Bayesian inference for inverses problems in signal and image processing

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Inverses problems in image processing

- General non linear inverse problem:

$$g(\boldsymbol{r}) = [\mathcal{H}f(\boldsymbol{r}')](\boldsymbol{r}) + \epsilon(\boldsymbol{r}), \quad \boldsymbol{r} = (x, y) \in \mathcal{R}, \quad \boldsymbol{r}' = (x', y') \in \mathcal{R}'$$

- Linear model:

$$g(\boldsymbol{r}) = \int_{\mathcal{R}'} f(\boldsymbol{r}') h(\boldsymbol{r}, \boldsymbol{r}') \,\mathrm{d}\boldsymbol{r}' + \epsilon(\boldsymbol{r})$$

- Linear and translation invariante (convolution) model:

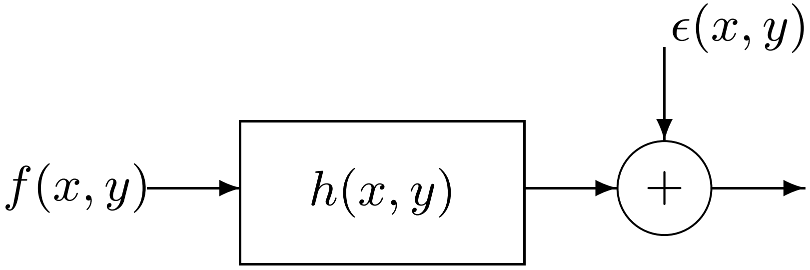
$$g(\boldsymbol{r}) = \int_{\mathcal{R}'} f(\boldsymbol{r}') h(\boldsymbol{r} - \boldsymbol{r}') \,\mathrm{d}\boldsymbol{r}' + \epsilon(\boldsymbol{r}) = h(\boldsymbol{r}) * f(\boldsymbol{r}) + \epsilon(\boldsymbol{r})$$

- Discretized version

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

where $\boldsymbol{g} = \{g(\boldsymbol{r}),\ \boldsymbol{r} \in \mathcal{R}\}$, $\boldsymbol{\epsilon} = \{\epsilon(\boldsymbol{r}),\ \boldsymbol{r} \in \mathcal{R}\}$ and $\boldsymbol{f} = \{f(\boldsymbol{r}'),\ \boldsymbol{r}' \in \mathcal{R}'\}$

Single channel image restoration



$$g(x,y) = f(x,y) * h(x,y) + \epsilon(x,y)$$

Observation model : $\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$

?

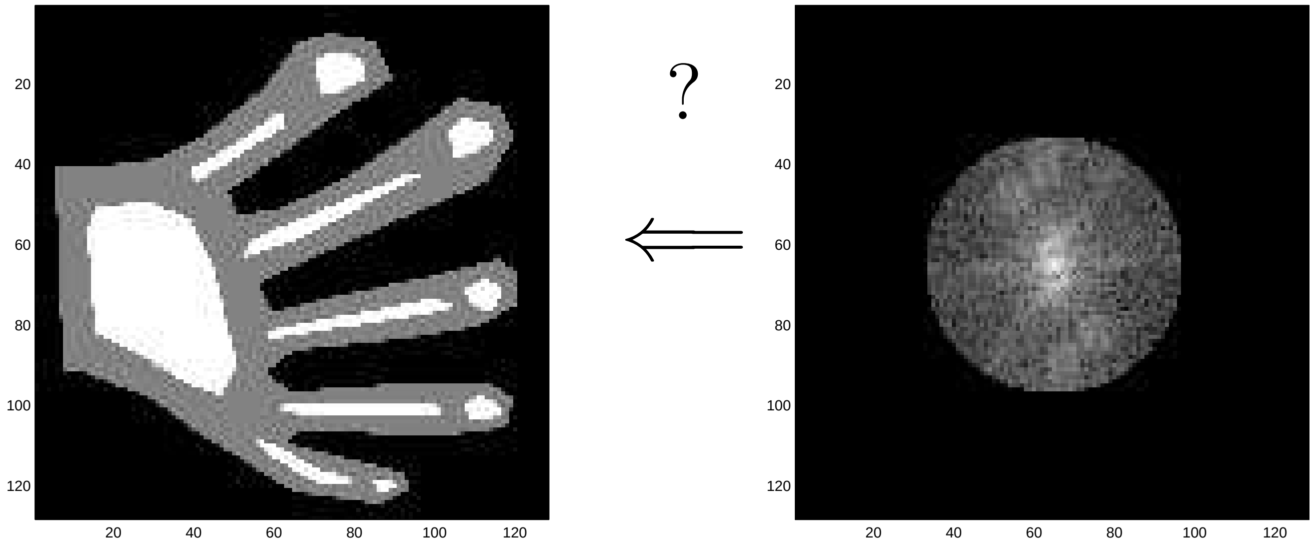
$\Longleftarrow$

Fourier synthesis in optical imaging

$$g(\boldsymbol{\omega}) = \int f(\boldsymbol{r}) \exp\left[-j\boldsymbol{\omega}^t\boldsymbol{r}\right] \, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{\omega})$$

- Coherent imaging: $\mathcal{G}(g) = g \quad \longrightarrow \quad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$
- Non coherent imaging: $\mathcal{G}(g) = |g| \quad \longrightarrow \quad \boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) + \boldsymbol{\epsilon}$

$\boldsymbol{g} = \{g(\boldsymbol{\omega}),\ \boldsymbol{\omega} \in \Omega\}, \quad \boldsymbol{\epsilon} = \{\epsilon(\boldsymbol{\omega}),\ \boldsymbol{\omega} \in \Omega\}$ and $\boldsymbol{f} = \{f(\boldsymbol{r}),\ \boldsymbol{r} \in \mathcal{R}\}$



Multi sensor image processing problems

- Disjoint multi sensors system:

$$\boldsymbol{g}_i = \boldsymbol{H}_i \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M$$

- General multi input multi output (MIMO) system:

$$\boldsymbol{g}_i = \sum_{j=1}^{N} \boldsymbol{H}_{ij} \boldsymbol{f}_j + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M$$

- General unknown mixing gain MIMO system:

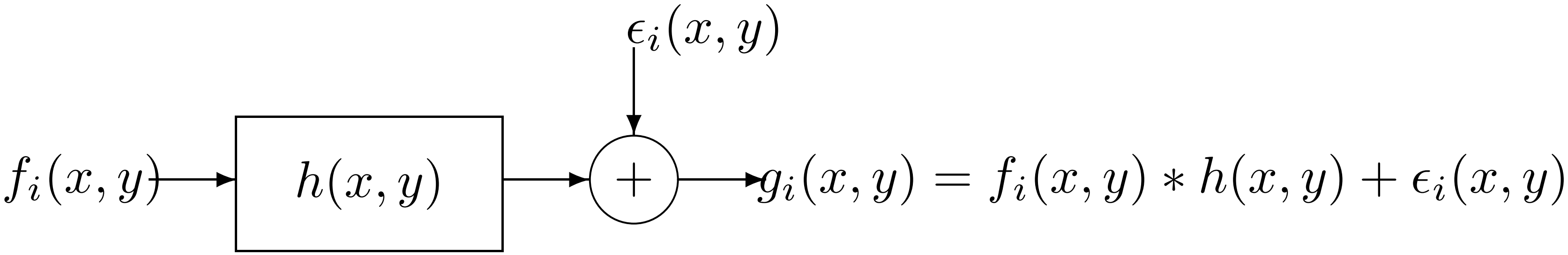
$$\boldsymbol{g}_i = \sum_{j=1}^{N} A_{ij} \boldsymbol{H}_j \boldsymbol{f}_j + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M$$

- Blind Sources Separation (BSS) problem:

$$\boldsymbol{g}_i = \sum_{j=1}^{N} A_{ij} \boldsymbol{f}_j + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M$$

  where $\boldsymbol{A} = \{A_{ij}, i = 1, \cdots, M,\ j = 1, \cdots, N\}$ is an unknown mixing matrix.

