A Bayesian approach to change point analysis of discrete time series

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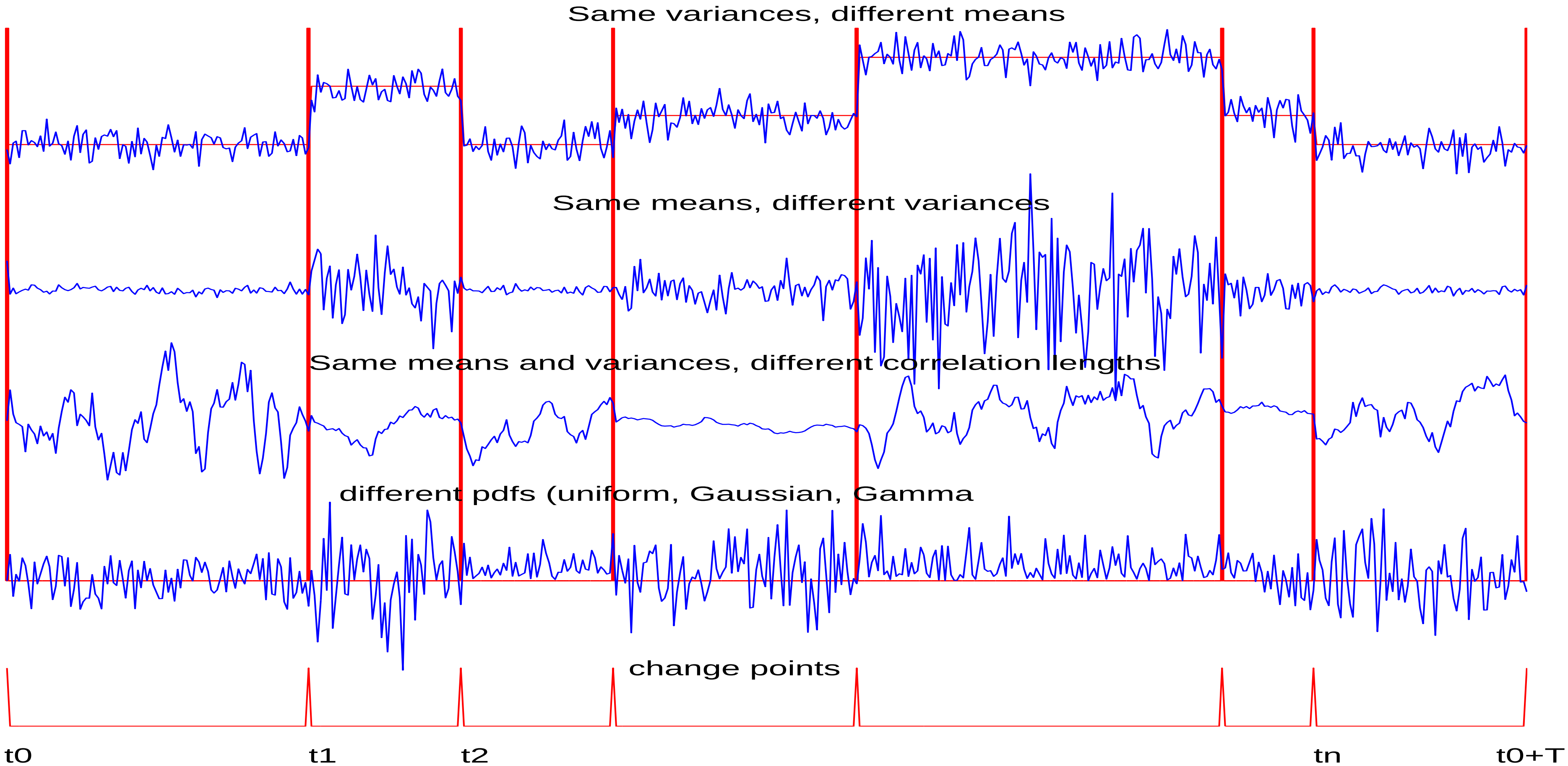
1 Introduction



