Final Exam of the Cours EE 671 : "Detection-Estimation"
Professor: Ali Mohammad-Djafari
Exam date: 05/04/1998 10:30 -12:30 Exam location: DBRT 301

Exercise 1: Recursive parameter estimation

Assume $Z_i = X_i + N_i, i = 1, \ldots, n$ where $N_i, i = 1, \ldots, n$ is a sequence of i.i.d. Gaussian random variables, each with zero mean and variance σ^2, and $X_i, i = 1, \ldots, n$ are defined by

$$X_0 = \Theta, \quad X_i = \alpha X_{i-1}, \quad i = 1, \ldots, n$$

where α is known and Θ is a Gaussian random parameter with zero mean and variance q^2.

1. Assuming that Θ and $\underline{N}$ are independent, find the MMSE estimation of Θ based on $Z_1, \ldots, Z_n$.
2. For each $n = 1, 2, \ldots$, let $\widehat{\theta}_n$ denote the MMSE estimate of Θ based on $Z_1, \ldots, Z_n$. Show that $\widehat{\theta}_n$ can be computed recursively by

   $$\widehat{\theta}_n = K_n^{-1} \left[K_{n-1} \widehat{\theta}_{n-1} + \alpha^n \, Z_n \right], \quad n = 1, 2, \ldots$$

   where $\widehat{\theta}_0 = 0$ and

   $$K_0 = \frac{\sigma^2}{q^2} \quad \text{and} \quad K_n = K_{n-1} + \alpha^{2n}$$

3. Draw a block diagram of this implementation.
4. Find an expression for the MSE $e_n = \mathrm{E}\left\{ (\widehat{\theta}_n - \Theta)^2 \right\}$.
5. For cases $\alpha < 1$; $\alpha = 1$ and $\alpha > 1$, what happens when $n \mapsto \infty$; $q^2 \mapsto \infty$; and $\sigma^2 \mapsto 0$?

Exercise 2: Parameter estimation

Assume $Z_i = A \sin(i\,\pi/2 + \Phi) + N_i, i = 1, \ldots, n$ where $N_i, i = 1, \ldots, n$ is a sequence of i.i.d. Gaussian random variables, each with zero mean and variance σ^2, and n is even.

1. Suppose A and Φ are non random with $A > 0$ and $\Phi \in [-\pi, \pi]$. Find their ML estimates.
2. Suppose A and Φ are random and independent with priors

   $$\pi(\phi) = \begin{cases} 1/\pi, & -\pi \le \phi \le \pi \\ 0, & \text{otherwise} \end{cases}$$

   $$\pi(a) = \begin{cases} \frac{a}{\beta^2} \exp\left[-\frac{a^2}{2\beta^2} \right], & a \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

   where β is known. Assume also that A and Φ are independent of $\underline{N}$.

   2.1 Give the expression of the posterior law $\pi(A, \Phi | Z_1, \ldots, Z_n)$;

   2.2 Find the joint MAP estimates of A and Φ;

   2.3 Give the expression of the posterior laws $\pi(A | Z_1, \ldots, Z_n)$ and $\pi(\Phi | Z_1, \ldots, Z_n)$, and find the MAP estimates of A and Φ

3. Under what conditions are the estimates from (1) and (2) approximately equal?

Exercise 3: Discrete periodic deconvolution

Assume $Z_i = S_i + N_i$ where

$$S_i = \sum_{k=0}^{n-1} h_k \, X_{i-k}$$

and where $\underline{h} = [h_0, h_1, \ldots, h_{n-1}]^t$ represents the finite impulse response of a channel. Assume also that X_i and S_i are periodic sequences with known period n, *i.e.* $X_{n\pm k} = X_i$ and $S_{n\pm k} = S_i$ for any integer k and that we observe n samples $\underline{Z} = [Z_0, \ldots, Z_{n-1}]^t$. We want estimate the input sequence $\underline{X} = [X_0, \ldots, X_{n-1}]^t$. Finally, we assume that $\underline{N} \sim \mathcal{N}\left(\underline{0}, \theta \boldsymbol{I}\right)$ and independent of $\underline{X}$.

1. Construct the matrices $\boldsymbol{H}$ and $\boldsymbol{X}$ in such a way that $\underline{Z} = \boldsymbol{H}\underline{X} + \underline{N}$ and $\underline{Z} = \boldsymbol{X}\underline{h} + \underline{N}$.

2. Assume that $\underline{h}$ and $\underline{X}$ are known. Design a Bayesian optimal detector with the uniform prior and the uniform cost coefficients.

3. Assume now that $\underline{h}$ is known and we want to estimate $\underline{X}$. Find the ML estimate $\widehat{\underline{X}}_{ML}(\underline{Z})$ of $\underline{X}$.

