SHAPE RECONSTRUCTION IN TOMOGRAPHY FROM A FEW NUMBER OF PROJECTIONS

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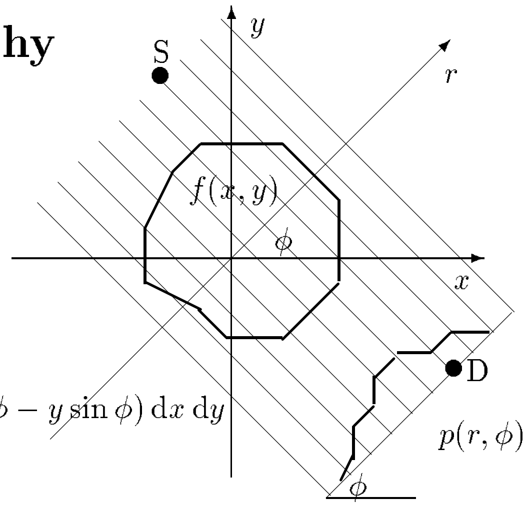
1 X-ray tomography



2D case :

$$p(r,\phi) = \int_{L_{r,\phi}} f(x,y)\,\mathrm{d}l$$

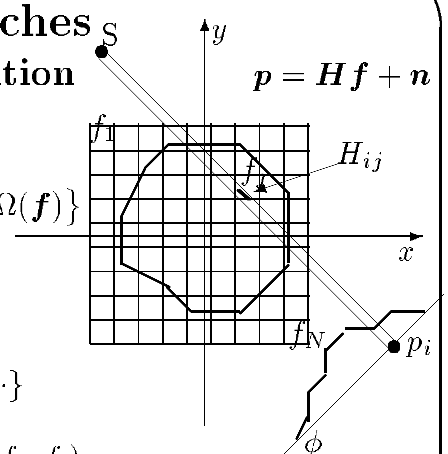
$$p(r,\phi) = \iint f(x,y)\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

3D Case :

$$p(r,\phi,\theta) = \int_{L_{r,\phi,\theta}} f(x,y,z)\,\mathrm{d}l$$

2 Main classical approaches

2.1 Pixel or voxel representation



$$\boldsymbol{p} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{n}$$

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \left\{ J(\boldsymbol{f}) = ||\boldsymbol{p} - \boldsymbol{H}\boldsymbol{f}||^2 + \lambda\Omega(\boldsymbol{f}) \right\}$$

Entropic priors : $\Omega(\boldsymbol{f}) = \sum_{j=1}^{N} v(f_j)$

with $v(x) = \{|x|^p,\ -x\log x,\ \log x,\ \cdots\}$

Markovien priors : $\Omega(\boldsymbol{f}) = \sum_{j=1}^{N} \sum_{i\in N_j} v(f_j, f_i)$

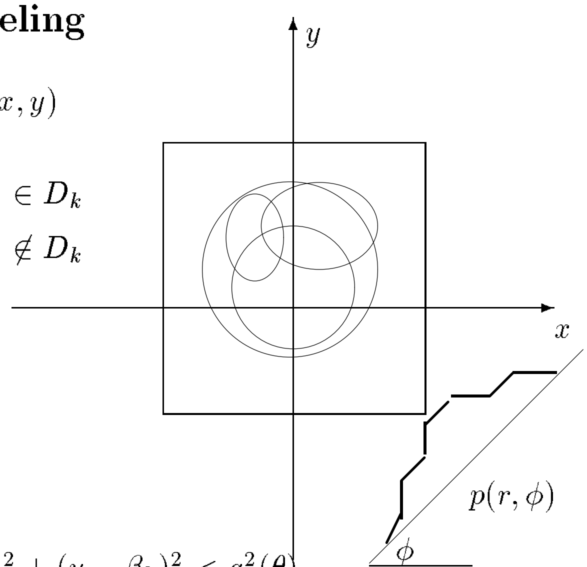
with $v(x,y) = \left\{ |x-y|^p, -|x-y|\log\frac{x}{y}, \log\cosh|x-y|, \cdots \right\}$

or $v(x,y) = \left\{ \min\{|x-y|^2, 1\}, \frac{-1}{1+|x-y|^2}, \cdots \right\}$

2.2 Parametric modeling

$$f(x,y) = \sum_{k=1}^{K} d_k f_k(x,y)$$

$$f_k(x,y) = \begin{cases} 1 & (x,y) \in D_k \\ 0 & (x,y) \notin D_k \end{cases}$$