2 Notations and Hypothesis

$\boldsymbol{x} = [x(t_0), \cdots, x(t_0+T)]'$ observed samples

$\boldsymbol{t} = [t_1, \cdots, t_N]'$ change points instants

$\boldsymbol{x} = [\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_N]'$ $\boldsymbol{x}_n = [x(t_n), x(t_n+1), \cdots, x(t_{n+1})]', \quad n = 0, \cdots, N$

$$\begin{aligned} p(x(t_n)) &= \mathcal{N}(\mu_n, \sigma_n^2) \\ p(x(t_n+l)|x(t_n+l-1)) &= \mathcal{N}(\rho_n\, x(t_n+l-1) + (1-\rho_n)\mu_n, \sigma_n^2(1-\rho_n^2)), \\ \text{with} \quad & l_n = t_{n+1} - t_n + 1 = \dim[\boldsymbol{x}_n], \quad l = 1, \cdots, l_n - 1 \\ p(\boldsymbol{x}_n) &= p(x(t_n)) \prod_{l=1}^{l_n} p(x(t_n+l)|x(t_n+l-1)) \\ &= \mathcal{N}(\mu_n \mathbf{1}, \Sigma_n) \;\; \text{with} \;\; \boldsymbol{\Sigma}_n = \sigma_n^2 \, \text{Toeplitz}([1, \rho_n, \rho_n^2, \cdots, \rho_n^{l_n}]) \end{aligned}$$

$$p(\boldsymbol{x}|\boldsymbol{t}, \boldsymbol{\theta}, N) = \prod_{n=0}^{N} \mathcal{N}(\mu_n \mathbf{1}, \boldsymbol{\Sigma}_n) \propto \exp\left[-\frac{1}{2} \sum_{n=0}^{N} (\boldsymbol{x}_n - \mu_n \mathbf{1})' \boldsymbol{\Sigma}_n^{-1} (\boldsymbol{x}_n - \mu_n \mathbf{1})\right]$$

where we noted $\boldsymbol{\theta} = \{\mu_n, \sigma_n, \rho_n, \; n = 0, \cdots, N\}$.

3 Bayesian Approach

Likelihood: $p(\boldsymbol{x}|\boldsymbol{t},\boldsymbol{\theta},N)$ + Priors: $p(\boldsymbol{t}|\lambda,N)$, $\quad p(\boldsymbol{\theta}|N) \longrightarrow$ Posterior: $p(\boldsymbol{t},\boldsymbol{\theta}|\boldsymbol{x},\lambda,N)$

Priors:

$$t_n = t_{n-1} + \epsilon_n \quad \text{with} \quad \epsilon_n \sim \mathcal{P}(\lambda),$$

$$p(\boldsymbol{t}|\lambda,N) = \prod_{n=1}^{N+1} \mathcal{P}(t_n - t_{n-1}|\lambda) = \prod_{n=1}^{N+1} e^{-\lambda} \frac{\lambda^{(t_n - t_{n-1})}}{(t_n - t_{n-1})!}$$

$$p(\boldsymbol{\theta}|N) = \prod_n p(\theta_n) \quad \left\{ \begin{array}{l} p(\mu_n) = \mathcal{N}(\mu_0, \sigma_0^2) \\ p(\sigma_n^2) = \mathcal{IG}(\alpha_0, \beta_0) \\ p(\rho_n) = \mathcal{U}([0,1]) \end{array} \right. \quad \text{Conjugate priors}$$

Posterior:

$$p(\boldsymbol{t},\boldsymbol{\theta}|\boldsymbol{x},\lambda,N) \propto p(\boldsymbol{x}|\boldsymbol{t},\boldsymbol{\theta},N)\, p(\boldsymbol{t}|\lambda,N)\, p(\boldsymbol{\theta}|N)$$

Inference: Estimate $\boldsymbol{t}$ and $\boldsymbol{\theta}$ using $p(\boldsymbol{t},\boldsymbol{\theta}|\boldsymbol{x},\lambda,N)$

MCMC Gibbs Sampling

Generate samples from posterior and average to estimate the mean values

Iterate until convergency

. sample $\boldsymbol{t}$ using $p(\boldsymbol{t}|\boldsymbol{x},\boldsymbol{\theta},N) \propto p(\boldsymbol{x}|\boldsymbol{t},\boldsymbol{\theta},N)\, p(\boldsymbol{t}|\lambda,N)$

. sample $\boldsymbol{\theta} = \{\mu_n, \sigma_n, \rho_n\}$ using $p(\boldsymbol{\theta}|\boldsymbol{x},\boldsymbol{t},N) \propto p(\boldsymbol{x}|\boldsymbol{t},\boldsymbol{\theta},N)\, p(\boldsymbol{\theta}|N)$