4. Find the MAP estimate $\widehat{\underline{X}}_{MAP}(\underline{Z})$ and the MMSE estimate $\widehat{\underline{X}}_{MMSE}(\underline{Z})$ if we assume that $\underline{X} \sim \mathcal{N}\left(\underline{0}, \sigma_x^2 \boldsymbol{I}\right)$.

5. Assume now that the input sequence $\underline{X}$ is known but $\underline{h}$ is unknown. Find the ML estimate $\widehat{\underline{h}}_{ML}(\underline{Z})$ of $\underline{h}$.

6. Find the MAP estimate $\widehat{\underline{h}}_{MAP}(\underline{Z})$ and the MMSE estimate $\widehat{\underline{h}}_{MMSE}(\underline{Z})$ if we assume that $\underline{h} \sim \mathcal{N}\left(\underline{0}, \sigma_h^2 \boldsymbol{I}\right)$.

7. Noting that $\boldsymbol{H}$ and $\boldsymbol{X}$ in (1) are circulant matrices and that we have the following relations:

$$\begin{array}{lll} \boldsymbol{H} = \boldsymbol{F}^t \boldsymbol{\Lambda}_h \boldsymbol{F}, & \boldsymbol{H}^t \boldsymbol{H} = \boldsymbol{F}^t \boldsymbol{\Lambda}_h^2 \boldsymbol{F}, & \boldsymbol{\Lambda}_h = \mathrm{diag}\left\{\mathrm{DFT}[h_0, \cdots, h_{n-1}]\right\} = \mathrm{diag}\left\{\underline{\tilde{h}}(\omega)\right\} \\ \boldsymbol{X} = \boldsymbol{F}^t \boldsymbol{\Lambda}_x \boldsymbol{F}, & \boldsymbol{X}^t \boldsymbol{X} = \boldsymbol{F}^t \boldsymbol{\Lambda}_x^2 \boldsymbol{F}, & \boldsymbol{\Lambda}_x = \mathrm{diag}\left\{\mathrm{DFT}[X_0, \cdots, X_{n-1}]\right\} = \mathrm{diag}\left\{\underline{\tilde{X}}(\omega)\right\} \\ \boldsymbol{F}^t \boldsymbol{F} = \boldsymbol{F}\boldsymbol{F}^t = \boldsymbol{I}, & \underline{\tilde{h}}(\omega) = \boldsymbol{F}\underline{h}, & \underline{\tilde{X}}(\omega) = \boldsymbol{F}\underline{X}, \quad \underline{h} = \boldsymbol{F}^t \underline{\tilde{h}}(\omega), \quad \underline{X} = \boldsymbol{F}^t \underline{\tilde{X}}(\omega) \end{array}$$

   find the expressions of $\underline{\tilde{X}}(\omega)_{ML}$ and $\underline{\tilde{h}}(\omega)_{ML}$ in (3) and (5) and $\underline{\tilde{X}}(\omega)_{MAP}$ and $\underline{\tilde{h}}(\omega)_{MAP}$ in (4) and (6).

8. Now assume that $\underline{h}$ and $\underline{X}$ are both unknown. Find the MAP and the MMSE estimates $\widehat{\underline{X}}_{MAP}(\underline{Z})$ and $\widehat{\underline{X}}_{MMSE}(\underline{Z})$ of $\underline{X}$ and $\widehat{\underline{h}}_{MAP}(\underline{Z})$ and $\widehat{\underline{h}}_{MMSE}(\underline{Z})$ of $\underline{h}$ if we assume that $\underline{X} \sim \mathcal{N}\left(\underline{0}, \sigma_x^2 \boldsymbol{I}\right)$ and $\underline{h} \sim \mathcal{N}\left(\underline{0}, \sigma_h^2 \boldsymbol{I}\right)$.

9. Assume now that X_i can only take the values $\{0, 1\}$. with $P(X_i = 0) = \pi_0$ and $P(X_i = 1) = 1 - \pi_0$, with known π_0, and that X_i are independent. Design an optimal detector for X_i assuming $\boldsymbol{h}$ to be known.

10. Assume now that $X_i, i = 1, \ldots, n$ can be modelled as a first order Markov chain with transition probabilites

$$\begin{array}{ll} P(X_i = 0, X_{i-1} = 0) = s & P(X_i = 0, X_{i-1} = 1) = 1 - s \\ P(X_i = 1, X_{i-1} = 1) = t & P(X_i = 1, X_{i-1} = 0) = 1 - t \end{array}$$

    Design an optimal detector for X_i.

