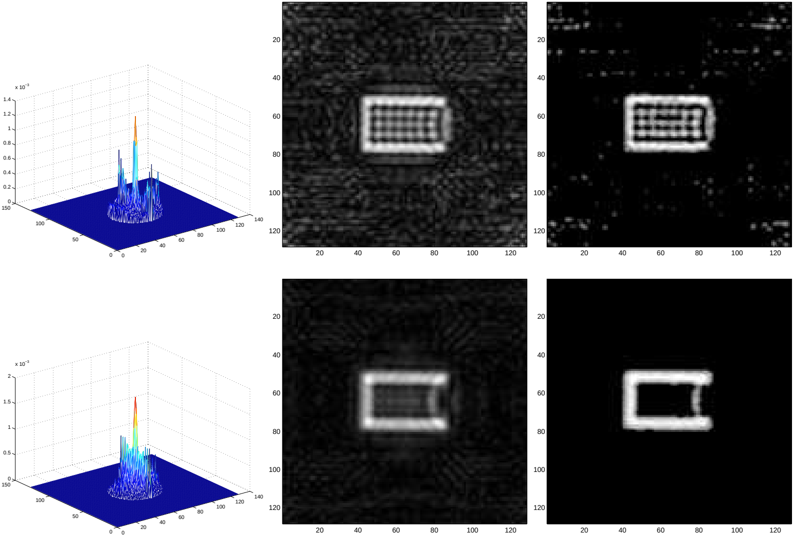
$$g(u,v) = \iint f(x,y) \exp\{-j(ux+vy)\} \, \mathrm{d}x \, \mathrm{d}y + \epsilon(u,v)$$

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon}$$



$f(x,y)$ $g(u,v)$ $\widehat{\mathbf{f}}$ IFT $\widehat{\mathbf{f}}$ Proposed method

Application in Microwave imaging



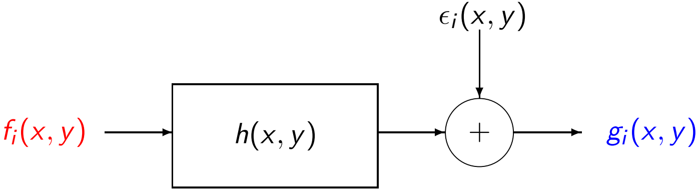
Conclusions

- Bayesian Inference for inverse problems
- Approximations (Laplace, MCMC, Variational)
- Gauss-Markov-Potts are useful prior models for images incorporating regions and contours
- Separable approximations for Joint posterior with Gauss-Markov-Potts priors
- Application in different CT (X ray, US, Microwaves, PET, SPECT)

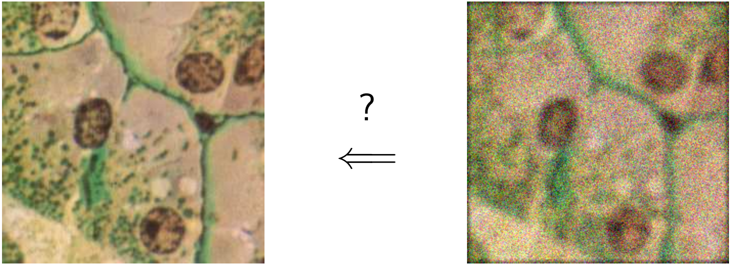
Perspectives :

- Efficient implementation in 2D and 3D cases
- Evaluation of performances and comparison with MCMC methods
- Application to other linear and non linear inverse problems: (PET, SPECT or ultrasound and microwave imaging)

Color (Multi-spectral) image deconvolution



Observation model : $\mathbf{g}_i = \mathbf{H}\mathbf{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, 2, 3$

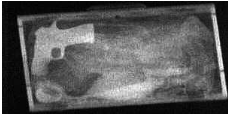


Images fusion and joint segmentation

(with O. Féron)