Multi-spectral image deconvolution



Observation model : $\quad \boldsymbol{g}_i = \boldsymbol{H}\boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, 2, 3$

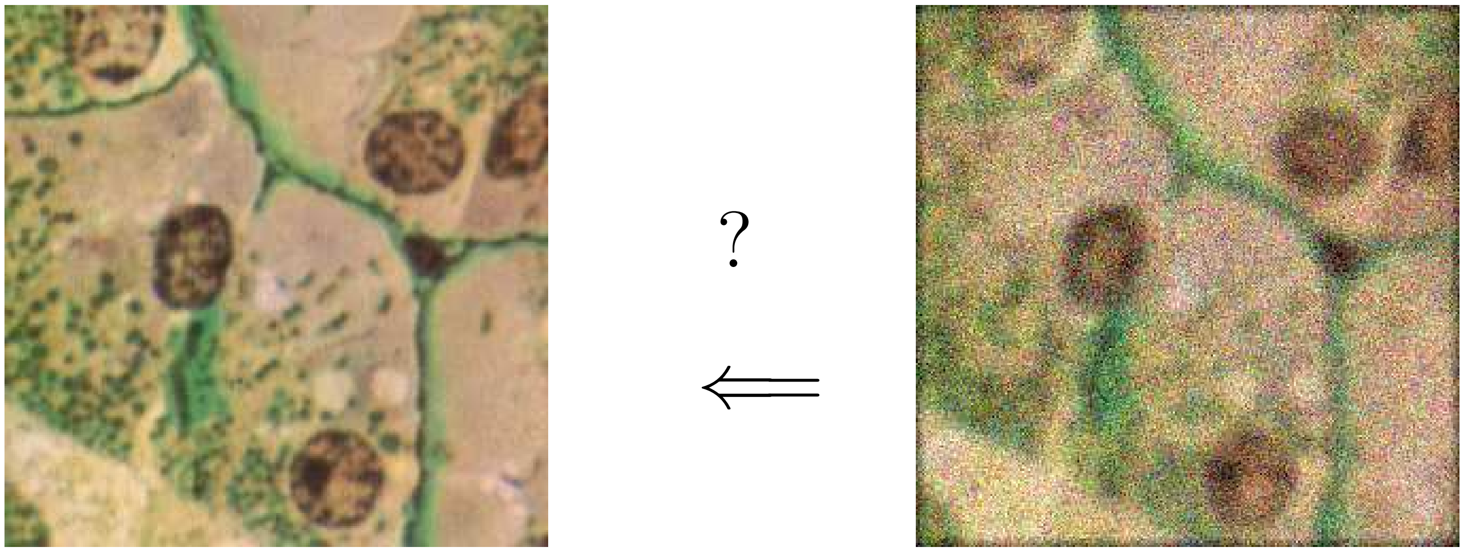
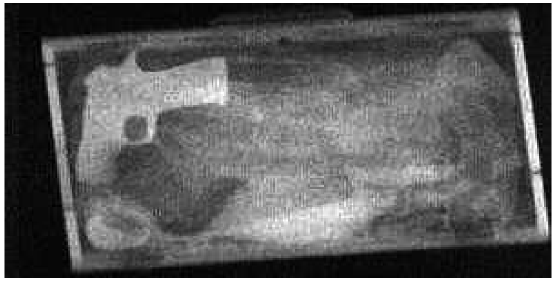
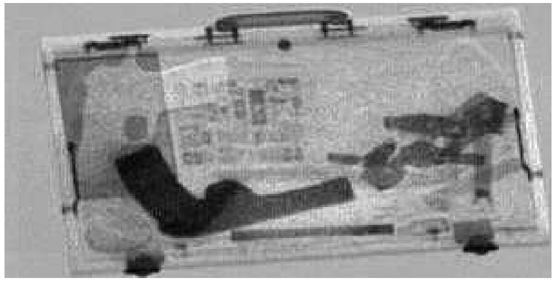


Image fusion and joint segmentation



Fusion ?

$\Longrightarrow$

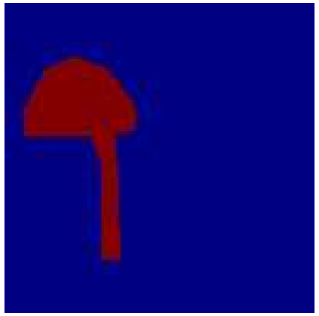


$$g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}), \quad i = 1, \cdots, M$$

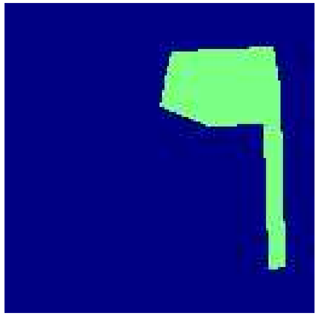
$$\underline{g}(\boldsymbol{r}) = \{g_i(\boldsymbol{r}),\ i = 1, M\}, \qquad \boldsymbol{g}_i = \{g_i(\boldsymbol{r}),\ \boldsymbol{r} \in \mathcal{R}\}, \qquad \underline{\boldsymbol{g}} = \{\boldsymbol{g}_i(\boldsymbol{r}),\ i = 1, M\}$$

$$\underline{g}(\boldsymbol{r}) = \underline{f}(\boldsymbol{r}) + \underline{\epsilon}(\boldsymbol{r}), \qquad \underline{\boldsymbol{g}} = \underline{\boldsymbol{f}} + \underline{\boldsymbol{\epsilon}}$$

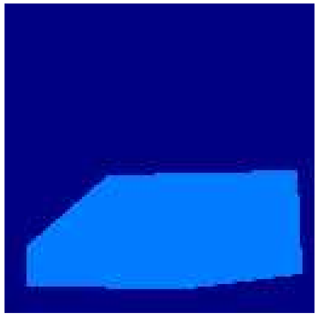
Blind image separation and joint segmentation



$f_1(\boldsymbol{r})$



$f_2(\boldsymbol{r})$

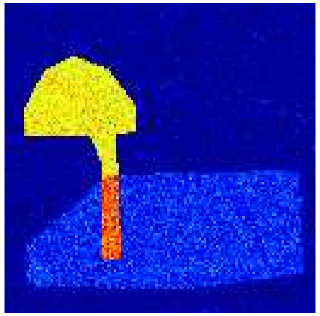


$f_3(\boldsymbol{r})$

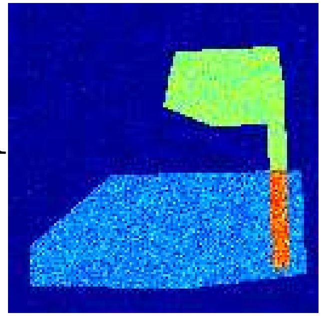
?

Separation

$\Longleftarrow$



$g_1(\boldsymbol{r})$



$g_2(\boldsymbol{r})$

$$g_i(\boldsymbol{r}) = \sum_{j=1}^{N} A_{ij} f_j(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r})$$

$$\underline{g}(\boldsymbol{r}) = \{g_i(\boldsymbol{r}),\ i = 1, M\}$$

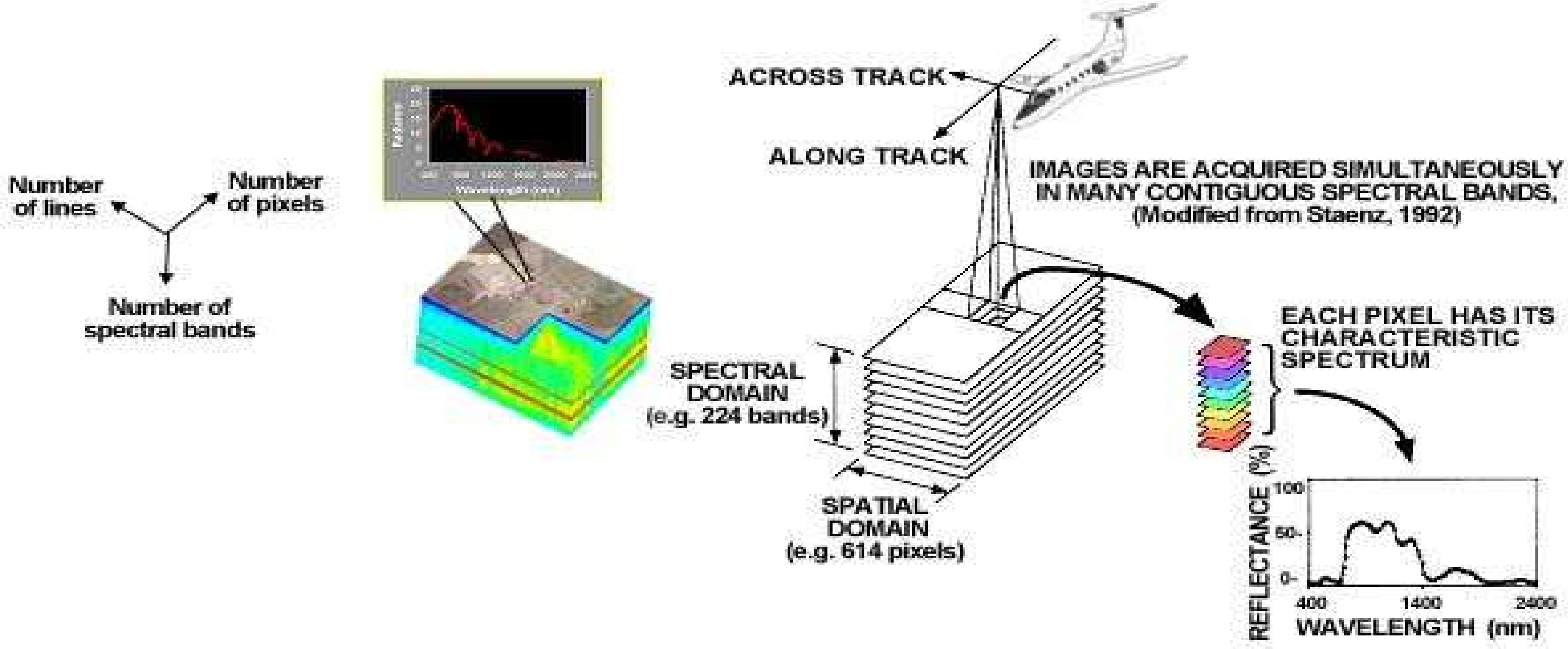
$$\underline{g}(\boldsymbol{r}) = \boldsymbol{A}\underline{f}(\boldsymbol{r}) + \underline{\epsilon}(\boldsymbol{r}),$$