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Exercise 1: Recursive parameter estimation

1. Replacing X_i in $Z_i = X_i + N_i$ we have

$$Z_i = \alpha^i \Theta + N_i \longrightarrow Z_i|\Theta = \theta \sim \mathcal{N}\left(\alpha^i\theta, \sigma^2\right)$$

$$f(\underline{Z}|\theta) = \prod_{i=1}^{n} (2\pi\sigma^2)^{1/2} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Z_i - \alpha^i\theta)^2\right]$$

$$\begin{aligned}
\pi(\theta|\underline{Z}) &\propto \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Z_i - \alpha^i\theta)^2 - \frac{1}{2q^2}\theta^2\right] \\
&\propto \exp\left[-\frac{1}{2\sigma^2}\theta^2\left[\sum_{i=1}^{n}\alpha^{2i} + \frac{\sigma^2}{q^2}\right] - \frac{1}{2\sigma^2}\sum_{i=1}^{n} Z_i\,\alpha^i\theta\right] \\
&\propto \exp\left[-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}\alpha^{2i} + \frac{\sigma^2}{q^2}\right]\left(\theta - \widehat{\theta}_{MAP}\right)^2\right] \\
&\propto \exp\left[-\frac{1}{2\,e_n}\left(\theta - \widehat{\theta}_{MAP}\right)^2\right]
\end{aligned}$$

$$\widehat{\theta}_{MAP} = \widehat{\theta}_{MMSE} = \frac{q^2\sum_{i=1}^{n}\alpha^i\,Z_i}{\sigma^2 + q^2\sum_{i=1}^{n}\alpha^{2i}}$$

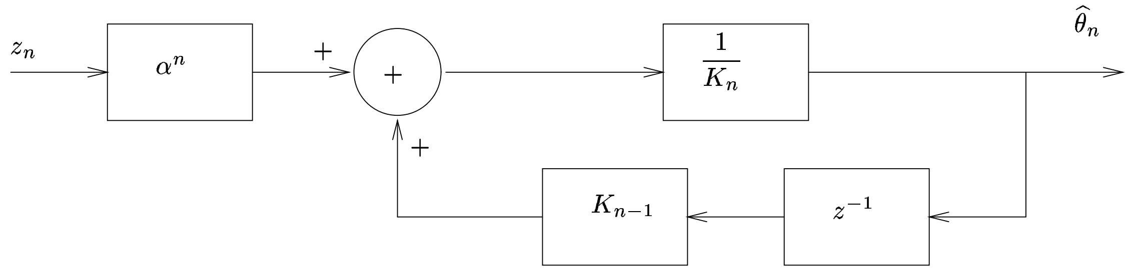
2. Noting

$$\widehat{\theta}_n = \frac{q^2\sum_{i=1}^{n}\alpha^i\,Z_i}{\sigma^2 + q^2\sum_{i=1}^{n}\alpha^{2i}} = \frac{\sum_{i=1}^{n}\alpha^i\,Z_i}{\frac{\sigma^2}{q^2} + \sum_{i=1}^{n}\alpha^{2i}}$$

it is easily seen that $\widehat{\theta}_0 = 0$, $K_n = \frac{\sigma^2}{q^2} + \sum_{i=1}^{n}\alpha^{2i}$, $K_0 = \frac{\sigma^2}{q^2}$, $K_n = K_{n-1} + \alpha^{2n}$ and

$$\widehat{\theta}_n = K_n^{-1}\left[K_{n-1}\widehat{\theta}_{n-1} + \alpha^n Z_n\right]$$

3. Block diagram



4. MSE is the variance of the posterior law $\pi(\theta|\underline{Z})$

$$e_n = \frac{\sigma^2}{\sum_{i=1}^n \alpha^{2i} + \frac{\sigma^2}{q^2}} = \sigma^2 K_n^{-1}$$

$$\begin{aligned}
K_n &= \frac{\sigma^2}{q^2} + \sum_{i=1}^n \alpha^{2i} \\
e_n &= q^2\sigma^2 \left(\sigma^2 + q^2 \sum_{i=1}^n \alpha^{2i}\right)^{-1} \\
\widehat{\theta}_n &= \frac{q^2 \sum_{i=1}^n \alpha^i Z_i}{\sigma^2 + q^2 \sum_{i=1}^n \alpha^{2i}}
\end{aligned}$$