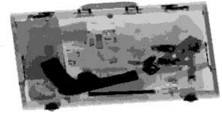
$$\left\{\begin{array}{l} g_i(\mathbf{r}) = f_i(\mathbf{r}) + \epsilon_i(\mathbf{r}) \\ p(f_i(\mathbf{r})|z(\mathbf{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma_{i\,k}^2) \\ p(\underline{\mathbf{f}}|\mathbf{z}) = \prod_i p(\mathbf{f}_i|\mathbf{z}) \end{array}\right.$$

$\mathbf{g}_1$

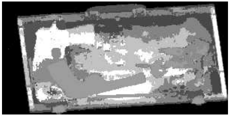


$\mathbf{g}_2$

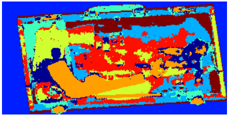
$\longrightarrow$



$\widehat{\mathbf{f}}_1$



$\widehat{\mathbf{f}}_2$

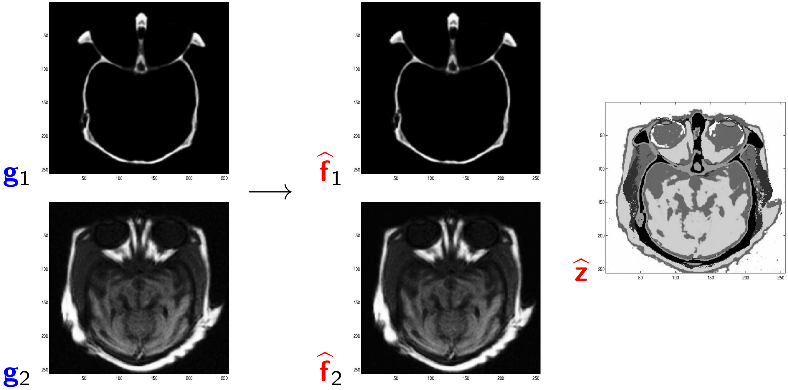


$\widehat{\mathbf{z}}$

Data fusion in medical imaging

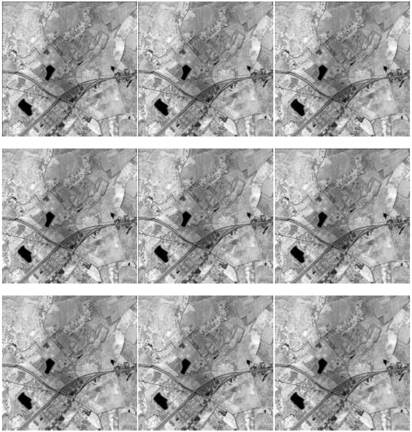
(with O. Féron)

$$\begin{cases} g_i(\mathbf{r}) = f_i(\mathbf{r}) + \epsilon_i(\mathbf{r}) \\ p(f_i(\mathbf{r})|z(\mathbf{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma_{i\,k}^2) \\ p(\underline{\mathbf{f}}|\mathbf{z}) = \prod_i p(\mathbf{f}_i|\mathbf{z}) \end{cases}$$



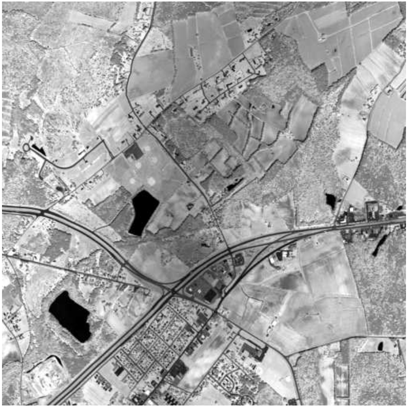
Super-Resolution

(with **F. Humblot**)



Low Resolution images

$\overset{?}{\Longrightarrow}$

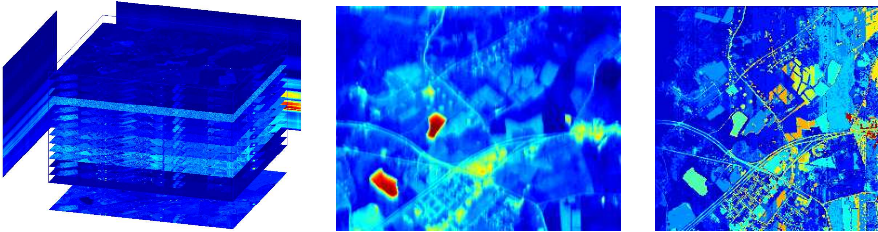


High Resolution image

Joint segmentation of hyper-spectral images

(with N. Bali & A. Mohammadpour)