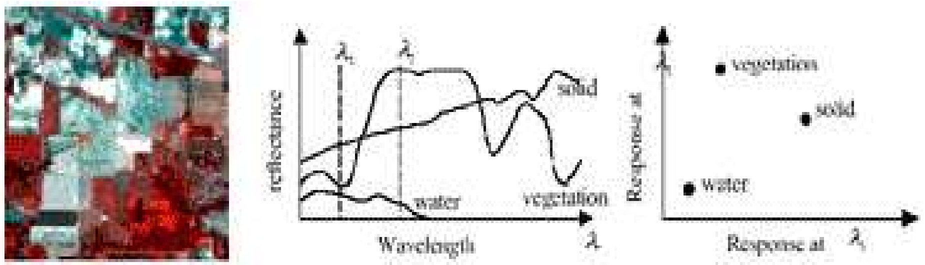
$$\underline{\boldsymbol{g}} = \{\boldsymbol{g}_i(\boldsymbol{r}),\ i = 1, M\}$$

$$\boldsymbol{g}_i = \{g_i(\boldsymbol{r}),\ \boldsymbol{r} \in \mathcal{R}\},$$

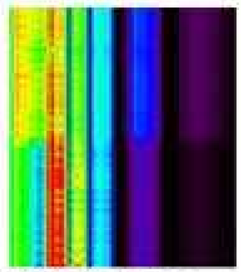
$$\underline{\boldsymbol{g}} = \boldsymbol{A}\underline{\boldsymbol{f}} + \underline{\boldsymbol{\epsilon}}$$

Segmentation of hyperspectral images

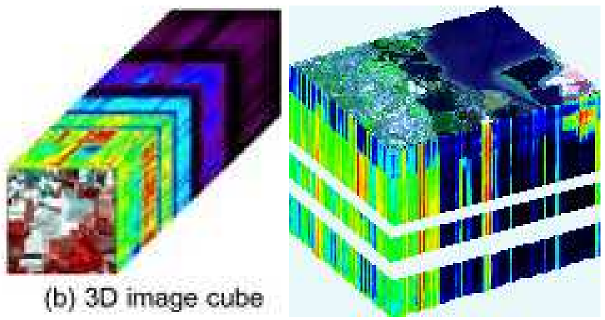




(a) Image Space (b) Spectral Space (c) Feature Space

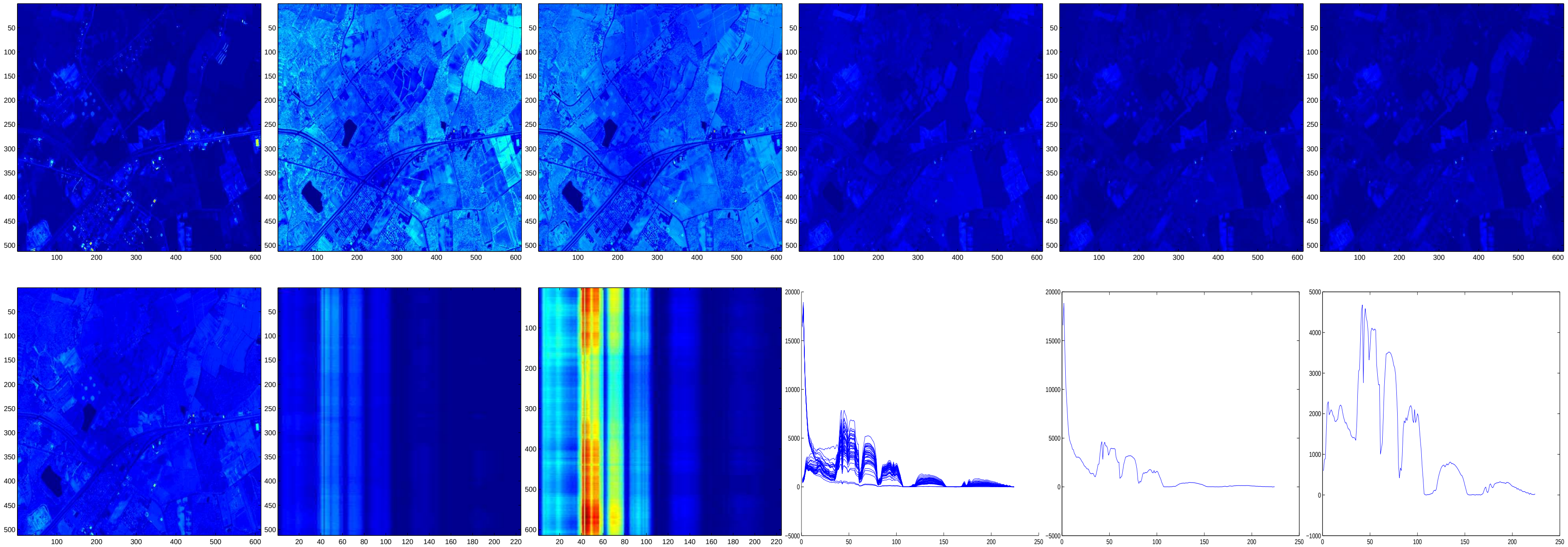


(a) Spectral slice



(b) 3D image cube

Segmentation of hyperspectral images

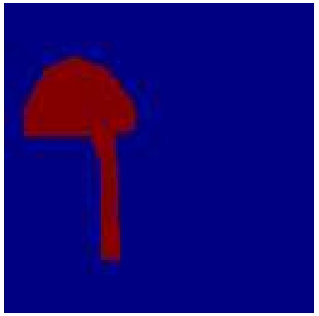


$$g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}), \quad i = 1, \cdots, M$$

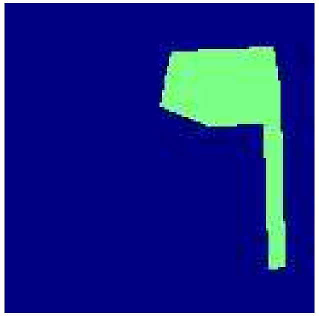
$$\underline{g}(\boldsymbol{r}) = \{g_i(\boldsymbol{r}),\ i = 1, M\}, \qquad \boldsymbol{g}_i = \{g_i(\boldsymbol{r}),\ \boldsymbol{r} \in \mathcal{R}\}, \qquad \underline{\boldsymbol{g}} = \{\boldsymbol{g}_i(\boldsymbol{r}),\ i = 1, M\}$$

$$\underline{g}(\boldsymbol{r}) = \underline{f}(\boldsymbol{r}) + \underline{\epsilon}(\boldsymbol{r}), \qquad \underline{\boldsymbol{g}} = \underline{\boldsymbol{f}} + \underline{\boldsymbol{\epsilon}}$$

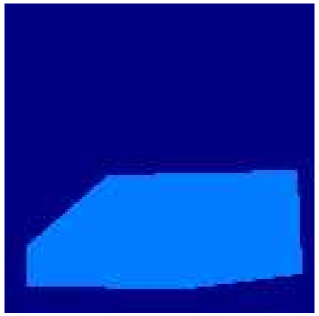
Blind image separation and joint segmentation



$f_1(\boldsymbol{r})$



$f_2(\boldsymbol{r})$

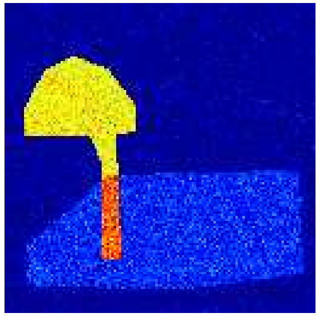


$f_3(\boldsymbol{r})$

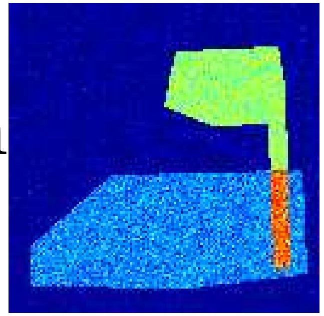
?

Separation

$\Longleftarrow$



$g_1(\boldsymbol{r})$



$g_2(\boldsymbol{r})$

$$g_i(\boldsymbol{r}) = \sum_{j=1}^{N} A_{ij} f_j(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r})$$

$$\underline{g}(\boldsymbol{r}) = \{g_i(\boldsymbol{r}),\ i = 1, M\}$$

$$\underline{g}(\boldsymbol{r}) = \boldsymbol{A}\underline{f}(\boldsymbol{r}) + \underline{\epsilon}(\boldsymbol{r}),$$

$$\underline{\boldsymbol{g}} = \{\boldsymbol{g}_i(\boldsymbol{r}),\ i = 1, M\}$$

$$\boldsymbol{g}_i = \{g_i(\boldsymbol{r}),\ \boldsymbol{r} \in \mathcal{R}\},$$

$$\underline{\boldsymbol{g}} = \boldsymbol{A}\underline{\boldsymbol{f}} + \underline{\boldsymbol{\epsilon}}$$

Bayesian estimation approach

$$\underline{\boldsymbol{g}} = \boldsymbol{A}\underline{\boldsymbol{f}} + \underline{\boldsymbol{\epsilon}}$$