5. Limit cases :

$$q^2 \mapsto \infty \longrightarrow \left\{ \begin{array}{lll} K_n^{-1} & \mapsto & 0 \\ e_n & \mapsto & 0 \\ \widehat{\theta}_n & \mapsto & 0 \end{array} \right. , \quad \sigma^2 \mapsto 0 \longrightarrow \left\{ \begin{array}{lll} K_n^{-1} & \mapsto & \left(\sum_{i=1}^n \alpha^{2i}\right)^{-1} \\ e_n & \mapsto & 0 \\ \widehat{\theta}_n & \mapsto & \frac{\sum_{i=1}^n \alpha^i Z_i}{q^2 \sum_{i=1}^n \alpha^{2i}} \end{array} \right.$$

$$K_\infty = \left\{ \begin{array}{llll} \sigma^2 + \frac{q^2\alpha^2}{1-\alpha^2} & \text{if} & \alpha & < 1 \\ \infty & \text{if} & \alpha & \geq 1 \end{array} \right.$$

$$e_\infty = \left\{ \begin{array}{llll} \frac{q^2\sigma^2}{\sigma^2 + \frac{q^2\alpha^2}{1-\alpha^2}} & \text{if} & \alpha & < 1 \\ 0 & \text{if} & \alpha & \geq 1 \end{array} \right.$$

$$\widehat{\theta}_\infty = \left\{ \begin{array}{llll} \frac{q^2 \sum_{i=1}^n \alpha^i Z_i}{\sigma^2 + \frac{q^2\alpha^2}{1-\alpha^2}} & \text{if} & \alpha & < 1 \\ 0 & \text{if} & \alpha & \geq 1 \end{array} \right.$$

Exercise 2: Parameter estimation

1. Joint ML estimate

$$Z_i|A,\Phi \sim \mathcal{N}\left(A\sin(i\pi/2+\Phi),\sigma^2\right)$$

$$L(A,\Phi) = f(\underline{Z}|A,\Phi) \propto \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}[Z_i - A\sin(i\pi/2+\Phi)]^2\right]$$

$$L(A,\Phi) = -\ln f(\underline{Z}|A,\Phi) = cte + \sum_{i=1}^{n}[Z_i - A\sin(i\pi/2+\Phi)]^2$$

$$(\widehat{A},\widehat{\Phi})_{ML} = \arg\min_{(A,\Phi)}\{L(A,\Phi)\}$$

$$\begin{aligned}
\frac{\partial L(A,\Phi)}{\partial A} &= 0 = -2\sum_{i=1}^{n}\sin(i\pi/2+\Phi)[Z_i - A\sin(i\pi/2+\Phi)] \\
\frac{\partial L(A,\Phi)}{\partial \Phi} &= 0 = -2A\sum_{i=1}^{n}\cos(i\pi/2+\Phi)[Z_i - A\sin(i\pi/2+\Phi)]
\end{aligned}$$

Knowing that n is even and expanding the sums

$$\sum_{i=1}^{n} X_n = \sum_{k=1}^{n/2} X_{2k-1} + \sum_{k=1}^{n/2} X_{2k}$$

and using

$$\begin{aligned}
\sum_{i=1}^{n}\sin^2(i\pi/2+\Phi) &= \sum_{i=1}^{n}\left[\frac{1}{2}-\frac{1}{2}\sin^2(i\pi+2\Phi)\right] = n/2, \\
\sum_{i=1}^{n}\sin(i\pi/2+\Phi)\cos(i\pi/2+\Phi) &= \frac{1}{2}\sum_{i=1}^{n}\sin(i\pi+\Phi) = 0
\end{aligned}$$

these two equations become

$$\begin{aligned}
A &= \frac{2}{n}\sum_{i=1}^{n} Z_i\,\sin(i\pi/2+\Phi) \\
\tan(\Phi) &= \sum_{k=1}^{n/2}(-1)^k Z_{2k} \Big/ \sum_{k=1}^{n/2}(-1)^k Z_{2k-1}
\end{aligned}$$

Another interesting way to do is:

$$\underline{X} = \begin{pmatrix} A\cos\phi \\ A\sin\phi \end{pmatrix}, \quad \underline{Z} = [Z_1,\ldots,Z_n]^t, \quad \underline{N} = [N_1,\ldots,N_n]^t,$$

$$\underline{Z} = \boldsymbol{H}\underline{X} + \underline{N}, \quad \text{where} \quad \boldsymbol{H} = \begin{pmatrix} \sin(\pi/2) & \cos(\pi/2) \\ \sin(\pi) & \cos(\pi) \\ \vdots & \vdots \\ \vdots & \vdots \\ \sin(n\pi/2) & \cos(n\pi/2) \end{pmatrix}$$