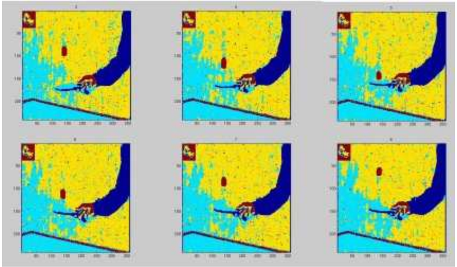
$$\begin{cases} g_i(\mathbf{r}) = f_i(\mathbf{r}) + \epsilon_i(\mathbf{r}) \\ p(f_i(\mathbf{r})|z(\mathbf{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma_{i\,k}^2), \quad k = 1, \cdots, K \\ p(\underline{\mathbf{f}}|\mathbf{z}) = \prod_i p(\mathbf{f}_i|\mathbf{z}) \\ m_{i\,k} \;\; \text{follow a Markovian model along the index} \;\; i \end{cases}$$



Segmentation of a video sequence of images

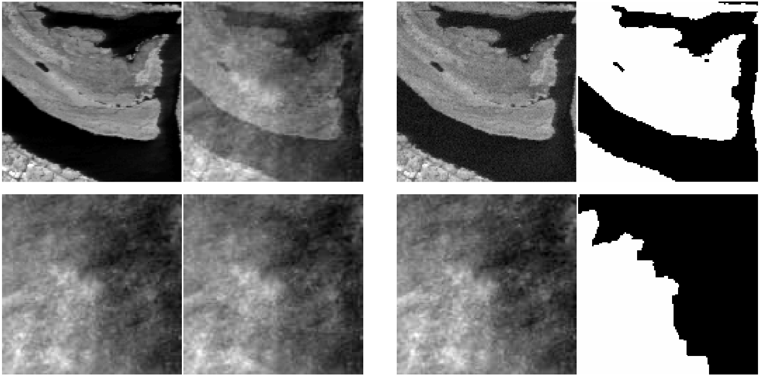
(with P. Brault)

$$\begin{cases} g_i(\mathbf{r}) = f_i(\mathbf{r}) + \epsilon_i(\mathbf{r}) \\ p(f_i(\mathbf{r})|z_i(\mathbf{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma^2_{i\,k}), \quad k = 1, \cdots, K \\ p(\underline{\mathbf{f}}|\mathbf{z}) = \prod_i p(\mathbf{f}_i|\mathbf{z}_i) \\ z_i(\mathbf{r}) \;\; \text{follow a Markovian model along the index} \;\; i \end{cases}$$



Source separation

(with H. Snoussi & M. Ichir) $\begin{cases} g_i(\mathbf{r}) = \sum_{j=1}^{N} A_{ij} f_j(\mathbf{r}) + \epsilon_i(\mathbf{r}) \\ p(f_j(\mathbf{r}) | z_j(\mathbf{r}) = k) = \mathcal{N}(m_{j\,k}, \sigma^2_{j\,k}) \\ p(A_{ij}) = \mathcal{N}(A_{0\,ij}, \sigma^2_{0\,ij}) \end{cases}$



$\underline{\mathbf{f}}$ $\underline{\mathbf{g}}$ $\underline{\widehat{\mathbf{f}}}$ $\underline{\widehat{\mathbf{z}}}$

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- D. Pougaza (Tomography and Copula)
- ———————
- Sh. Zhu (SAR Imaging)
- D. Fall (Emission Positon Tomography, Non Parametric Bayesian)

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- N. Gac & A. Rabanal (GPU Implementation)
- Th. Rodet (Tomography)
- ———————
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Questions and Discussions