For example:

$$f_k(x,y) = \begin{cases} 1 & \text{if} \quad (x-\alpha_k)^2 + (y-\beta_k)^2 < g_k^2(\theta) \\ 0 & \text{elsewhere} \end{cases}$$

A. Mohammad-Djafari Shape reconstruction in tomography June 10, 1997

$$g_k(\theta) = \sqrt{a_k^2 \cos^2\theta + b_k^2 \sin^2\theta}.$$

Noting $\boldsymbol{\theta} = \{d_k, \alpha_k, \beta_k, a_k, b_k, k = 1, \cdots, K\}$

$$p(r,\phi) = h(r,\phi;\boldsymbol{\theta}) + n(r,\phi) = \sum_{k=1}^{K} d_k p_k(r,\phi) + n(r,\phi)$$

with

$$p_k(r,\phi) = \begin{cases} \frac{2a_k b_k}{g_k^2(\phi)} \sqrt{g_k^2(\phi) - r^2)} & \text{if} \quad r < g_k(\phi), \\ 0 & \text{elsewhere} \end{cases}$$

The LS:

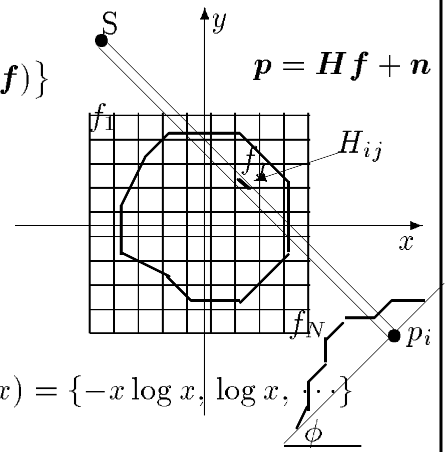
$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \left\{ \|p(r,\phi) - h(r,\phi;\boldsymbol{\theta})\|^2 \right\}$$

or, more generaly, the ML estimate:

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \left\{ -\ln p(\boldsymbol{p}|\boldsymbol{\theta}) \right\}$$

2.3 Binary objects image reconstruction

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \left\{ J(\boldsymbol{f}) = ||\boldsymbol{p} - \boldsymbol{H}\boldsymbol{f}||^2 + \lambda\Omega(\boldsymbol{f}) \right\}$$



$\Omega(\boldsymbol{f})$ has to be choosed to enforce the binary character of $\boldsymbol{f}$.

Examples:

- Beta like prior models:

$$\Omega(\boldsymbol{f}) = \sum_{j=1}^{N} v(f_j) + v(1 - f_j) \quad \text{with} \quad v(x) = \{-x \log x, \log x, \cdots\}$$

- Binary markovien models :

$$\Omega(\boldsymbol{f}) = \sum_{j=1}^{N} \sum_{i \in \mathcal{N}_j} v(f_j, f_i) \quad \text{with} \quad v(x,y) = \begin{cases} +\alpha & \text{if } x = y \\ -\alpha & \text{if } x \neq y \end{cases}$$

2.4 Binary objects and "Level Set" approach

- Object=the zero-crossing of a smooth function $u(x,y)$:

$$\partial D = \{(x,y) : u(x,y) = 0\} \quad \text{and} \quad f(x,y) = \begin{cases} c_1 & \text{if } u(x,y) > 0, \\ c_2 & \text{if } u(x,y) < 0 \end{cases},$$