ML estimate:

$$\underline{\widehat{X}} = (\boldsymbol{H}^t\boldsymbol{H})^{-1}\boldsymbol{H}^t\underline{Z} \quad \text{where} \quad \boldsymbol{H}^t\boldsymbol{H} = \begin{pmatrix} n/2 & 0 \\ 0 & n/2 \end{pmatrix} \longrightarrow \left\{ \begin{array}{ll} \hat{A}\cos\phi & = \frac{2}{n}\sum_{i=1}^{n} Z_i \, \sin(i\pi/2) \\ \hat{A}\sin\phi & = \frac{2}{n}\sum_{i=1}^{n} Z_i \, \cos(i\pi/2) \end{array} \right.$$

$$\begin{aligned} \hat{A}^2 &= \frac{2}{n}\sum_{i=1}^{n} Z_i\,[\sin(i\pi/2) + \cos(i\pi/2)] \\ \tan(\hat{\Phi}) &= \sum_{i=1}^{n} Z_i \, \sin(i\pi/2) / \sum_{i=1}^{n} Z_i \, \cos(i\pi/2) \end{aligned}$$

Note that, if we were asked to obtain the marginal MAP estimates of A or Φ, we had to calculate the likelihoods

$$\begin{aligned} L(A) &= -\ln f(\underline{Z}|A) = -\ln \int_{-\pi}^{\pi} f(\underline{Z}|A,\Phi)\,\mathrm{d}\Phi \\ L(\Phi) &= -\ln f(\underline{Z}|\Phi) = -\ln \int_{0}^{\infty} f(\underline{Z}|A,\Phi)\,\mathrm{d}A \end{aligned}$$

which are more difficult to do.

2. MAP estimate

$$\pi(A,\Phi|\underline{Z}) \propto \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}[Z_i - A\sin(i\pi/2+\Phi)]^2 - \frac{1}{2\beta^2}(A^2 - 2\ln A)\right], \quad A > 0, -\pi < \Phi < \pi$$

To obtain the expressions of $\pi(A|\underline{Z})$ and $\pi(\Phi|\underline{Z})$ we need the to calculate the following integrals

$$\begin{aligned} \pi(A|\underline{Z}) &\propto \int_{-\pi}^{\pi} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}[Z_i - A\sin(i\pi/2+\Phi)]^2 - \frac{1}{2\beta^2}(A^2 - 2\ln A)\right] \mathrm{d}\Phi \\ \pi(\Phi|\underline{Z}) &\propto \int_{0}^{\infty} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}[Z_i - A\sin(i\pi/2+\Phi)]^2 - \frac{1}{2\beta^2}(A^2 - 2\ln A)\right] \mathrm{d}A \end{aligned}$$

Unfortunately, none of these integrals has analytical solution.

$$(\widehat{A},\widehat{\Phi})_{MAP} = \arg\min_{(A,\Phi)} \{J(A,\Phi)\}$$

with

$$J(A,\Phi) = L(A,\Phi) + \frac{\sigma^2}{2\beta^2}(A^2 - 2\ln A) + cte$$

3. We can see that if $A = 1$ and if $\beta^2 >> \sigma^2$, then the terme $\frac{\sigma^2}{\beta^2}A^2$ can be neglected against $L(A,\Phi)$ and the MAP and the ML estimates will approximately equal.

Exercise 3: Discrete periodic deconvolution

1. Using the periodicity of h_i and X_i we obtain

$$\boldsymbol{H} = \begin{pmatrix} h_0 & h_{n-1} & & & h_1 \\ h_1 & h_0 & h_{n-1} & & h_2 \\ & \ddots & \ddots & \ddots & \\ & & & & h_{n-1} \\ h_{n-1} & & & h_1 & h_0 \end{pmatrix}, \quad \boldsymbol{X} = \begin{pmatrix} X_0 & X_{n-1} & & & X_1 \\ X_1 & X_0 & X_{n-1} & & X_2 \\ & \ddots & \ddots & \ddots & \\ & & & & X_{n-1} \\ X_{n-1} & & & X_1 & X_0 \end{pmatrix}$$

These are circulant matrices.

2. Detection

$$\begin{cases} H_0 & : \quad Z_i = N_i \\ H_1 & : \quad Z_i = S_i + N_i \end{cases} \longrightarrow \ln L(\underline{Z}) = \sum_{0=1}^{n-1} (Z_i - S_i)^2 - Z_i^2$$

$$\sum_{0=1}^{n-1} (Z_i - S_i)^2 - Z_i^2 = -2 \sum_{0=1}^{n-1} Z_i S_i + \sum_{0=1}^{n-1} S_i^2 \longrightarrow \sum_{0=1}^{n-1} Z_i S_i \begin{matrix} > \\ = \\ < \end{matrix} \frac{1}{2} \sum_{0=1}^{n-1} S_i^2$$

$$\delta(\underline{Z}) = \begin{cases} 1 & & > & \\ 0/1 & \text{if} \quad \sum_{i=1}^{n} Z_i S_i & = & \frac{1}{2} \sum_{i=1}^{n} S_i^2 \\ 1 & & < & \end{cases}$$