- Forward model and prior knowledge on the noise $\longrightarrow$ $p(\underline{\boldsymbol{g}}|\underline{\boldsymbol{f}})$
- Prior knowledge on $\underline{\boldsymbol{f}}$ $\longrightarrow$ $p(\underline{\boldsymbol{f}})$
- Bayes rule: $p(\underline{\boldsymbol{f}}|\underline{\boldsymbol{g}}) = p(\underline{\boldsymbol{g}}|\underline{\boldsymbol{f}})\; p(\underline{\boldsymbol{f}}) \,/\, p(\underline{\boldsymbol{g}}) \propto p(\underline{\boldsymbol{g}}|\underline{\boldsymbol{f}})\; p(\underline{\boldsymbol{f}})$
- Infer on $\underline{\boldsymbol{f}}$ via $p(\underline{\boldsymbol{f}}|\underline{\boldsymbol{g}})$:
  - MAP estimator:

$$\widehat{\underline{\boldsymbol{f}}} = \arg\max_{\underline{\boldsymbol{f}}} \left\{p(\underline{\boldsymbol{f}}|\underline{\boldsymbol{g}})\right\} = \arg\min_{\boldsymbol{f}} \left\{-\ln p(\underline{\boldsymbol{f}}|\underline{\boldsymbol{g}})\right\}$$

  - MSE estimator $\widehat{\underline{\boldsymbol{f}}} = \int \underline{\boldsymbol{f}}\; p(\underline{\boldsymbol{f}}|\underline{\boldsymbol{g}})\, \mathrm{d}\underline{\boldsymbol{f}}$ posterior mean

Bayesian blind sources separation (BBSS)

$$\underline{\boldsymbol{g}} = \boldsymbol{A}\underline{\boldsymbol{f}} + \underline{\boldsymbol{\epsilon}}$$

- Likelihood $\quad p(\underline{\boldsymbol{g}}|\underline{\boldsymbol{f}}, \boldsymbol{A}, \underline{\boldsymbol{\theta}}_1)$
- Priors $\quad p(\underline{\boldsymbol{f}}|\underline{\boldsymbol{\theta}}_2)$, $p(\boldsymbol{A}|\boldsymbol{\theta}_3)$ and $p(\boldsymbol{\theta})$ $\quad$ hyperparameters $\boldsymbol{\theta} = \{\underline{\boldsymbol{\theta}}_1\, \underline{\boldsymbol{\theta}}_2, \boldsymbol{\theta}_3\}$
- Bayes rule

$$p(\underline{\boldsymbol{f}}, \boldsymbol{A}, \boldsymbol{\theta}|\underline{\boldsymbol{g}}) \propto p(\underline{\boldsymbol{g}}|\underline{\boldsymbol{f}}, \boldsymbol{A}, \underline{\boldsymbol{\theta}}_1)\, p(\underline{\boldsymbol{f}}|\underline{\boldsymbol{\theta}}_2)\, p(\boldsymbol{A}|\boldsymbol{\theta}_3)\, p(\boldsymbol{\theta})$$

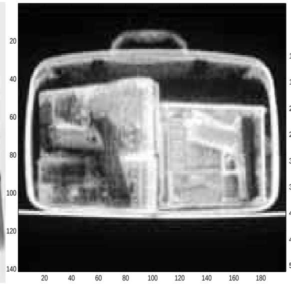
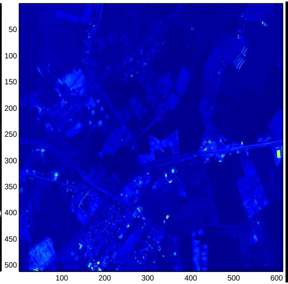
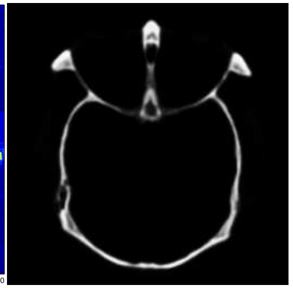
- Generalized a posteriori EM-ML estimator

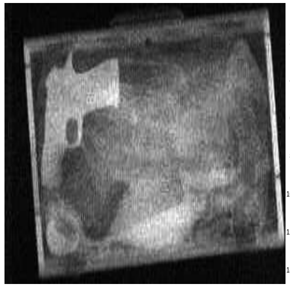
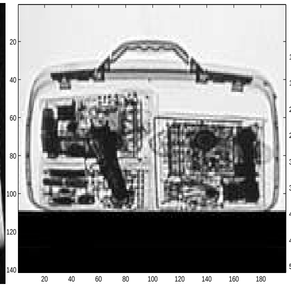
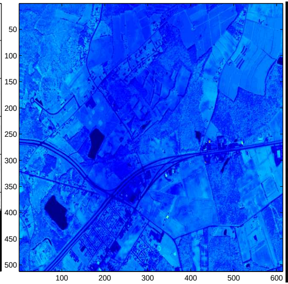
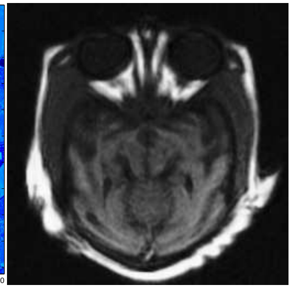
$$(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{A},\boldsymbol{\theta})} \{p(\boldsymbol{A}, \boldsymbol{\theta}|\underline{\boldsymbol{g}})\} \quad \longrightarrow \quad \widehat{\underline{\boldsymbol{f}}} = \arg\max_{\underline{\boldsymbol{f}}} \left\{p(\underline{\boldsymbol{f}}|\underline{\boldsymbol{g}}, \widehat{\boldsymbol{A}}, \widehat{\boldsymbol{\theta}})\right\}$$

- MSE estimator which corresponds to the posterior mean using MCMC

$$\{\widehat{\underline{\boldsymbol{f}}}, \widehat{\boldsymbol{A}}, \widehat{\boldsymbol{\theta}}\} = \int \{\widehat{\underline{\boldsymbol{f}}}, \widehat{\boldsymbol{A}}, \widehat{\boldsymbol{\theta}}\}\, p(\underline{\boldsymbol{f}}, \boldsymbol{A}, \boldsymbol{\theta}|\underline{\boldsymbol{g}})\, \mathrm{d}\{\underline{\boldsymbol{f}}, \widehat{\boldsymbol{A}}, \widehat{\boldsymbol{\theta}}\}.$$

HMM modeling of images

What they share ?

Borders & Regions

Segmentation

Hidden variable $z(\boldsymbol{r})$

$z(\boldsymbol{r}) = k, \; k = 1, \cdots, K$

$\mathcal{R}_k = \{\boldsymbol{r} \; : \; z(\boldsymbol{r}) = k\}, \; \mathcal{R} = \cup_k \mathcal{R}_k$

Homogeneity in regions

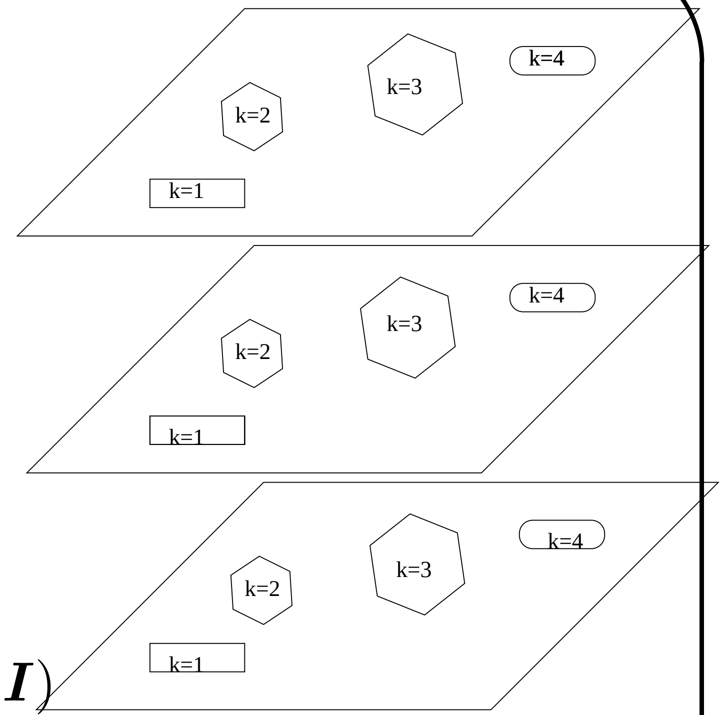
$p(f(\boldsymbol{r}) | z(\boldsymbol{r}) = k) = \mathcal{N}(m_k, \sigma_k^2)$

$$R_{j_k} = \{\boldsymbol{r} \; : \; z_j(\boldsymbol{r}) = k\}, \qquad \mathcal{R}_j = \cup_k R_{j_k}$$

$$\boldsymbol{f}_{j_k} = \{f_j(\boldsymbol{r}) \; : \; \boldsymbol{r} \in R_{j_k}\}, \quad \boldsymbol{f}_j = \cup_k \boldsymbol{f}_{j_k}.$$

$$p(f_j(\boldsymbol{r}), \boldsymbol{r} \in \mathcal{R}) = \prod_{k=1}^{K_j} p(f_j(\boldsymbol{r}), \boldsymbol{r} \in R_{j_k}) = \prod_{k=1}^{K_j} \mathcal{N}(m_{j_k}\mathbf{1}, \sigma_{j_k}^2)$$

Main hypothesis:



- Pixels values in different regions of an image are independent.
- For pixels values in a given region of an image, two possibilities:
  - i.i.d.: $p(f_j(\boldsymbol{r})|z_j(\boldsymbol{r})=k) = \mathcal{N}(m_{j\,k}, \sigma^2_{j\,k})$
    $p(f_j(\boldsymbol{r}), \boldsymbol{r} \in R_{j\,k}) = \mathcal{N}(m_{j\,k}\mathbf{1}, \sigma^2_{j\,k}\boldsymbol{I})$
  - Markovien: $p(f_j(\boldsymbol{r}), \boldsymbol{r} \in R_{j\,k}) = \mathcal{N}(m_{j\,k}\mathbf{1}, \boldsymbol{\Sigma}_{j\,k})$
- For pixels values in different images but in a given common region two possibilities:
  - i.i.d.: $(f_j(\boldsymbol{r})|z(\boldsymbol{r})=k)$ independent of $(f_i(\boldsymbol{r})|z(\boldsymbol{r})=k), \quad i \neq j$
  - Markovien: $p(f_j(\boldsymbol{r})|z(\boldsymbol{r})=k, \underline{\boldsymbol{f}}(\boldsymbol{r})) = p(f_j(\boldsymbol{r})|z(\boldsymbol{r})=k, f_{j-1}(\boldsymbol{r}))$

Modeling the labels

$$p(f_j(\boldsymbol{r})|z_j(\boldsymbol{r})=k)=\mathcal{N}(m_{j\,k},\sigma_{j\,k}^2)\longrightarrow p(f_j(\boldsymbol{r}))=\sum_k P(z_j(\boldsymbol{r})=k)\,\mathcal{N}(m_{j\,k},\sigma_{j\,k}^2)$$

- Independent Gaussian Mixture model (IGM), where $\boldsymbol{z}_j=\{z_j(\boldsymbol{r}),\boldsymbol{r}\in\mathcal{R}\}$ are assumed to be independent and

$$P(z_j(\boldsymbol{r})=k)=p_k,\quad \text{with} \quad \sum_k p_k=1 \;\text{ and }\; p(\boldsymbol{z}_j)=\prod_k p_k$$

- Contextual Gaussian Mixture model (CGM): $\boldsymbol{z}_j$ Markovien

$$p(\boldsymbol{z}_j)\propto\exp\left[\alpha\sum_{\boldsymbol{r}\in\mathcal{R}}\sum_{\boldsymbol{s}\in\mathcal{V}(\boldsymbol{r})}\delta(z_j(\boldsymbol{r})-z_j(\boldsymbol{s}))\right]$$

which is the Potts Markov random feild (PMRF).
The parameter α controls the mean value of the regions' sizes.

Expressions of likelihood, prior and posterior laws

$$\boldsymbol{g}_i = \sum_{j=1}^{N} A_{i,j} \boldsymbol{f}_j + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M \quad \longrightarrow \quad \underline{\boldsymbol{g}} = \boldsymbol{A}\underline{\boldsymbol{f}} + \underline{\boldsymbol{\epsilon}}$$

- Likelihood:

$$p(\underline{\boldsymbol{g}}|\boldsymbol{A}, \underline{\boldsymbol{f}}, \boldsymbol{\theta}_1) = \prod_{i=1}^{M} p(\boldsymbol{g}_i|\boldsymbol{A}, \underline{\boldsymbol{f}}, \boldsymbol{\Sigma}_{\epsilon i}) = \prod_{i=1}^{M} \mathcal{N}(\boldsymbol{A}\underline{\boldsymbol{f}}, \boldsymbol{\Sigma}_{\epsilon i})$$

$$\boldsymbol{\Sigma}_{\epsilon i} = \sigma_{\epsilon i}^2 \boldsymbol{I}, \quad \longrightarrow \quad \boldsymbol{\theta}_1 = \{\sigma_{\epsilon i}^2, i = 1, \cdots, M\}$$

- HMM for the images:

$$p(\underline{\boldsymbol{f}}|\underline{\boldsymbol{z}}, \boldsymbol{\theta}_2) = \prod_{j=1}^{N} p(\boldsymbol{f}_j|\boldsymbol{z}_j, \boldsymbol{m}_j, \boldsymbol{\Sigma}_j)$$

where $\underline{\boldsymbol{z}} = \{\boldsymbol{z}_j, j = 1, \cdots, N\}$ and where we assumed that $\boldsymbol{f}_j|\boldsymbol{z}_j$ are independent. $\longrightarrow \boldsymbol{\theta}_2 = \{(m_{i\,k}, \sigma_{j\,k}^2), j = 1, \cdots, N\}$

- PMRF for the labels:

$$p(\underline{\boldsymbol{z}}) \propto \prod_{j=1}^{N} \exp\left[\alpha \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{s}\in\mathcal{V}(\boldsymbol{r})} \delta(z_j(\boldsymbol{r}) - z_j(\boldsymbol{s}))\right]$$

- Conjugate priors for the hyperparameters $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$:
$\boldsymbol{\theta} = \{\{\sigma_{\epsilon_i}^2, i = 1, \cdots, M\}, \{(m_{i\,k}, \sigma_{j\,k}^2), j = 1, \cdots, M, k = 1, \cdots, K\}\}$

$$\begin{aligned} p(m_{j\,k}) &= \mathcal{N}(m_{j\,k0}, \sigma_{j\,k0}^2) \\ p(\sigma_{j\,k}^2) &= \mathcal{IG}(\alpha_{j0}, \beta_{j0}) \\ p(\boldsymbol{\Sigma}_{j\,k}) &= \mathcal{IW}(\alpha_{j0}, \Lambda_{j0}) \\ p(\sigma_{\epsilon i}) &= \mathcal{IG}(\alpha_{i0}, \beta_{i0}) \end{aligned}$$

- Joint posterior law of $\underline{\boldsymbol{f}}$, $\underline{\boldsymbol{z}}$ and $\underline{\boldsymbol{\theta}}$

$$p(\underline{\boldsymbol{f}}, \underline{\boldsymbol{z}}, \underline{\boldsymbol{\theta}}|\underline{\boldsymbol{g}}) \propto p(\underline{\boldsymbol{g}}|\underline{\boldsymbol{f}}, \boldsymbol{\theta}_1)\; p(\underline{\boldsymbol{f}}|\underline{\boldsymbol{z}}, \boldsymbol{\theta}_2)\; p(\underline{\boldsymbol{z}}|\boldsymbol{\theta}_2)\; p(\underline{\boldsymbol{\theta}})$$

General MCMC sampling scheme

$$p(\underline{\boldsymbol{f}}, \underline{\boldsymbol{z}}, \underline{\boldsymbol{\theta}} | \underline{\boldsymbol{g}}) \propto p(\underline{\boldsymbol{g}} | \underline{\boldsymbol{f}}, \boldsymbol{\theta}_1)\, p(\underline{\boldsymbol{f}} | \underline{\boldsymbol{z}}, \boldsymbol{\theta}_2)\, p(\underline{\boldsymbol{z}} | \boldsymbol{\theta}_2)\, p(\underline{\boldsymbol{\theta}})$$

Gibbs sampling:

- Generate samples $(\underline{\boldsymbol{f}}, \underline{\boldsymbol{z}}, \underline{\boldsymbol{\theta}})^{(1)}, \cdots, (\underline{\boldsymbol{f}}, \underline{\boldsymbol{z}}, \underline{\boldsymbol{\theta}})^{(N)}$ using
  - $\underline{\boldsymbol{f}} \sim p(\underline{\boldsymbol{f}} | \underline{\boldsymbol{g}}, \underline{\boldsymbol{z}}, \underline{\boldsymbol{\theta}}),$
  - $\underline{\boldsymbol{z}} \sim p(\underline{\boldsymbol{z}} | \underline{\boldsymbol{g}}, \underline{\boldsymbol{f}}, \underline{\boldsymbol{\theta}}),$
  - $\underline{\boldsymbol{\theta}} \sim p(\underline{\boldsymbol{\theta}} | \underline{\boldsymbol{g}}, \underline{\boldsymbol{f}}, \underline{\boldsymbol{z}}),$
- Compute any statistics such as mean, median, variance, ...

Difficulties:

- No mixture, No convolution: $\boldsymbol{g}_i = \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M$
- Mixture but No convolution: $\boldsymbol{g}_i = \sum_{j=1}^{N} A_{ij}\, \boldsymbol{f}_j + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M$
- Convolution but No mixture: $\boldsymbol{g}_i = \boldsymbol{H}_i\, \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M$
- Mixture and Convolution: $\boldsymbol{g}_i = \sum_{j=1}^{N} A_{ij}\, \boldsymbol{H}_{ij} \boldsymbol{f}_j + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M$

Examples of applications

- No mixture, No convolution applications:
  - Multi channel image fusion and joint segmentation

$$\boldsymbol{g}_i = \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M, \qquad \boldsymbol{f}_i | \boldsymbol{z} \ \text{ independent}$$

  - Hyperspectral image segmentation

$$\boldsymbol{g}_i = \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M, \qquad \boldsymbol{f}_i | \boldsymbol{z} \ \text{ dependent}$$

  - Video movie segmentation with motion estimation

$$\boldsymbol{g}_i = \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M, \qquad \boldsymbol{f}_i | \boldsymbol{z}_i \ \text{ independent}$$