- Time evolution for u and consequently for ∂D and function f:

$$\partial D(t) = \{(x,y) : u(x,y,t) = 0\}$$

such that when $t \mapsto \infty$ the function $u(x,y)$ and its associated $f(x,y)$ is a solution to problem in a LS sense:

$$J(f) = \|p - R(f)\|^2$$

- The function $u(x,y)$ is assimilated to a surface (of heat front wave)

• The evolution of the contours ∂D is constraint to be perpendicular to the surface:

$$(\delta x, \delta y) = \alpha(x, y, t) \frac{\nabla u}{|\nabla u|}$$

• Originaly developed by Osher and Sethian (Osher 88) for problems involving curves and surfaces motion,

• Adapted, used and refered as *snakes* or *active contour models* by many authors in computer vision (Catte 92, Malladi 95) and

• recently for inverse problems (Santosa 96).

• We are presently working on this approach trying to
– extend it for minimizing a regularized criterion in place of the the LS criterion and
– implementing it in 2D and 3D tomographic image reconstruction.

2.5 Compact binary object modelling

$$f(x,y) = \begin{cases} 1 & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$$

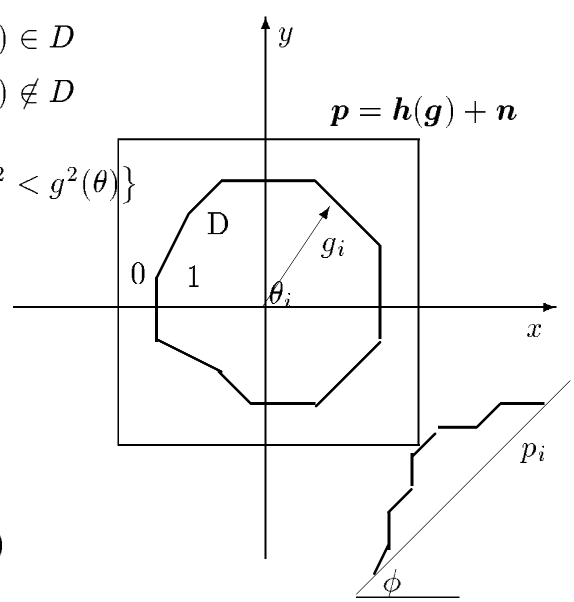
$$D = \left\{(x,y) : \rho^2(\theta) = x^2 + y^2 < g^2(\theta)\right\}$$

$$\boldsymbol{g} = [g_1, \cdots, g_n]$$

$$\boldsymbol{p} = [p_1, \cdots, p_m]$$

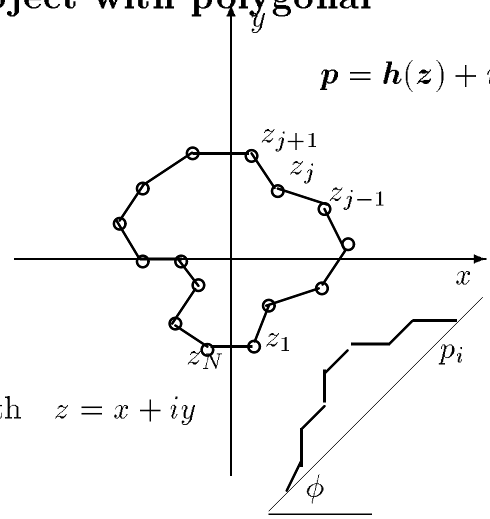
$$\boldsymbol{p} = \boldsymbol{h}(\boldsymbol{g}) + \boldsymbol{n}$$

$$J(\boldsymbol{g}) = ||\boldsymbol{p} - \boldsymbol{h}(\boldsymbol{g})||^2 + \lambda \Omega(\boldsymbol{g})$$



2.6 Compact binary object with polygonal contours modelling

$$\boldsymbol{p} = \boldsymbol{h}(\boldsymbol{z}) + \boldsymbol{n}$$



Mains relations:

1. $\mu_{pq} = \iint_P f(x,y) x^p y^q \,\mathrm{d}x\,\mathrm{d}y$
2. $c_k = \iint f(x,y) z^k \,\mathrm{d}x\,\mathrm{d}y \quad \text{with} \quad z = x + iy$
3. $c_k = \sum_{j=0}^{k} \binom{k}{j} i^j \mu_{k-j,j}$
4. $h_k(\phi) = \int_{-1}^{1} p(r,\phi) r^k \,\mathrm{d}r$

Using the relation

$$\int_{-1}^{1} p(r,\phi) F(r)\,\mathrm{d}r = \iint_P f(x,y) F(x\cos\phi + y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

with $F(r) = r^k$ we obtain some relations between μ_{pq} and $h_k(\phi)$.

$$\begin{aligned} h_k(\phi) &= \sum_{j=0}^{k} \binom{k}{j} \cos^{k-j}(\phi)\,\sin^j(\phi)\,\mu_{k-j,j} \\ \tau_k &= \sum_{j=0}^{N} a_j z_j^k \quad \text{with} \quad \tau_k = k(k-1)c_{k-2} \\ a_j &= \frac{i}{2}\left(\frac{\bar{z}_{j-1} - \bar{z}_j}{z_{j-1} - z_j} - \frac{\bar{z}_j - \bar{z}_{j+1}}{z_j - z_{j+1}}\right). \end{aligned}$$

Reconstruction scheme:

1- Calculate $h_k(\phi)$ from the projections $p(r, \phi)$;

2- Calculate $\mu_{k,j}$ from $h_k(\phi)$;

3- Calculate c_k, then τ_k from $\mu_{k-j,j}$;

4- Calculate z_j from τ_k

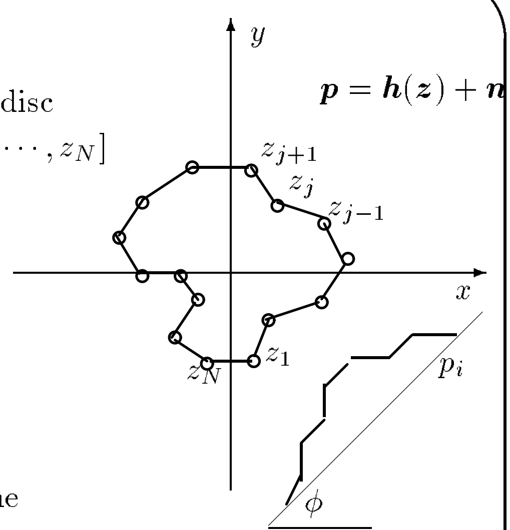
Note:

steps 1 and 3 are directe, but steps 2 and 4 needs matrix inversion

Link with antenna array processing

	τ_k	z_j	a_j
Image Reconstruction	Harmonic Moments	Vertices Coordinates	?
Antenna Array Processing	Data	Direction of Arrivals (DOA)	Source Amplitudes

3 Proposed method



• Object is modeled as a polygonal disc caracterized by it's vertices $\boldsymbol{z} = [z_1, \cdots, z_N]$

• $\widehat{\boldsymbol{z}} = \arg\min_{\boldsymbol{z}} \{J(\boldsymbol{z})\}$ with

$$J(\boldsymbol{z}) = ||\boldsymbol{p} - \boldsymbol{h}(\boldsymbol{z})||^2 + \lambda\Omega(\boldsymbol{z})$$

$$\Omega(\boldsymbol{z}) = \sum_{j=1}^{N} |z_{j-1} - 2z_j + z_{j+1}|^2$$

• $|z_{j-1} - 2z_j + z_{j+1}|^2$ is the Euclidian distance between the point z_j and the midpoint of the line segment passing through z_{j-1} and z_{j+1}.

• This choice favors a shape whose local curvature is limited.