3. ML estimate of $\underline{X}$

$$f(\underline{Z}|\underline{X}) = \mathcal{N}\left(\boldsymbol{H}\underline{X}, \sigma^2 \boldsymbol{I}\right) \longrightarrow -\ln L(\underline{X}) = cte + \|\underline{Z} - \boldsymbol{H}\underline{X}\|^2 \longrightarrow \widehat{\underline{X}}_{ML} = (\boldsymbol{H}^t \boldsymbol{H})^{-1} \boldsymbol{H}^t \underline{Z}$$

4. MAP estimate of $\underline{X}$

$$\pi(\underline{X}|\underline{Z}) \propto \exp\left[-\frac{1}{2\sigma^2} [\|\underline{Z} - \boldsymbol{H}\underline{X}\|^2 + \sigma_x^2 \|\underline{X}\|^2\right] \longrightarrow \widehat{\underline{X}}_{MAP} = \widehat{\underline{X}}_{MMSE} = (\boldsymbol{H}^t \boldsymbol{H} + \frac{\sigma^2}{\sigma_x^2} \boldsymbol{I})^{-1} \boldsymbol{H}^t \underline{Z}$$

5. ML estimate of $\underline{h}$

$$f(\underline{Z}|\underline{h}) = \mathcal{N}\left(\boldsymbol{X}\underline{h}, \sigma^2 \boldsymbol{I}\right) \longrightarrow -\ln L(\underline{h}) = cte + \|\underline{Z} - \boldsymbol{X}\underline{h}\|^2 \longrightarrow \widehat{\underline{h}}_{ML} = (\boldsymbol{X}^t \boldsymbol{X})^{-1} \boldsymbol{X}^t \underline{Z}$$

6. MAP estimate of $\underline{h}$

$$\pi(\underline{h}|\underline{Z}) \propto \exp\left[-\frac{1}{2\sigma^2} [\|\underline{Z} - \boldsymbol{X}\underline{h}\|^2 + \sigma_h^2 \|\underline{h}\|^2\right] \longrightarrow \widehat{\underline{h}}_{MAP} = \widehat{\underline{h}}_{MMSE} = (\boldsymbol{X}^t \boldsymbol{X} + \frac{\sigma^2}{\sigma_h^2} \boldsymbol{I})^{-1} \boldsymbol{X}^t \underline{Z}$$

7. Expressions of $\tilde{\underline{X}}(\omega)$ and $\tilde{\underline{h}}(\omega)$:
Starting by the ML estimate and

$$\boldsymbol{F}\widehat{\underline{X}} = \boldsymbol{F}(\boldsymbol{H}^t \boldsymbol{H})^{-1} \boldsymbol{H}^t \underline{Z}$$

and replacing for $\boldsymbol{H}$ and $\boldsymbol{H}^t$ we obtain

$$\tilde{\underline{X}}(\omega) = \boldsymbol{F}(\boldsymbol{\Lambda}_h^2)^{-1} \boldsymbol{\Lambda}_h^t \tilde{\underline{Z}}(\omega) \longrightarrow \tilde{\underline{X}}(\omega) = \frac{\tilde{\underline{h}}^*(\omega)}{|\tilde{\underline{h}}(\omega)|^2} \tilde{\underline{Z}}(\omega) = \frac{1}{\tilde{\underline{h}}(\omega)} \tilde{\underline{Z}}(\omega)$$

Similarly

$$\tilde{\underline{h}}(\omega) = \frac{\tilde{\underline{X}}^*(\omega)}{|\tilde{\underline{X}}(\omega)|^2} \tilde{\underline{Z}}(\omega) = \frac{1}{\tilde{\underline{X}}(\omega)} \tilde{\underline{Z}}(\omega)$$

Similarly for the MAP estimates, we obtain

$$\begin{aligned}\underline{\tilde{X}}(\omega) &= \frac{\underline{\tilde{h}}^*(\omega)}{|\underline{\tilde{h}}(\omega)|^2+\sigma^2}\,\underline{\tilde{Z}}(\omega) = \frac{1}{\underline{\tilde{h}}(\omega)}\,\frac{|\underline{\tilde{h}}(\omega)|^2}{|\underline{\tilde{h}}(\omega)|^2+\frac{\sigma^2}{\sigma_x^2}}\,\underline{\tilde{Z}}(\omega)\\ \underline{\tilde{h}}(\omega) &= \frac{\underline{\tilde{X}}^*(\omega)}{|\underline{\tilde{X}}(\omega)|^2+\sigma^2}\,\underline{\tilde{Z}}(\omega) = \frac{1}{\underline{\tilde{X}}(\omega)}\,\frac{|\underline{\tilde{X}}(\omega)|^2}{|\underline{\tilde{X}}(\omega)|^2+\frac{\sigma^2}{\sigma_h^2}}\,\underline{\tilde{Z}}(\omega)\end{aligned}$$