μ_n using $p(\mu_n|\boldsymbol{x},\boldsymbol{t},N)$

σ_n^2 using $p(\sigma_n^2|\boldsymbol{x},\boldsymbol{t},N)$

ρ_n using $p(\rho_n|\boldsymbol{x},\boldsymbol{t},N)$

$$p(\mu_n|\boldsymbol{x},\boldsymbol{t}) = \mathcal{N}(\widehat{\mu}_n,\widehat{\sigma}_n^2) \text{ with } \left\{ \begin{array}{l} \widehat{\mu}_n = \widehat{\sigma}_n^2 \left[\frac{\mu_0}{\sigma_0^2} + \mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{x}_n\right] \\ \widehat{\sigma}_n^2 = \left(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1} + \frac{1}{\sigma_0^2}\right)^{-1} \end{array} \right.$$

$$p(\sigma_n^2|\boldsymbol{x},\boldsymbol{t}) = \mathcal{IG}(\widehat{\alpha}_n,\widehat{\beta}_n) \text{ with } \left\{ \begin{array}{l} \widehat{\alpha}_n = \alpha_0 + \frac{l_n}{2} \\ \widehat{\beta}_n = \beta_0 + \frac{1}{2}(\boldsymbol{x}_n - \mu_n\mathbf{1})\boldsymbol{\Sigma}_n^{-1}(\boldsymbol{x}_n - \mu_n\mathbf{1}), \end{array} \right.$$

- $p(\rho_n|\boldsymbol{x},\boldsymbol{t})$ is not a classical law.

Its expression is given by :

$$p(\rho_n|\boldsymbol{x},\boldsymbol{t},N) = \prod_{n=0}^{N} p(\rho_n|\boldsymbol{x}_n,\boldsymbol{t},N)$$
$$\propto \left(\frac{1}{\sigma_n^2(1-\rho_n^2)}\right)^{\frac{ln}{2}} \exp\left\{-\frac{1}{2\sigma_n^2(1-\rho_n^2)}(\boldsymbol{x}_n-\mu_n\mathbf{1})'\boldsymbol{\Sigma}_n^{-1}(\boldsymbol{x}_n-\mu_n\mathbf{1})\right\}$$
$$\propto \left(\frac{1}{\sigma_n^2(1-\rho_n^2)}\right)^{\frac{ln}{2}} \exp\left\{-\frac{1}{2\sigma_n^2(1-\rho_n^2)}\sum_{l=1}^{ln}(x(t_n+l)-\rho_n x(t_n+l-1)-(1-\rho_n)\mu_n)^2\right\}$$

- Hasting-Metropolis MCMC

- Instrumental density: A Gaussian obtained by Laplace approximation

$$m_{\rho_n} \longrightarrow \int_0^1 \rho_n \quad p(\rho_n|\boldsymbol{x},\boldsymbol{t},N)$$
$$\sigma_{\rho_n}^2 \longrightarrow \int_0^1 \rho_n^2 \quad p(\rho_n|\boldsymbol{x},\boldsymbol{t},N) - m_{\rho_n}^2$$

Sampling of $p(\boldsymbol{t}|\boldsymbol{x}, \boldsymbol{\theta})$

- $x_{t:s} = [x(t), x(t+1), \ldots, x(s)]$

$$\begin{aligned} R(t,s) &= p(x_{t:s}|t, s \text{ in the same segment}) \\ Q(t) &= p(x_{t:s}| \text{ changepoint at } t-1), \quad Q(1) = p(\boldsymbol{x}) \end{aligned}$$

- $g(t_j - t_{j-1})$ the a priori density of the interval between two changepoints, and $G(t)$ its associated distribution function.

- Then:

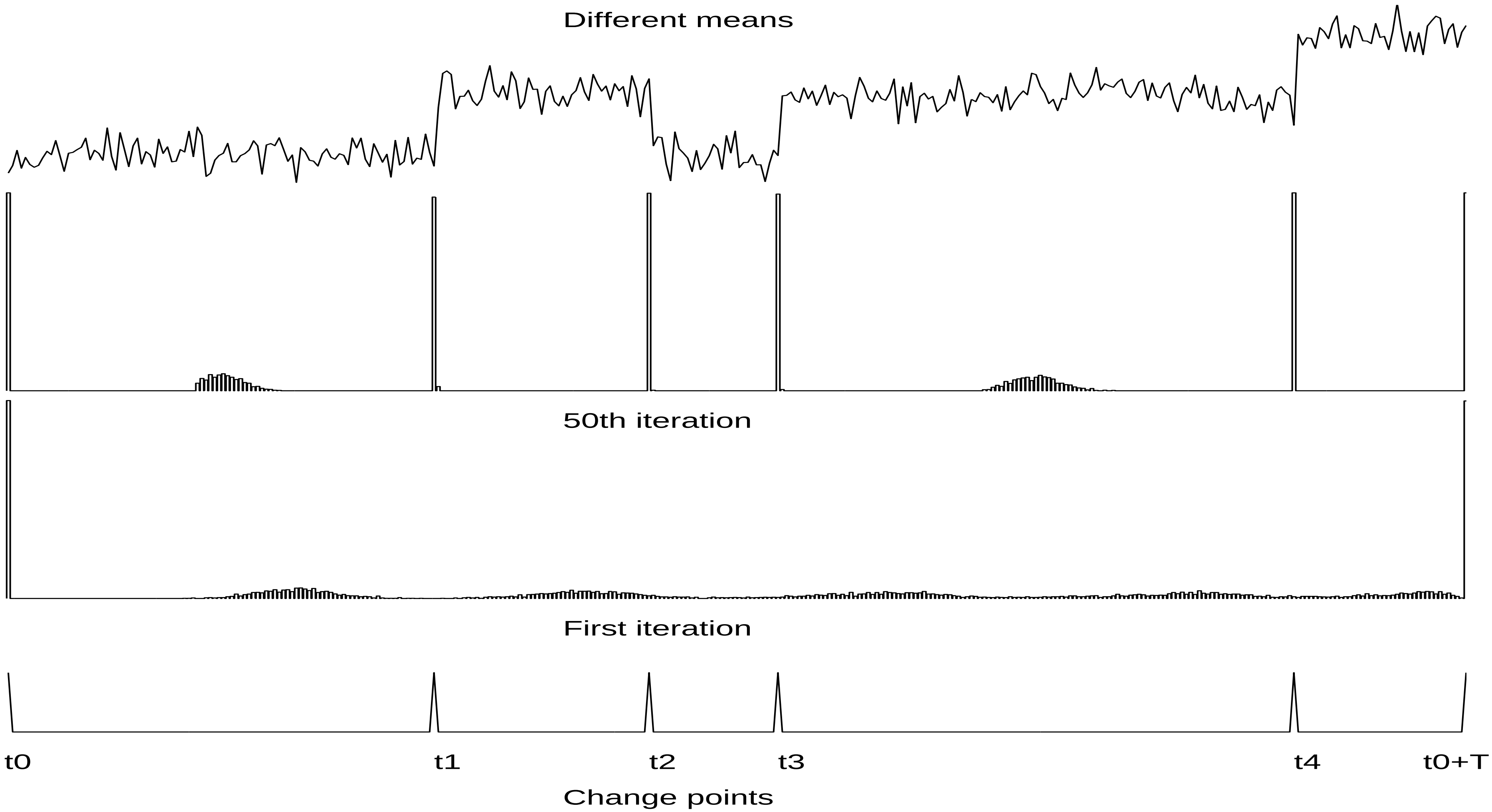
$$p(t_j|t_{j-1}, \boldsymbol{x}) = \frac{R(t_{j-1}, t_j) Q(t_j+1) g(t_j - t_{j-1})}{Q(t_{j-1})}$$

and

$$p(t_j = T|t_{j-1}, \boldsymbol{x}) = \frac{P(t_{j-1}, T)(1 - G(T - t_{j-1} - 1))}{Q(t_{j-1})}$$