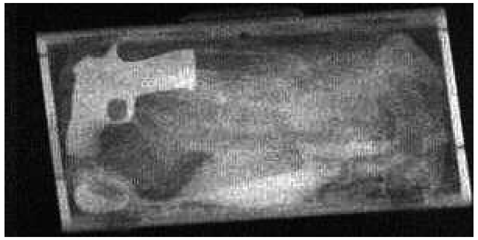
- Mixture, No convolution applications:
  - Blind source (image) separation (BSS) and joint segmentation
    $\boldsymbol{g}_i = \sum_{j=1}^{N} A_{ij}\, \boldsymbol{f}_j + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M, \quad \boldsymbol{f}_i | \boldsymbol{z}_i$ independent
- Convolution but No mixture applications:
  - Fourier synthesis in optical imaging $\quad \boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) + \boldsymbol{\epsilon}, \quad \boldsymbol{f} | \boldsymbol{z}$
  - Single channel image restoration $\quad \boldsymbol{g} = \boldsymbol{H} \boldsymbol{f} + \boldsymbol{\epsilon}, \quad \boldsymbol{f} | \boldsymbol{z}$

Images fusion and joint segmentation

(Olivier FÉRON)

$$\left\{\begin{array}{l} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma_{i\,k}^2) \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}) \end{array}\right.$$

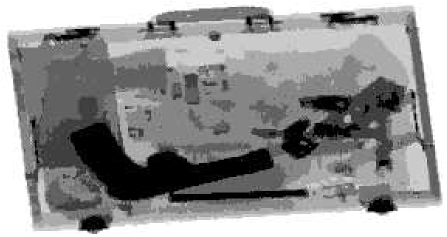
$\boldsymbol{g}_1$

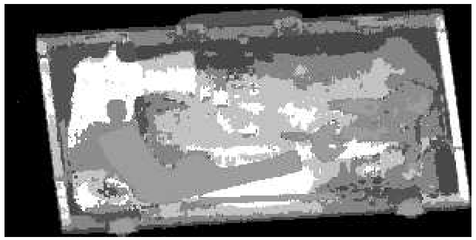


$\boldsymbol{g}_2$

$\longrightarrow$

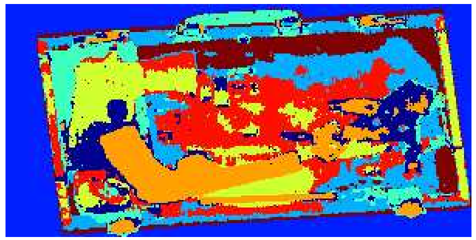
$\widehat{\boldsymbol{f}}_1$





$\widehat{\boldsymbol{f}}_2$

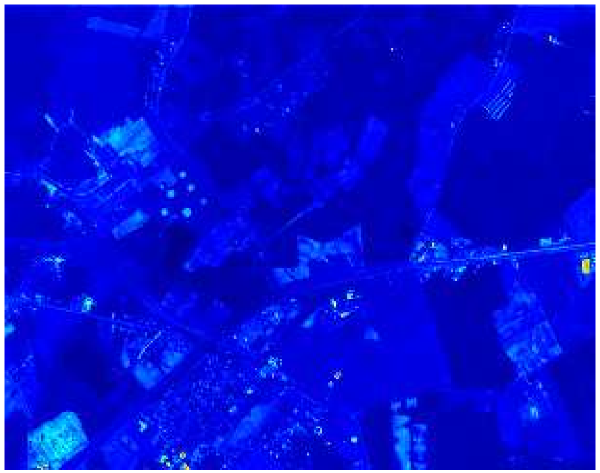
$\widehat{\boldsymbol{z}}$

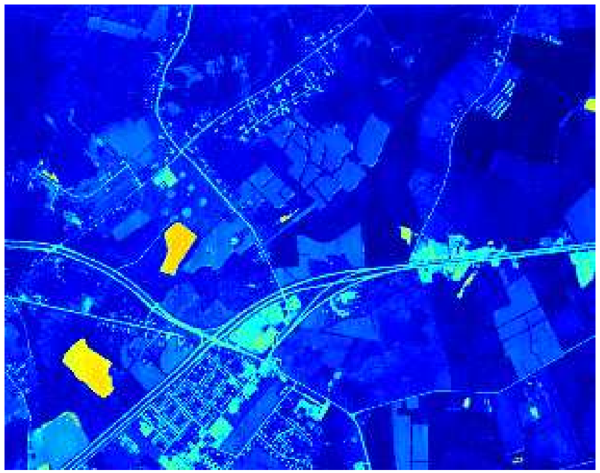


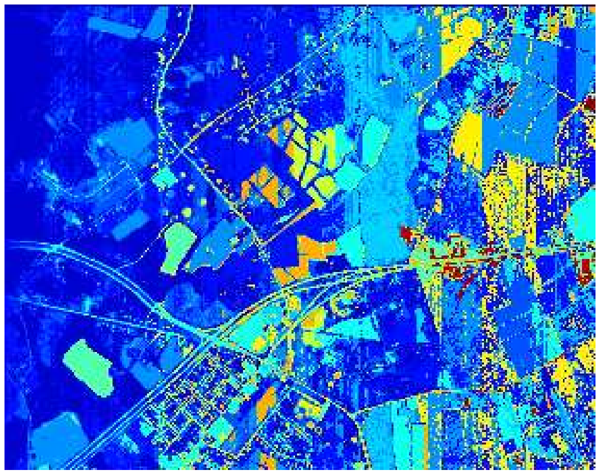
Joint segmentation of hyper-spectral images

(Adel MOHAMMADPOUR)

$$\begin{cases} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}), \quad \boldsymbol{r} \in \mathcal{R}, \text{ or } \boldsymbol{g}_i = \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M \\ p(f_i(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma^2_{i\,k}), \quad k = 1, \cdots, K \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}) \\ m_{i\,k} \text{ follow a Markovian model along the index } i \end{cases}$$







Segmentation of a video sequence of images

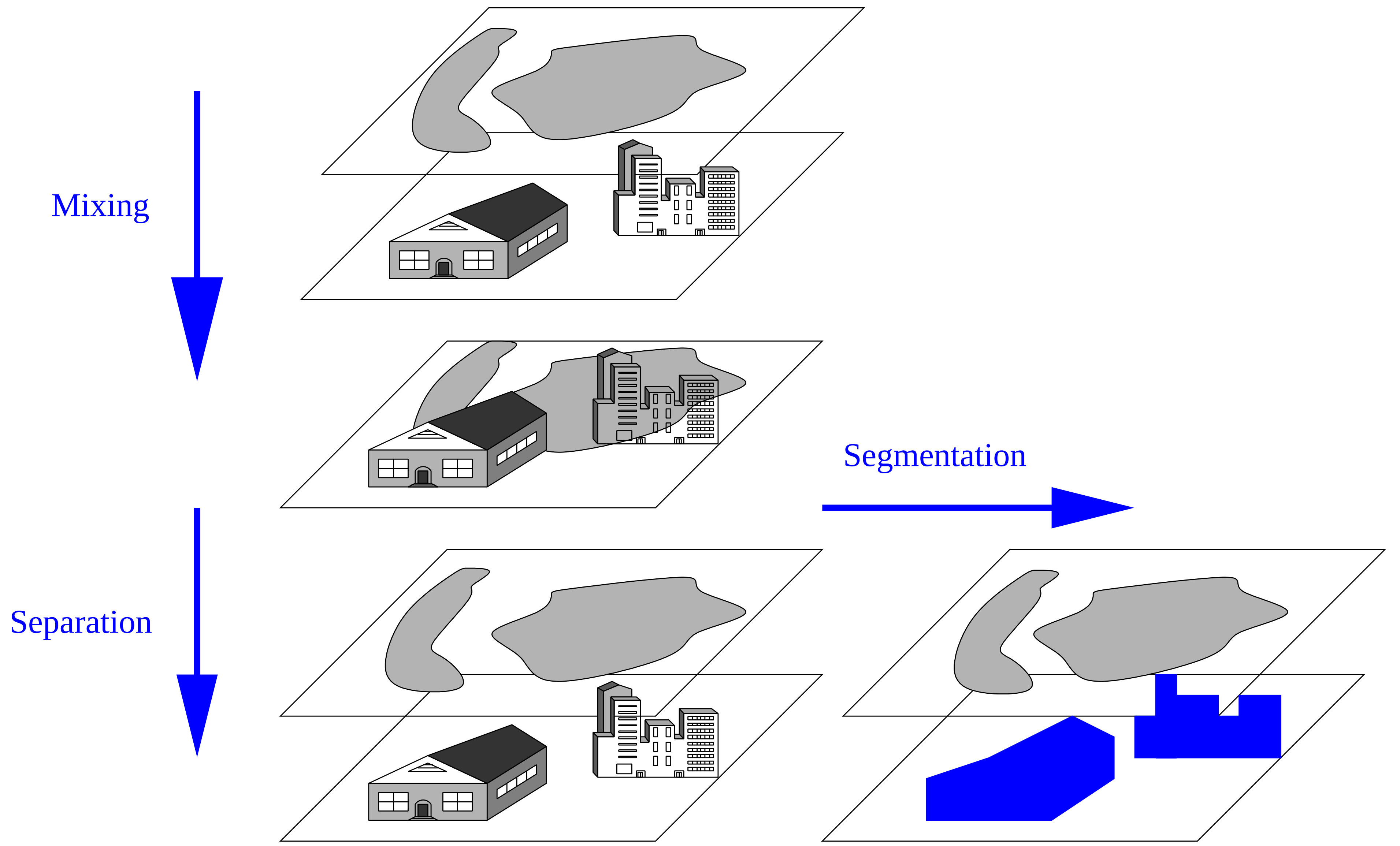
(Patrice BRAULT)