8. Blind deconvolution

$$\begin{aligned}f(\underline{Z}|\underline{X},\underline{h}) &= \mathcal{N}\left(\boldsymbol{H}\underline{X},\sigma^2\boldsymbol{I}\right) =\sim \mathcal{N}\left(\boldsymbol{X}\underline{h},\sigma^2\boldsymbol{I}\right)\\ \pi(\underline{X},\underline{h}|\underline{Z}) &\propto \exp\left[-\frac{1}{2\sigma^2}\left[\|\underline{Z}-\boldsymbol{H}\underline{X}\|^2+\frac{\sigma^2}{\sigma_h^2}\|\underline{h}\|^2+\frac{\sigma^2}{\sigma_x^2}\|\underline{X}\|^2\right]\right]\\ &\propto \exp\left[-\frac{1}{2\sigma^2}\left[\|\underline{Z}-\boldsymbol{X}\underline{h}\|^2+\frac{\sigma^2}{\sigma_h^2}\|\underline{h}\|^2+\frac{\sigma^2}{\sigma_x^2}\|\underline{X}\|^2\right]\right]\end{aligned}$$

We can try the joint MAP

$$(\underline{\widehat{X}},\underline{\widehat{h}})_{MAP} = \arg\min_{(\underline{X},\underline{h})}\left\{J(\underline{X},\underline{h})\right\}$$

with

$$J(\underline{X},\underline{h}) = \|\underline{Z}-\boldsymbol{H}\underline{X}\|^2+\frac{\sigma^2}{\sigma_h^2}\|\underline{h}\|^2+\frac{\sigma^2}{\sigma_h^2}\|\underline{X}\|^2 = \|\underline{Z}-\boldsymbol{X}\underline{h}\|^2+\frac{\sigma^2}{\sigma_h^2}\|\underline{h}\|^2+\frac{\sigma^2}{\sigma_h^2}\|\underline{X}\|^2$$

using an iterative algorithm such as

$$\begin{aligned}\underline{\widehat{X}}^{(k+1)} &= \arg\min_{\underline{X}}\left\{J(\underline{X},\underline{\widehat{h}}^{(k)})\right\} = (\boldsymbol{H}^t\boldsymbol{H}+\frac{\sigma^2}{\sigma_x^2}\boldsymbol{I})^{-1}\boldsymbol{H}^t\underline{Z} \quad\text{with}\quad \boldsymbol{H}=\widehat{\boldsymbol{H}}^{(k)}\\ \underline{\widehat{h}}^{(k+1)} &= \arg\min_{\underline{h}}\left\{J(\underline{X}^{(k)},\underline{\widehat{h}})\right\} = (\boldsymbol{X}^t\boldsymbol{X}+\frac{\sigma^2}{\sigma_h^2}\boldsymbol{I})^{-1}\boldsymbol{X}^t\underline{Z} \quad\text{with}\quad \boldsymbol{X}=\widehat{\boldsymbol{X}}^{(k)}\end{aligned}$$

We can also try to find the posteriors

$$\begin{aligned}\pi(\underline{X}|\underline{Z}) &= \int \pi(\underline{X}|\underline{Z})\,\mathrm{d}\underline{X} = \int \exp\left[-\frac{1}{2\sigma^2}\left[\|\underline{Z}-\boldsymbol{X}\underline{h}\|^2+\frac{\sigma^2}{\sigma_h^2}\|\underline{h}\|^2+\frac{\sigma^2}{\sigma_x^2}\|\underline{X}\|^2\right]\right]\,\mathrm{d}\underline{h}\\ \pi(\underline{h}|\underline{Z}) &= \int \pi(\underline{X}|\underline{Z})\,\mathrm{d}\underline{X} = \int \exp\left[-\frac{1}{2\sigma^2}\left[\|\underline{Z}-\boldsymbol{H}\underline{X}\|^2+\frac{\sigma^2}{\sigma_h^2}\|\underline{h}\|^2+\frac{\sigma^2}{\sigma_x^2}\|\underline{X}\|^2\right]\right]\,\mathrm{d}\underline{X}\end{aligned}$$