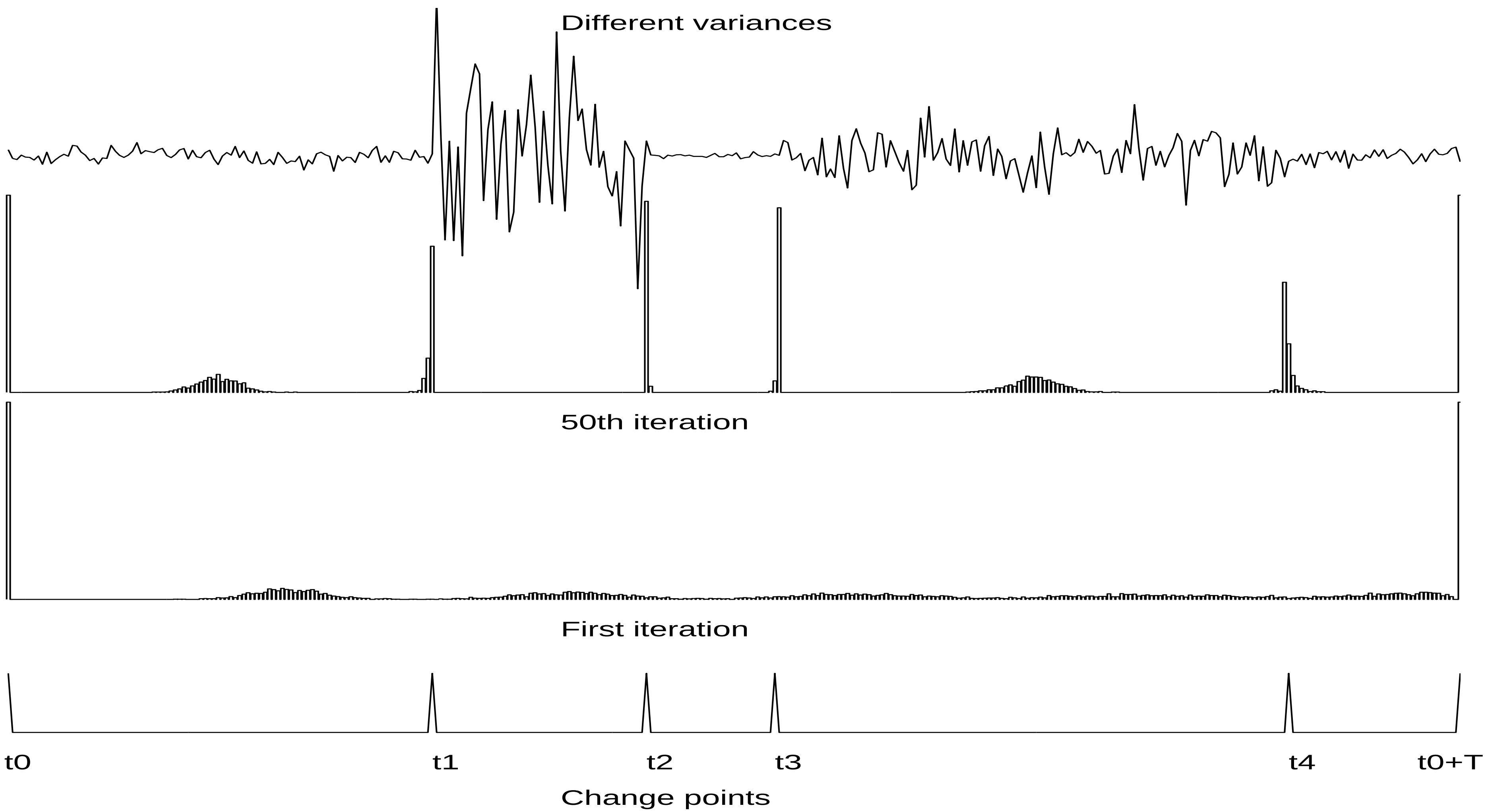
4 Simulation results

Change of the means



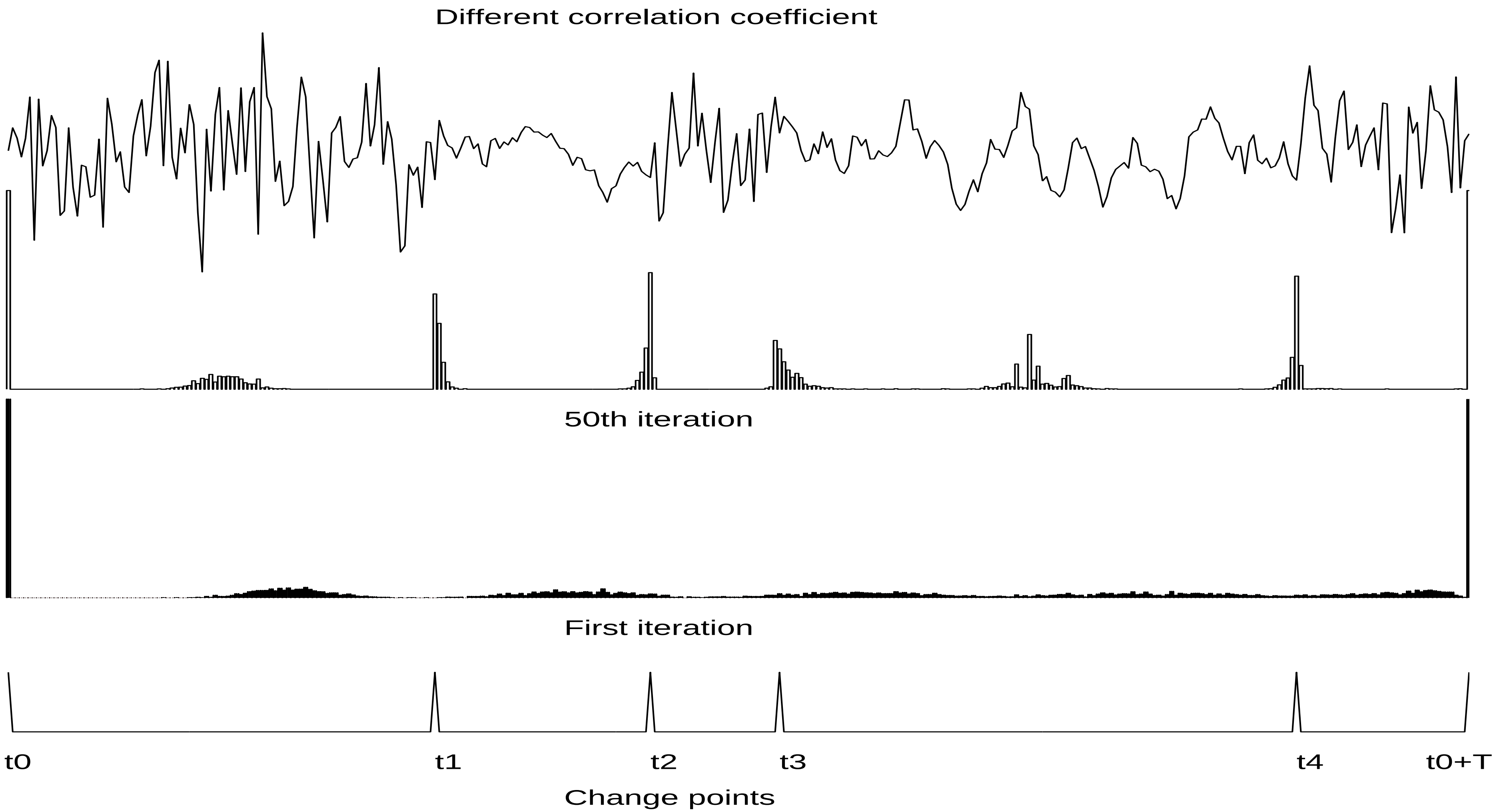
m	$\hat{m}\|\boldsymbol{x},\boldsymbol{t}$	$\hat{\sigma}^2\|\boldsymbol{x},\boldsymbol{t}$	$\hat{m}\|\boldsymbol{x}$	$\hat{\sigma}^2\|\boldsymbol{x}$
1.5	1.4966	0.0015	1.4969	0.0013
1.7	1.7084	0.0017	1.7013	0.0038
1.5	1.4912	0.0020	1.5015	0.0045
1.7	1.6940	0.0014	1.6929	0.0016
1.9	1.9012	0.0015	1.8915	0.0039

Change in the variances



σ^2	$\hat{\sigma}^2 \mid \boldsymbol{x}, \boldsymbol{t}$	$\hat{\sigma}^2 \mid \boldsymbol{x}$
0.01	0.0083	0.0081
1	0.9918	0.9598
0.001	0.0007	0.0026
0.1	0.0945	0.0940
0.01	0.0079	0.0107

Change in the correlation coefficient



a	$\hat{a}\|\boldsymbol{x}$
0	0.0988
0.9	0.7875
0.1	0.3737
0.8	0.8071
0.2	0.1710

5 Conclusions and Perspectives

- A Bayesian approach for estimating change points in time series is presented
- Detection of change points due to changes in the mean is easier than those due to changes in variances or changes in correlation coefficient.
- In this work, first we assumed to know the number N of change points.
- We also studied the role of the a priori parameter λ on the results.
- We are investigating the estimation of the number of change points in the same framework.
- Other modeling using other hidden variables than change point time instants are possible and are under investigation.
- We are also investigating the extension of this work to image processing (2D signals) where the change points are contours.