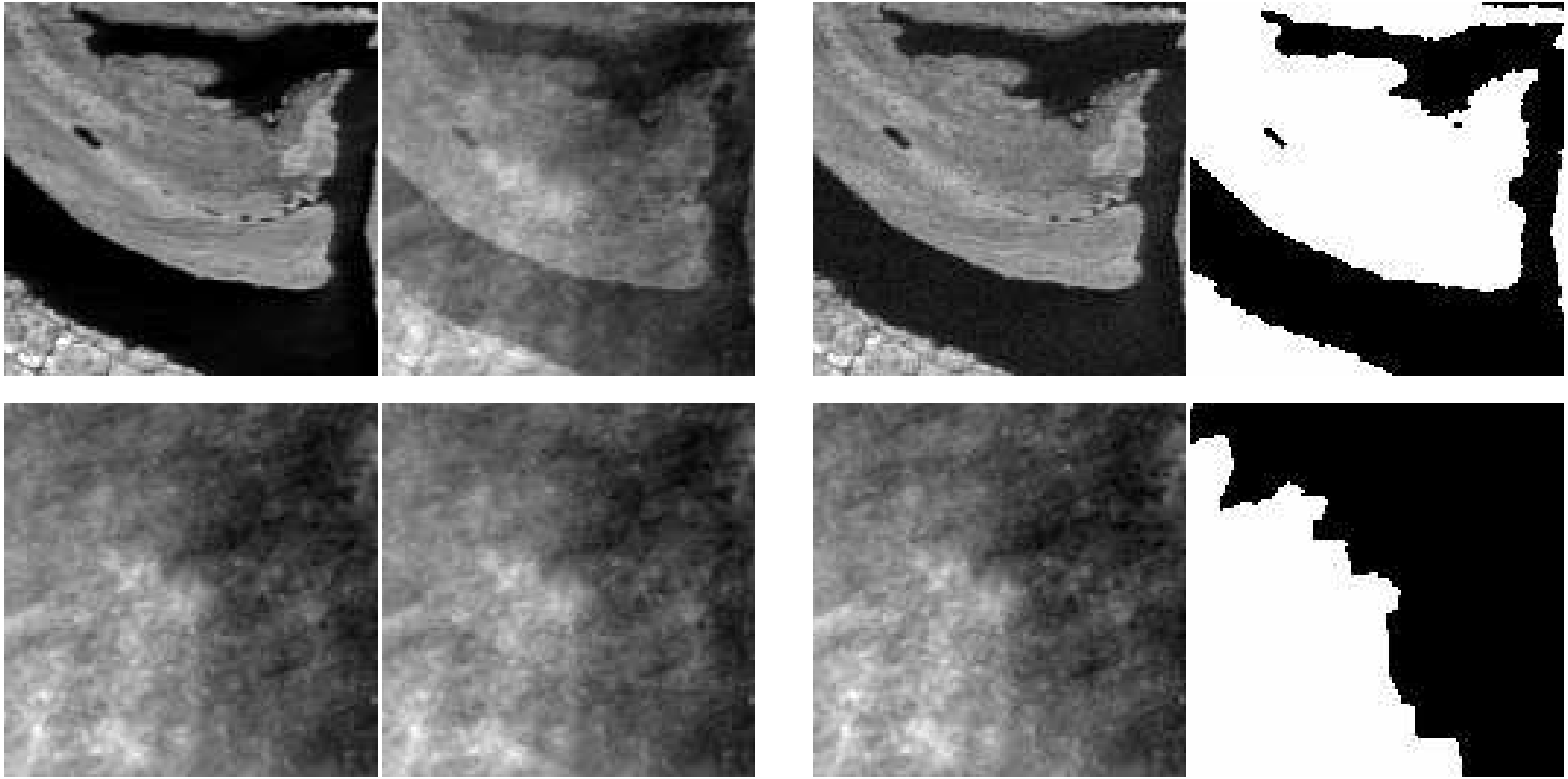
$$
\left\{
\begin{array}{l}
g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}), \quad \boldsymbol{r} \in \mathcal{R}, \quad \text{or} \quad \boldsymbol{g}_i = \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad \text{or} \quad \underline{\boldsymbol{g}} = \underline{\boldsymbol{f}} + \underline{\boldsymbol{\epsilon}} \\
p(f_i(\boldsymbol{r}) | z_i(\boldsymbol{r}) = k) = \mathcal{N}(m_{i\,k}, \sigma_{i\,k}^2), \quad k = 1, \cdots, K \\
p(\underline{\boldsymbol{f}} | \boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i | \boldsymbol{z}_i) \\
z_i(\boldsymbol{r}) \ \text{ follow a Markovian model along the index } \ i
\end{array}
\right.
$$

Blind image separation and joint segmentation

$$\left\{\begin{array}{l}
g_i(\boldsymbol{r}) = \sum_{j=1}^{N} A_{ij} f_j(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \longrightarrow \boldsymbol{g}(\boldsymbol{r}) = \boldsymbol{A}\boldsymbol{f}(\boldsymbol{r}) + \boldsymbol{\epsilon}(\boldsymbol{r}) \longrightarrow \underline{\boldsymbol{g}} = \boldsymbol{A}\underline{\boldsymbol{f}} + \underline{\boldsymbol{\epsilon}} \\
p(\underline{\boldsymbol{g}}|\underline{\boldsymbol{f}}, \boldsymbol{A}, \boldsymbol{\Sigma_\epsilon}) = \prod_{\boldsymbol{r}\in\mathcal{R}} p(\boldsymbol{g}(\boldsymbol{r})|\boldsymbol{f}(\boldsymbol{r}), \boldsymbol{A}) = \prod_{\boldsymbol{r}\in\mathcal{R}} \mathcal{N}(\boldsymbol{A}\boldsymbol{f}(\boldsymbol{r}), \boldsymbol{\Sigma_\epsilon}) \\
p(f_j(\boldsymbol{r})|z_j(\boldsymbol{r}) = k) = \mathcal{N}(m_{j\,k}, \sigma^2_{j\,k}), \\
p(\underline{\boldsymbol{f}}|\underline{\boldsymbol{z}}) = \prod_{\boldsymbol{r}\in\mathcal{R}} \prod_j p(f_j(\boldsymbol{r})|z_j(\boldsymbol{r})) \\
p(A_{ij}) = \mathcal{N}(A_{0\,ij}, \sigma^2_{0\,ij}) \quad \text{or} \quad p(\text{vect}(\boldsymbol{A})) = \mathcal{N}(\text{vect}(\boldsymbol{A}_0), \sigma^2_{0\,ij}\boldsymbol{I})
\end{array}\right.$$

$$\left\{\begin{array}{l}
p(\underline{\boldsymbol{f}}|\underline{\boldsymbol{g}}, \underline{\boldsymbol{z}}, \underline{\boldsymbol{\theta}}, \boldsymbol{A}) = \sum_{\boldsymbol{r}\in\mathcal{R}} \mathcal{N}(\widehat{\boldsymbol{f}}(\boldsymbol{r}), \widehat{\boldsymbol{\Sigma}}(\boldsymbol{r})) \quad \text{with} \\
\widehat{\boldsymbol{\Sigma}}(\boldsymbol{r}) = (\boldsymbol{A}^t \boldsymbol{\Sigma_\epsilon}^{-1} \boldsymbol{A} + \boldsymbol{\Sigma}_z{}^{-1}(\boldsymbol{r}))^{-1} \quad \text{and} \\
\widehat{\boldsymbol{f}}(\boldsymbol{r}) = \widehat{\boldsymbol{\Sigma}}(\boldsymbol{r}) \left(\boldsymbol{A}^t \boldsymbol{\Sigma_\epsilon}^{-1} \boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{\Sigma}_z(\boldsymbol{r})^{-1} \boldsymbol{m}_z(\boldsymbol{r})\right) \\
p(\boldsymbol{z}(\boldsymbol{r})|\boldsymbol{g}(\boldsymbol{r}), \boldsymbol{\theta}, \boldsymbol{A}) \propto (p(\boldsymbol{g}(\boldsymbol{r})|\boldsymbol{z}(\boldsymbol{r}), \boldsymbol{\theta}, \boldsymbol{A}))\ p(\boldsymbol{z}(\boldsymbol{r})) \quad \text{with} \\
p(\boldsymbol{g}(\boldsymbol{r})|\boldsymbol{z}(\boldsymbol{r}), \underline{\boldsymbol{\theta}}) = \mathcal{N}(\boldsymbol{A}\boldsymbol{m}_{z(\boldsymbol{r})}, \boldsymbol{A}\boldsymbol{\Sigma}_{z(\boldsymbol{r})}\boldsymbol{A}^t + \boldsymbol{\Sigma_\epsilon}) \\
p(\boldsymbol{A}, \boldsymbol{\Sigma_\epsilon}|\underline{\boldsymbol{f}}, \underline{\boldsymbol{g}}, \underline{\boldsymbol{\theta}}) = \mathcal{N}(\boldsymbol{A};\ \boldsymbol{A}_p, \boldsymbol{\Sigma}_p)\ \mathcal{W}(\boldsymbol{\Sigma_\epsilon}^{-1};\ \nu_p, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}p})
\end{array}\right.$$