These integrals have explicite solutions using

$$\begin{aligned}\int \exp\left[-\frac{1}{2\sigma^2}x^2\right]\,\mathrm{d}x &= (2\pi\sigma^2)^{1/2}\\ \int \exp\left[-\frac{1}{2\sigma^2}\|\underline{X}\|^2\right]\,\mathrm{d}\underline{X} &= (2\pi\sigma^2)^{n/2}\\ \int \exp\left[-\frac{1}{2}(\underline{X}^t\boldsymbol{P}^{-1}\underline{X})\right]\,\mathrm{d}\underline{X} &= (2\pi)^{n/2}|\boldsymbol{P}|^{1/2}, \quad\text{where}\quad |\boldsymbol{P}| = \text{dét}\left\{\boldsymbol{P}\right\}\end{aligned}$$

9. Binary and independent

$$\pi(\underline{x}) = P\{\underline{X} = \underline{x}\} = (\pi_0)^{n-\sum x_i}\,(\pi_1)^{\sum x_i}$$

$$\ln \pi(\underline{x}) = (n - \sum x_i)\ln(\pi_0) + \sum x_i\,\ln(\pi_1) = n\,\ln \pi_0 + \sum x_i\,\ln \frac{1-\pi_0}{\pi_0}$$

$$\ln \pi(\underline{x}|\underline{Z}) = cte - \frac{1}{2\sigma^2}\|\underline{Z} - \boldsymbol{H}\underline{x}\|^2 - \sum x_i\,\ln \frac{\pi_0}{1-\pi_0}$$

$$\widehat{\underline{X}} = \arg\min_{\underline{x}} \left\{ J(\underline{x}) = \|\underline{Z} - \boldsymbol{H}\underline{x}\|^2 + \alpha \sum x_i \right\} \quad \text{with} \quad \alpha = 2\sigma^2\,\ln \frac{\pi_0}{1-\pi_0}$$

Noting that

$$\frac{\partial J}{\partial x_i} = \left[-2\boldsymbol{H}^t(\underline{Z} - \boldsymbol{H}\underline{x})\right]_i + \alpha$$

an iterative algorithm can be proposed

$$\widehat{x}_i^{(k+1)} = \begin{cases} 1 & & > & \\ \widehat{x}_i^{(k)} & \text{if} \quad [\boldsymbol{H}^t\boldsymbol{H}\widehat{\underline{x}}^{(k)}]_i & = & [\boldsymbol{H}^t\underline{Z}]_i + \alpha \\ 0 & & < & \end{cases}$$

10. Binary and first order Markov chain:

Noting that

$$\begin{aligned} \text{Num. of transitions} \quad 1-1 \quad &= \sum_{|i-j|=1} x_i\,x_j \\ \text{Num. of transitions} \quad 0-0 \quad &= \sum_{|i-j|=1} (1-x_i)(1-x_j) \\ \text{Num. of transitions} \quad 0-1 \quad &= \frac{1}{2}\sum_{|i-j|=1} (x_i - x_j)^2 \\ \text{Num. of transitions} \quad 1-0 \quad &= \frac{1}{2}\sum_{|i-j|=1} (x_i - x_j)^2 \end{aligned}$$

we have

$$\pi(\underline{x}) = P\{\underline{X} = \underline{x}\} \propto (s)^{\sum_{|i-j|=1} x_i\,x_j}(t)^{\sum_{|i-j|=1}(1-x_i)(1-x_j)}(1-s)^{\sum_{|i-j|=1}(x_i-x_j)^2}(1-t)^{\sum_{|i-j|=1}(x_i-x_j)^2}$$

Simplifying and noting that $\sum x_i^2 = \sum x_i$ and noting $\alpha = \frac{s^2}{(1-s)(1-t)}$ and $\beta = \frac{st}{(1-s)(1-t)}$ we obtain

$$\ln \pi(\underline{x}) = -(\ln\alpha)\sum x_i - (\ln\beta)\sum_{|i-j|=1} x_i\,x_j$$

$$\ln \pi(\underline{x}|\underline{Z}) = cte - \frac{1}{2\sigma^2}\|\underline{Z} - \boldsymbol{H}\underline{x}\|^2 - (\ln\alpha)\sum x_i - (\ln\beta)\sum_{|i-j|=1} x_i\,x_j$$

$$\widehat{\underline{x}} = \arg\min_{\underline{x}} \left\{ J(\underline{x}) = \|\underline{Z} - \boldsymbol{H}\underline{x}\|^2 + (2\sigma^2\ln\alpha)\sum x_i + (2\sigma^2\ln\beta)\sum_{|i-j|=1} x_i\,x_j \right\}$$

Again, based on

$$\frac{\partial J}{\partial x_i} = \left[-2\boldsymbol{H}^t(\underline{Z} - \boldsymbol{H}\underline{x})\right]_i + 2\sigma^2\ln\alpha + 2\sigma^2\ln\beta\sum_{|i-j|=1} x_j$$

an iterative algorithm can be proposed.