Probabilistic interpretation:

Bayesian MAP estimation :

$$\widehat{\boldsymbol{z}} = \arg\max_{\boldsymbol{z}} \{p(\boldsymbol{z}|\boldsymbol{p})\} = \arg\min_{\boldsymbol{z}} \{J(\boldsymbol{z})\} \text{ with } J(\boldsymbol{z}) = \|\boldsymbol{p}-\boldsymbol{h}(\boldsymbol{z})\|^2+\lambda\Omega(\boldsymbol{z})$$

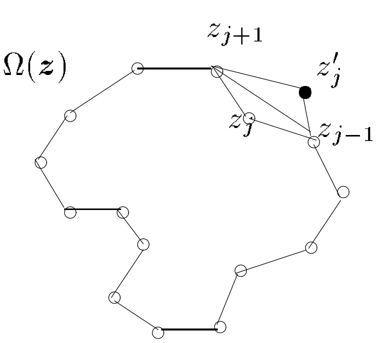
with the following assumptions :

- Noise is assumed Gaussian : $p(\boldsymbol{p}|\boldsymbol{z}) \propto \exp\left[-\frac{1}{2\sigma_b^2}\|\boldsymbol{p}-\boldsymbol{h}(\boldsymbol{z})\|^2\right]$
- z_j are modeled as as random variables with the following Markovian law:

$p(z_j|\boldsymbol{z}) = p(z_j|z_{j-1}, z_{j+1}) \propto \exp\left[-\frac{1}{2\sigma^2}||z_{j-1} - 2z_j + z_{j+1}||^2\right]$

$$p(\boldsymbol{z}) =\propto \exp\left[-\frac{1}{2\sigma^2}\sum_{j=1}^{N}||z_{j-1} - 2z_j + z_{j+1}||^2\right]$$

$$J(\boldsymbol{z}) = \|\boldsymbol{p}-\boldsymbol{h}(\boldsymbol{z})\|^2+\lambda\Omega(\boldsymbol{z})$$



- $J(\boldsymbol{z})$ is multimodal.
- There is no analytic expression for $J(\boldsymbol{z})$ and for it's gradient.
- When one coordinate z_j is changed it is possible to calculated rapidly δJ.

Three methods:

- Simulated annealing
- Deterministic relaxation techniques such as ICM or ICD
- Local descent algorithms with good initialization.

4 Simulation results and conclusions

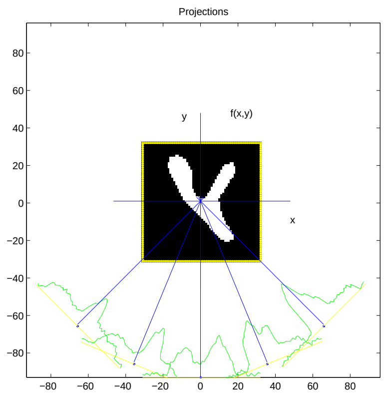


Fig. 1: Original image and simulated noisy projections.

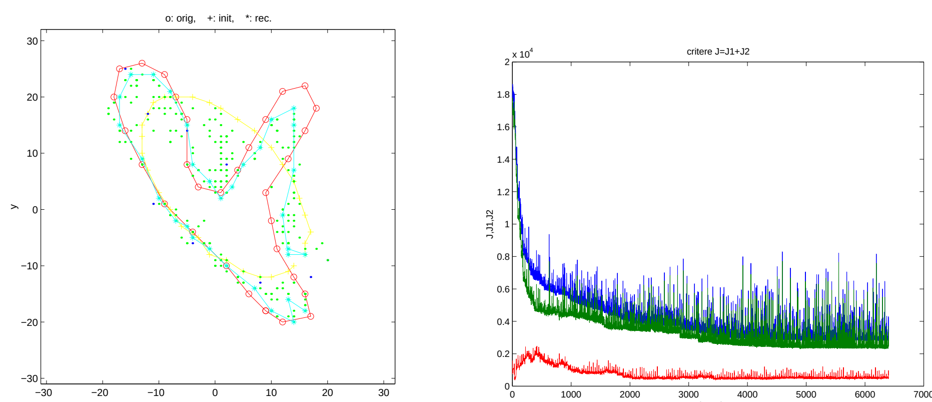


Fig. 2: Reconstruction using simulated annealing.
a) Original (o), Initialization (+) and Reconstructed objects (x)
b) Evolution of $J = J_1 + \lambda J_2$ where $J1 = Q(\boldsymbol{z})$ and $J_2 = \Omega(\boldsymbol{z})$

