Bayesian Approach for Inverse Problems in Imaging

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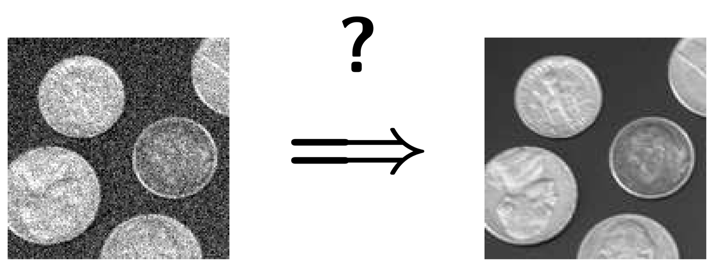
Absract

Inverse problems arise in almost all the imaging systems: denoising, segmentation, deconvolution and restoration, joint restoration, segmentation and contour detection. We proposed and developped methods based on the Bayesian inference for all these problems. In particular, we use a family of Gauss-Markov-Potts prior models within the Bayesian framework which gives us the possibility to perform jointly denoising or restoration, segmentation and contours detection in an optimal way.

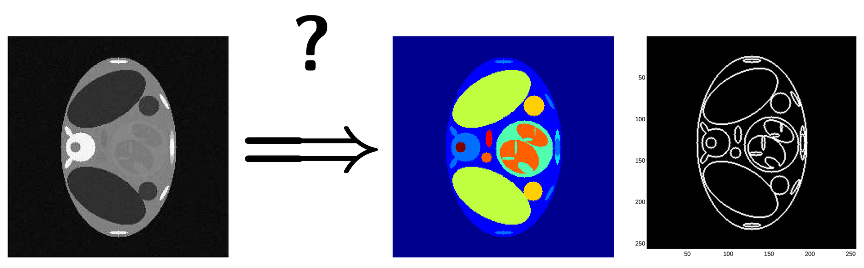
Inverse Problems in Imaging Systems

- Denoising:



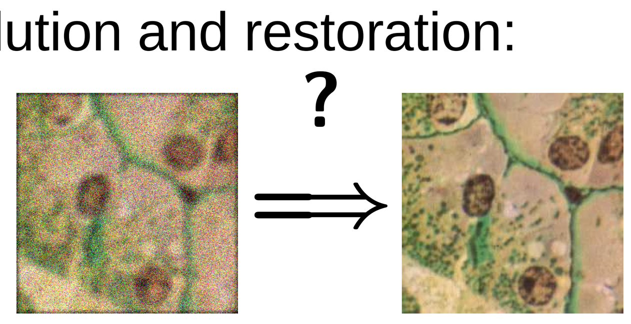
$g(r)$ $f(r)$

- Segmentation and contour detection:



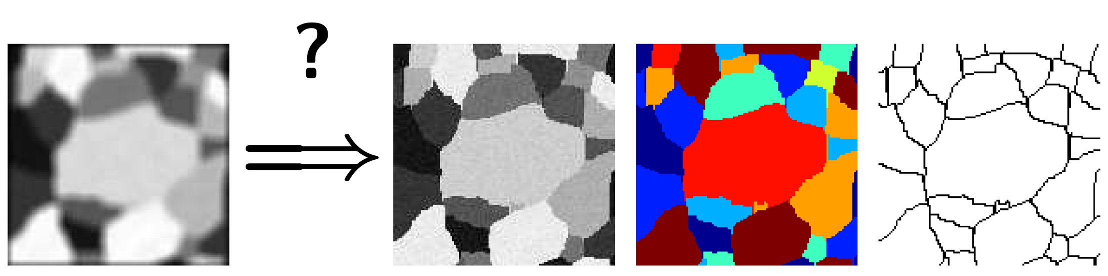
$g(r)$ $z(r)$ $q(r)$

- Deconvolution and restoration:



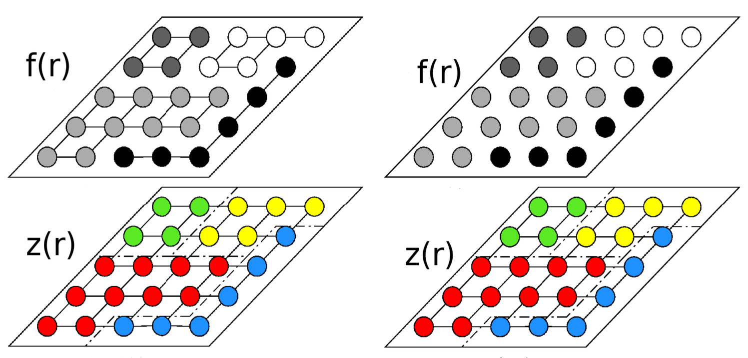
$g(r)$ $f(r)$

- Joint Restoration, segmentation and contour estimation:



$g(r)$ $f(r)$ $z(r)$ $q(r)$

A family of Hierarchical Gauss-Markov-Potts prior models



$$p(f(r)|z(r)=k) = \mathcal{N}(m_k, v_k),\ k = 1,\cdots,K$$

$$p(z) \propto \exp\left\{-\sum_{r\in\mathcal{R}}\sum_k \alpha_k \delta(z(r)-k) - \sum_{r\in\mathcal{R}} \gamma \sum_{s\in\mathcal{V}(r)} \delta(z(r)-z(s))\right\}$$

$$p(f_k|z(r)=k) = \mathcal{N}(m_k \mathbf{1}, \Sigma_k)$$

$$p(f|z) = \prod_{k=1}^K p(f_k) = \prod_{k=1}^K \mathcal{N}(m_k \mathbf{1}, \Sigma_k).$$

$$p(f,z) = p(f|z)\,p(z)$$

Model Parameters:

$$\theta = \{(\alpha_k, m_k, v_k), k=1,\cdots,K,\gamma\}$$

Bayesian Estimation Approach

- Forward model

$$\mathcal{M}: \quad g = Hf + \epsilon$$

- Likelihood: Observation model $\mathcal{M}$ + Hypothesis on the noise $\epsilon \longrightarrow$

$$p(g|f;\mathcal{M}) = p_\epsilon(g - Hf)$$

- A priori information $\quad p(f|\mathcal{M})$
- Bayes :

$$p(f|g;\mathcal{M}) = \frac{p(g|f;\mathcal{M})\,p(f|\mathcal{M})}{p(g|\mathcal{M})}$$

- Estimators:
  - Mode (Maximum A Posteriori)
  - Mean (Posterior Mean)
  - Marginal modes
  - ...

Full Bayesian approach

$$\mathcal{M}: \quad g = Hf + \epsilon$$

- Forward & errors model: $\longrightarrow p(g|f,\theta_1;\mathcal{M})$
- Prior models $\longrightarrow p(f|\theta_2;\mathcal{M})$
- Hyperparameters $\theta = (\theta_1,\theta_2) \longrightarrow p(\theta|\mathcal{M})$
- Bayes:

$$p(f,\theta|g;\mathcal{M}) = \frac{p(g|f,\theta;\mathcal{M})p(f|\theta;\mathcal{M})p(\theta|\mathcal{M})}{p(g|\mathcal{M})}$$

- Joint MAP:

$$(\widehat{f},\widehat{\theta}) = \arg\max_{(f,\theta)} \{p(f,\theta|g;\mathcal{M})\}$$

- Marginalization:

$$\begin{cases} p(f|g;\mathcal{M}) = \int p(f,\theta|g;\mathcal{M})\,\mathrm{d}\theta \\ p(\theta|g;\mathcal{M}) = \int p(f,\theta|g;\mathcal{M})\,\mathrm{d}f \end{cases}$$

- Posterior means:

$$\begin{cases} \widehat{f} = \int f\, p(f,\theta|g;\mathcal{M})\,\mathrm{d}f\,\mathrm{d}\theta \\ \widehat{\theta} = \int \theta\, p(f,\theta|g;\mathcal{M})\,\mathrm{d}f\,\mathrm{d}\theta \end{cases}$$

- Evidence of the model:

$$p(g|\mathcal{M}) = \iint p(g|f,\theta;\mathcal{M})\,p(f|\theta;\mathcal{M})\,p(\theta|\mathcal{M})\ \mathrm{d}f\ \mathrm{d}\theta$$

Bayesian estimation with Gauss-Markov-Potts prior

Unsupervised:

$$p(f,z,\theta|g) \propto p(g|f,v_\epsilon)\ p(f|z,m,v)\ p(z|\gamma,\alpha)\ p(\theta)$$

$$p(g|f,v_\epsilon) = \mathcal{N}(Hf, v_\epsilon)$$

$$\theta = \{v_\epsilon, (\alpha_k, m_k, v_k), k = 1,\cdot,K\}$$

$p(\theta)$ Conjugate priors

Bayesian Computation

- Direct computation and use of $p(f,z,\theta|g;\mathcal{M})$ is too complex
- Possible approximations :
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)
- Main idea in Variational Bayesian methods: Approximate

