SHAPE RECONSTRUCTION IN TOMOGRAPHY
FROM A FEW NUMBER OF PROJECTIONS

Ali MOHAMMAD-DJAFARI
Laboratoire des Signaux et Systèmes (CNRS-ESE-UPS)
Supélec, Plateau de Moulon
91192 Gif-sur-Yvette Cedex, FRANCE.

e-mail: djafari@lss.supelec.fr

Contents:

1. Image reconstruction in X-ray tomography
2. Main classical approaches
   • Pixel or voxel representation and general
     regularization methods
   • Parametric modelisation and LS, ML or MAP
     parameter estimation
   • Binary objects and Markov modelling
   • Binary objects and “Level set” approach
   • Compact binary object modelling and shape
     reconstruction
   • Compact binary object with polygonal contours
3. Proposed method
4. Simulation results and conclusions
1 X-ray tomography

2D case:
\[ p(r, \phi) = \int_{L_{r, \phi}} f(x, y) \, dl \]
\[ p(r, \phi) = \int f(x, y) \delta(r - x \cos \phi - y \sin \phi) \, dx \, dy \]

3D Case:
\[ p(r, \phi, \theta) = \int_{L_{r, \phi, \theta}} f(x, y, z) \, dl \]

2 Main classical approaches

2.1 Pixel or voxel representation

\[ p = H f + n \]
\[ \hat{f} = \text{arg min} \{ J(f) = ||p - Hf||^2 + \lambda \Omega(f) \} \]

Entropic priors: \( \Omega(f) = \sum_{j=1}^{N} v(f_j) \)
with \( v(x) = \{|x|^p, -x \log x, \log x, \cdots\} \)

Markovian priors: \( \Omega(f) = \sum_{j=1}^{N} \sum_{i \in N_j} v(f_j, f_i) \)
with \( v(x, y) = \{|x - y|^p, -|x - y| \log \frac{x}{y}, \log \cosh |x - y|, \cdots\} \)
or \( v(x, y) = \{ \min\{|x - y|^2, 1\}, \frac{-1}{1+|x-y|^2}, \cdots\} \)
2.2 Parametric modeling

\[ f(x, y) = \sum_{k=1}^{K} d_k f_k(x, y) \]

\[ f_k(x, y) = \begin{cases} 
1 & (x, y) \in D_k \\
0 & (x, y) \notin D_k 
\end{cases} \]

For example:

\[ f_k(x, y) = \begin{cases} 
1 & \text{if } (x - \alpha_k)^2 + (y - \beta_k)^2 < g_k^2(\theta) \\
0 & \text{elsewhere} 
\end{cases} \]

\[ g_k(\theta) = \sqrt{a_k^2 \cos^2 \theta + b_k^2 \sin^2 \theta}. \]

Noting \( \theta = \{d_k, \alpha_k, \beta_k, a_k, b_k, k = 1, \ldots, K\} \)

\[ p(r, \phi) = h(r, \phi; \theta) + n(r, \phi) = \sum_{k=1}^{K} d_k p_k(r, \phi) + n(r, \phi) \]

with

\[ p_k(r, \phi) = \begin{cases} 
\frac{2a_k b_k}{g_k^2(\phi)} \sqrt{g_k^2(\phi) - r^2} & \text{if } r < g_k(\phi), \\
0 & \text{elsewhere} 
\end{cases} \]

The LS:

\[ \hat{\theta} = \arg \min_{\theta} \{\|p(r, \phi) - h(r, \phi; \theta)\|^2\} \]

or, more generally, the ML estimate:

\[ \hat{\theta} = \arg \min_{\theta} \{-\ln p(p|\theta)\} \]
2.3 Binary objects image reconstruction

$$\hat{f} = \arg \min_f \{ J(f) = \|p - Hf\|^2 + \lambda \Omega(f) \}$$

$$\Omega(f)$$ has to be chosen to enforce the binary character of $$f$$.