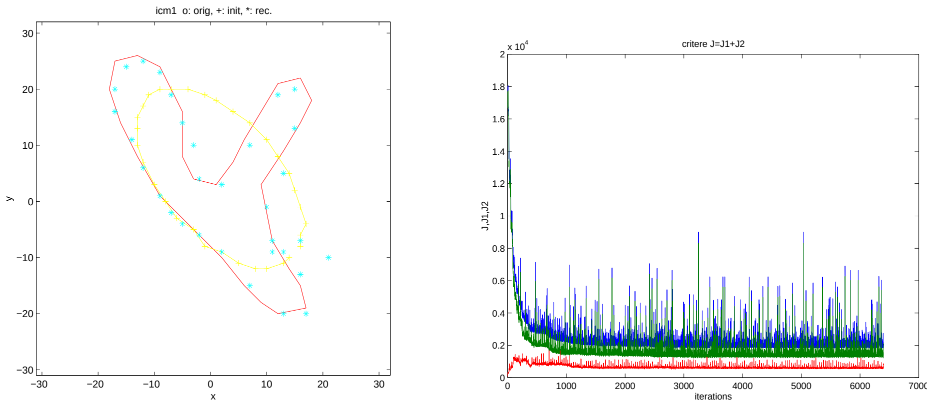
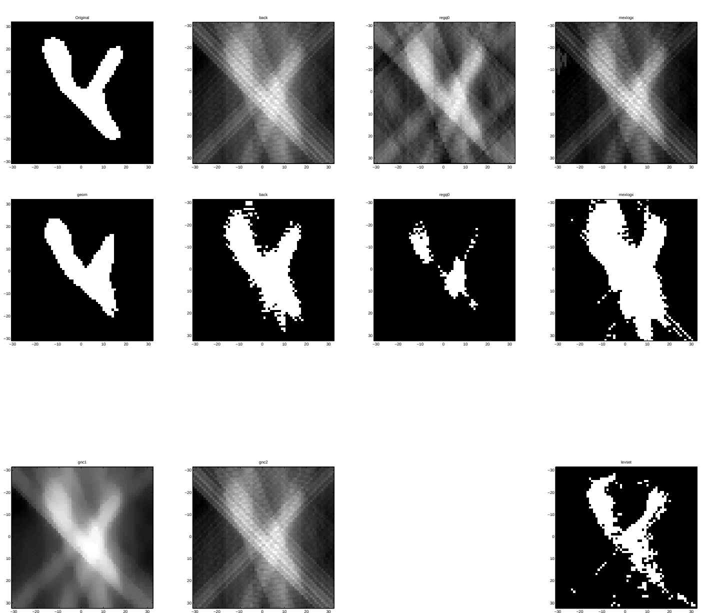


Fig. 3: Reconstruction using a local minimizer.

a) Original, Initialization and Reconstructed objects

b) Evolution of the criterion $J = J_1 + \lambda J_2$ during the iterations.



A comparison with backprojection and some other classical methods :

a) Original, b) Proposed method, c) Backprojection and its binary threshold (d),

e) Gaussian Markov modeling MAP reconstruction and its binary threshold (f),

g) Maximum entropy regularized reconstruction and its binary threshold (h),

i) Compound Markov modeling and GNC optimization algorithm using truncated quadratic potential and Lorentzian potential function (j),

h,k) Two results obtained with Level Set approach