$$p(f,z,\theta|g;\mathcal{M})$$

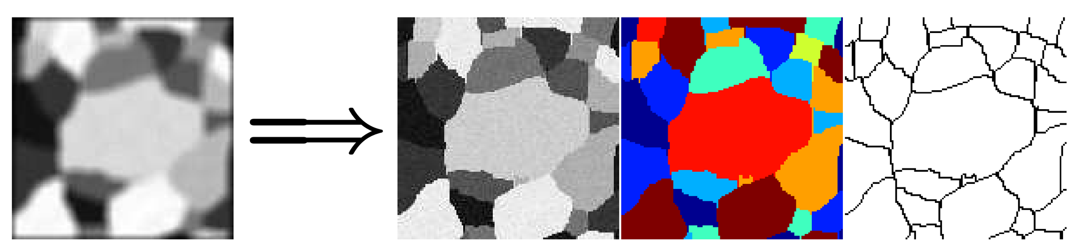
by

$$q(f,z,\theta) = q_1(f|z)\ q_2(z)\ q_3(\theta)$$

  - Choice of approximation criterion : $KL(q:p)$

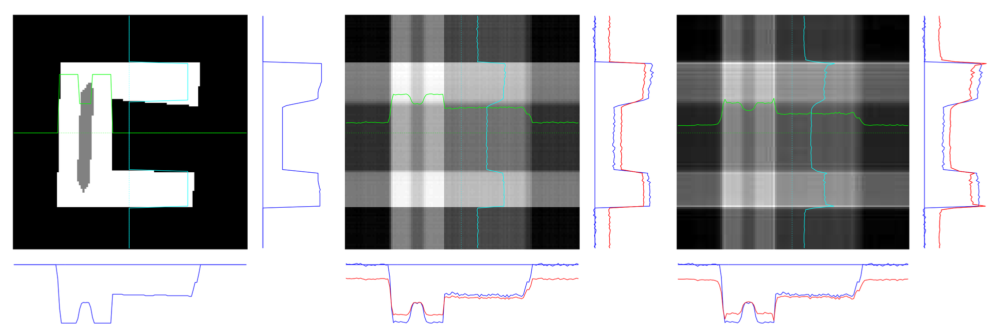
Examples of applications

- **Joint Restoration-Segmentation**

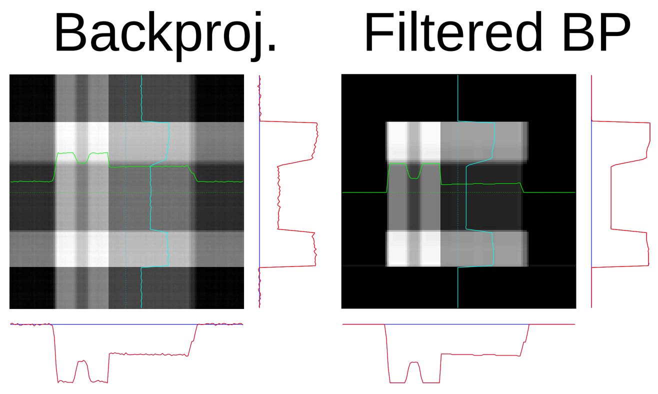


$g(r)$ $\widehat{f}(r)$ $\widehat{z}(r)$ $\widehat{q}(r)$

- **Joint Compted Tomography Reconstruction-Segmentation**

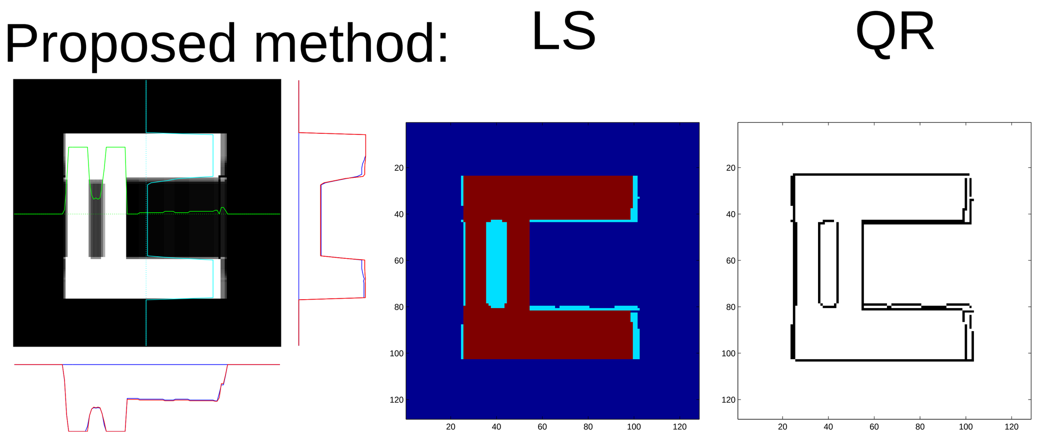


Original Backproj. Filtered BP



LS QR

Proposed method:



$\widehat{f}(r)$ $\widehat{z}(r)$ $\widehat{q}(r)$

References

A. Mohammad-Djafari, Gauss-Markov-Potts Priors for Images in Computer Tomography Resulting to Joint Optimal Reconstruction and segmentation, *International Journal of Tomography & Statistics*, Vol. 11:W09, pp. 76-92, 2008.

A. Mohammad-Djafari, Super-Resolution: A short review, a new method based on hidden Markov modeling of HR image and future challenges, *The Computer Journal*, doi:10,1093/comjnl/bxn005: (2008).