Examples:
- Beta like prior models:
  $$\Omega(f) = \sum_{j=1}^{N} v(f_j) + v(1 - f_j)$$ with $$v(x) = \{-x \log x, \log x, \ldots\}$$
- Binary markovien models:
  $$\Omega(f) = \sum_{j=1}^{N} \sum_{i \in \mathcal{N}_j} v(f_j, f_i)$$ with $$v(x, y) = \begin{cases} +\alpha & \text{if } x = y \\ -\alpha & \text{if } x \neq y \end{cases}$$

2.4 Binary objects and “Level Set” approach

- Object=the zero-crossing of a smooth function $$u(x, y)$$:
  $$\partial D = \{(x, y) : u(x, y) = 0\}$$ and $$f(x, y) = \begin{cases} c_1 & \text{if } u(x, y) > 0, \\ c_2 & \text{if } u(x, y) < 0 \end{cases}$$

- Time evolution for $$u$$ and consequently for $$\partial D$$ and function $$f$$:
  $$(\partial D(t) = \{(x, y) : u(x, y, t) = 0\}$$

such that when $$t \to \infty$$ the function $$u(x, y)$$ and its associated $$f(x, y)$$ is a solution to problem in a LS sense:

$$J(f) = \|p - R(f)\|^2$$

- The function $$u(x, y)$$ is assimilated to a surface (of heat front wave)
• The evolution of the contours $\partial D$ is constraint to be perpendicular to the surface:

$$ (\delta x, \delta y) = \alpha(x, y, t) \frac{\nabla u}{|\nabla u|} $$

• Originally developed by Osher and Sethian (Osher 88) for problems involving curves and surfaces motion,

• Adapted, used and referred as snakes or active contour models by many authors in computer vision (Catte 92, Malladi 95) and

• recently for inverse problems (Santosa 96).

• We are presently working on this approach trying to
  - extend it for minimizing a regularized criterion in place of the LS criterion and
  - implementing it in 2D and 3D tomographic image reconstruction.

2.5 Compact binary object modelling

$$ f(x, y) = \begin{cases} 
1 & (x, y) \in D \\
0 & (x, y) \notin D 
\end{cases} $$

$$ D = \{(x, y) : \rho^2(\theta) = x^2 + y^2 < g^2(\theta)\} $$

$$ g = [g_1, \cdots, g_n] $$

$$ p = [p_1, \cdots, p_m] $$

$$ p = h(g) + n $$

$$ J(g) = ||p - h(g)||^2 + \lambda \Omega(g) $$
2.6 Compact binary object with polygonal contours modelling

Mains relations:

1. $\mu_{pq} = \iint_{p} f(x, y) x^p y^q \, dx \, dy$

2. $c_k = \iint f(x, y) z^k \, dx \, dy \quad \text{with} \quad z = x + iy$

3. $c_k = \sum_{j=0}^{k} \binom{k}{j} i^j \mu_{k-j,j}$

4. $h_k(\phi) = \int_{-1}^{1} p(r, \phi) r^k \, dr$

Using the relation

$$\int_{-1}^{1} p(r, \phi) F(r) \, dr = \iint_{p} f(x, y) F(x \cos \phi + y \sin \phi) \, dx \, dy$$

with $F(r) = r^k$ we obtain some relations between $\mu_{pq}$ and $h_k(\phi)$.

$h_k(\phi) = \sum_{j=0}^{k} \binom{k}{j} \cos^{k-j}(\phi) \sin^j(\phi) \mu_{k-j,j}$

$\tau_k = \sum_{j=0}^{N} a_j z_j^k \quad \text{with} \quad \tau_k = k(k-1)c_{k-2}$

$a_j = \frac{i}{2} \left( \frac{z_j - z_{j-1}}{z_j - z_{j+1}} - \frac{z_{j+1} - z_j}{z_j - z_{j+1}} \right)$.
Reconstruction scheme:
1- Calculate $h_k(\phi)$ from the projections $p(r, \phi)$;
2- Calculate $\mu_{k,j}$ from $h_k(\phi)$;
3- Calculate $e_k$, then $\tau_k$ from $\mu_{k-j,j}$;
4- Calculate $z_j$ from $\tau_k$

Note:
steps 1 and 3 are directe, but steps 2 and 4 needs matrix inversion

Link with antenna array processing

<table>
<thead>
<tr>
<th>Image Reconstruction</th>
<th>$\tau_k$</th>
<th>$z_j$</th>
<th>$\alpha_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Harmonic Moments</td>
<td></td>
<td>Vertices Coordinates</td>
<td>$?$</td>
</tr>
<tr>
<td>Antenna Array</td>
<td>Data</td>
<td>Direction of Arrivals (DOA)</td>
<td>Source Amplitudes</td>
</tr>
</tbody>
</table>

3 Proposed method

- Object is modeled as a polygonal disc characterized by it’s vertices $z = [z_1, \cdots, z_N]$
- $\hat{z} = \arg\min_z \{ J(z) \}$ with
  $$J(z) = ||p - h(z)||^2 + \lambda \Omega(z)$$
- $\Omega(z) = \sum_{j=1}^{N} |z_{j-1} - 2z_j + z_{j+1}|^2$
- $|z_{j-1} - 2z_j + z_{j+1}|^2$
  is the Euclidian distance between the point $z_j$ and the midpoint of the line segment passing through $z_{j-1}$ and $z_{j+1}$.
- This choice favors a shape whose local curvature is limited.
Probabilistic interpretation:
Bayesian MAP estimation:
\[ \hat{z} = \arg \max_z \{ p(z|p) \} = \arg \min_z \{ J(z) \} \text{ with } J(z) = \| p - h(z) \|^2 + \lambda \Omega(z) \]

with the following assumptions:
- Noise is assumed Gaussian: \( p(p|z) \propto \exp \left[ -\frac{1}{2\sigma^2} \| p - h(z) \| ^2 \right] \)
- \( z_j \) are modeled as as random variables with the following Markovian law:
\[
p(z_j|z) = p(z_j|z_{j-1}, z_{j+1}) \propto \exp \left[ -\frac{1}{2\sigma^2} \| z_{j-1} - 2z_j + z_{j+1} \| ^2 \right]
\]
\[
p(z) = \propto \exp \left[ -\frac{1}{2\sigma^2} \sum_{j=1}^{N} \| z_{j-1} - 2z_j + z_{j+1} \| ^2 \right]
\]

- \( J(z) \) is multimodal.
- There is no analytic expression
for \( J(z) \) and for it’s gradient.
- When one coordinate \( z_j \) is changed
it is possible to calculated rapidly \( \delta J \).

Three methods:
- Simulated annealing
- Deterministic relaxation techniques such as ICM or ICD
- Local descent algorithms with good initialization.
4 Simulation results and conclusions

![Simulated noisy projections](image1)

Fig. 1: Original image and simulated noisy projections.

![Reconstruction results](image2)

Fig. 2: Reconstruction using simulated annealing.

a) Original (o), Initialization (+) and Reconstructed objects (x)

b) Evolution of $J = J_1 + \lambda J_2$ where $J_1 = Q(z)$ and $J_2 = \Omega(z)$
Fig. 3: Reconstruction using a local minimizer.
   a) Original, Initialization and Reconstructed objects
   b) Evolution of the criterion $J = J_1 + \lambda J_2$ during the iterations.
a) Original, b) Proposed method, c) Backprojection and its binary threshold (d),

e) Gaussian Markov modeling MAP reconstruction and its binary threshold (f),
g) Maximum entropy regularized reconstruction and its binary threshold (h),
i) Compound Markov modeling and GNC optimization algorithm using truncated quadratic potential and Lorentzian potential function (j),
h,k) Two results obtained with Level Set approach.