Mixing
Separation
Segmentation



$\underline{f}$ $\underline{g}$ $\underline{\widehat{f}}$ $\underline{\widehat{z}}$

Single channel image restoration

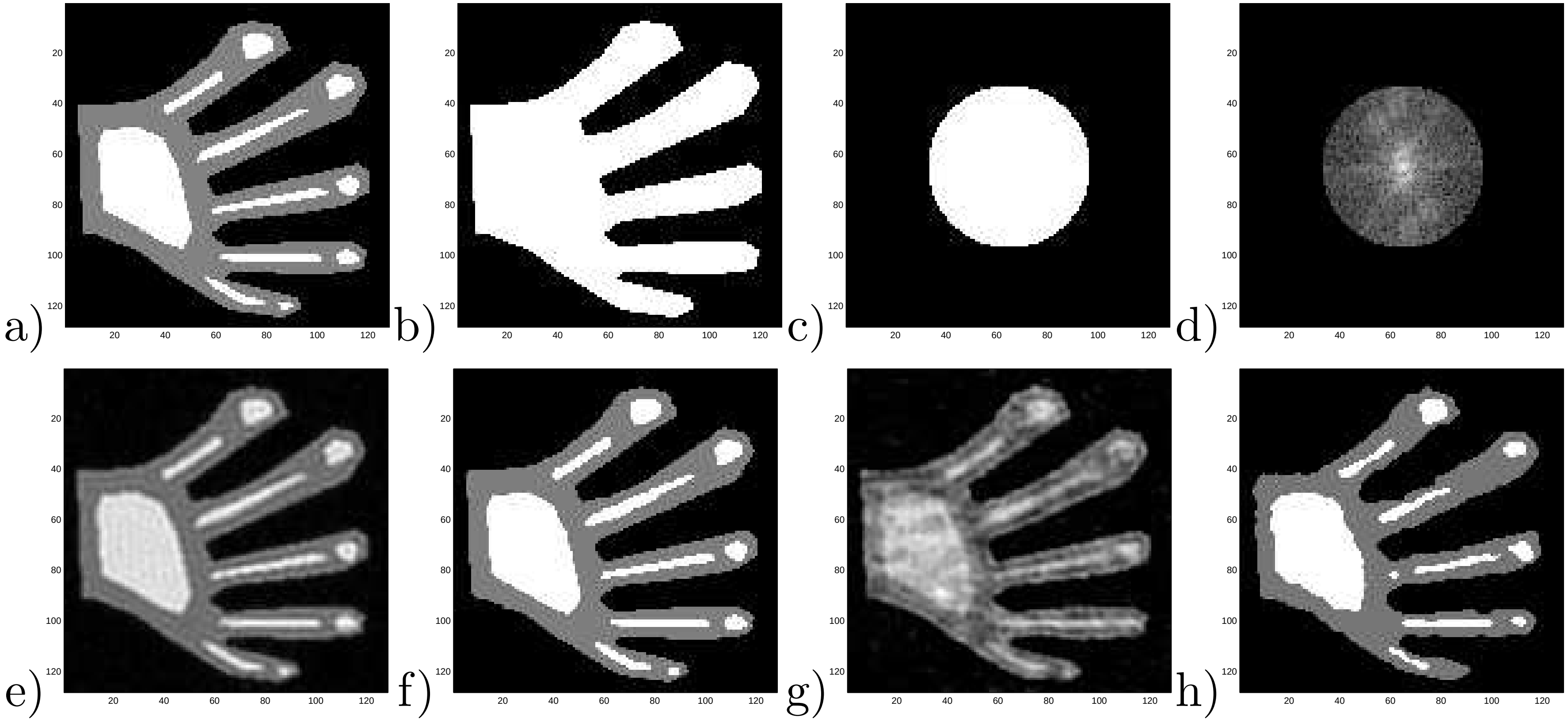
$$g(\boldsymbol{r}') = \int h(\boldsymbol{r}' - \boldsymbol{r})\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{r}') \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

Fourier synthesis inverse problem

$$g(\boldsymbol{\omega}) = \int \exp\left[-j(\boldsymbol{\omega}.\boldsymbol{r})\right]\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{\omega}) \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

$$\left\{\begin{array}{l}
p(\boldsymbol{\epsilon}) = \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}) \longrightarrow p(\boldsymbol{g}|\boldsymbol{f}) = \mathcal{N}(\boldsymbol{H}\boldsymbol{f}, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}) \text{ with } \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} = {\sigma_\epsilon}^2 \boldsymbol{I} \\
p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_k, \sigma_k^2), \quad k = 1, \cdots, K \\
p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta}, \boldsymbol{g}) = \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\Sigma}}) \text{ with} \\
\widehat{\boldsymbol{\Sigma}} = (\boldsymbol{H}^t {\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}}^{-1} \boldsymbol{H} + {\boldsymbol{\Sigma}_z}^{-1})^{-1} \text{ and } \widehat{\boldsymbol{f}} = \widehat{\boldsymbol{\Sigma}} \left(\boldsymbol{H}^t {\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}}^{-1} \boldsymbol{g} + {\boldsymbol{\Sigma}_z}^{-1} \boldsymbol{m}_z\right) \\
\text{Compute } \widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \left\{p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta}, \boldsymbol{g})\right\} = \arg\min_{\boldsymbol{f}} \left\{J(\boldsymbol{f})\right\} \text{ with} \\
J(\boldsymbol{f}) = \frac{1}{\sigma_\epsilon^2} \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \sum_k \frac{\|\boldsymbol{f}_k - m_k \boldsymbol{1}\|^2}{\sigma_k^2} \\
p(\boldsymbol{z}|\boldsymbol{g}, \boldsymbol{\theta}) \propto p(\boldsymbol{g}|\boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{z}) \text{ with} \\
p(\boldsymbol{g}|\boldsymbol{z}, \boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{H}\boldsymbol{m}_z, \boldsymbol{\Sigma}_{\boldsymbol{g}}) \text{ with } \boldsymbol{\Sigma}_{\boldsymbol{g}} = \boldsymbol{H}\boldsymbol{\Sigma}_z \boldsymbol{H}^t + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} \\
\text{Use } p(\boldsymbol{z}|\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{\theta}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{z})
\end{array}\right.$$

Simulation results

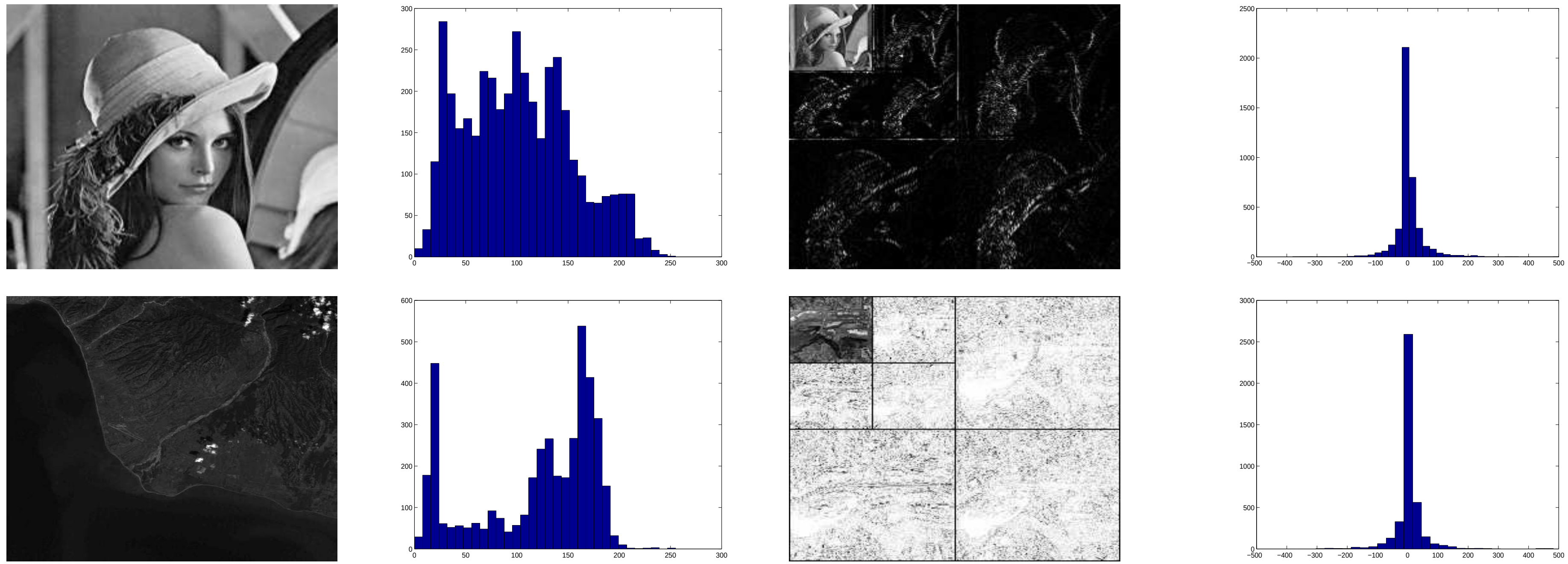


a) object, b) exact known support, c) support of the data, d) measured data, e) and f) Results when phase is measured: e) IFT and f) proposed method, g) and h) Results when the phase is not measured but we know the support of the object: g) by Gerchberg-Saxton h) by the proposed method.

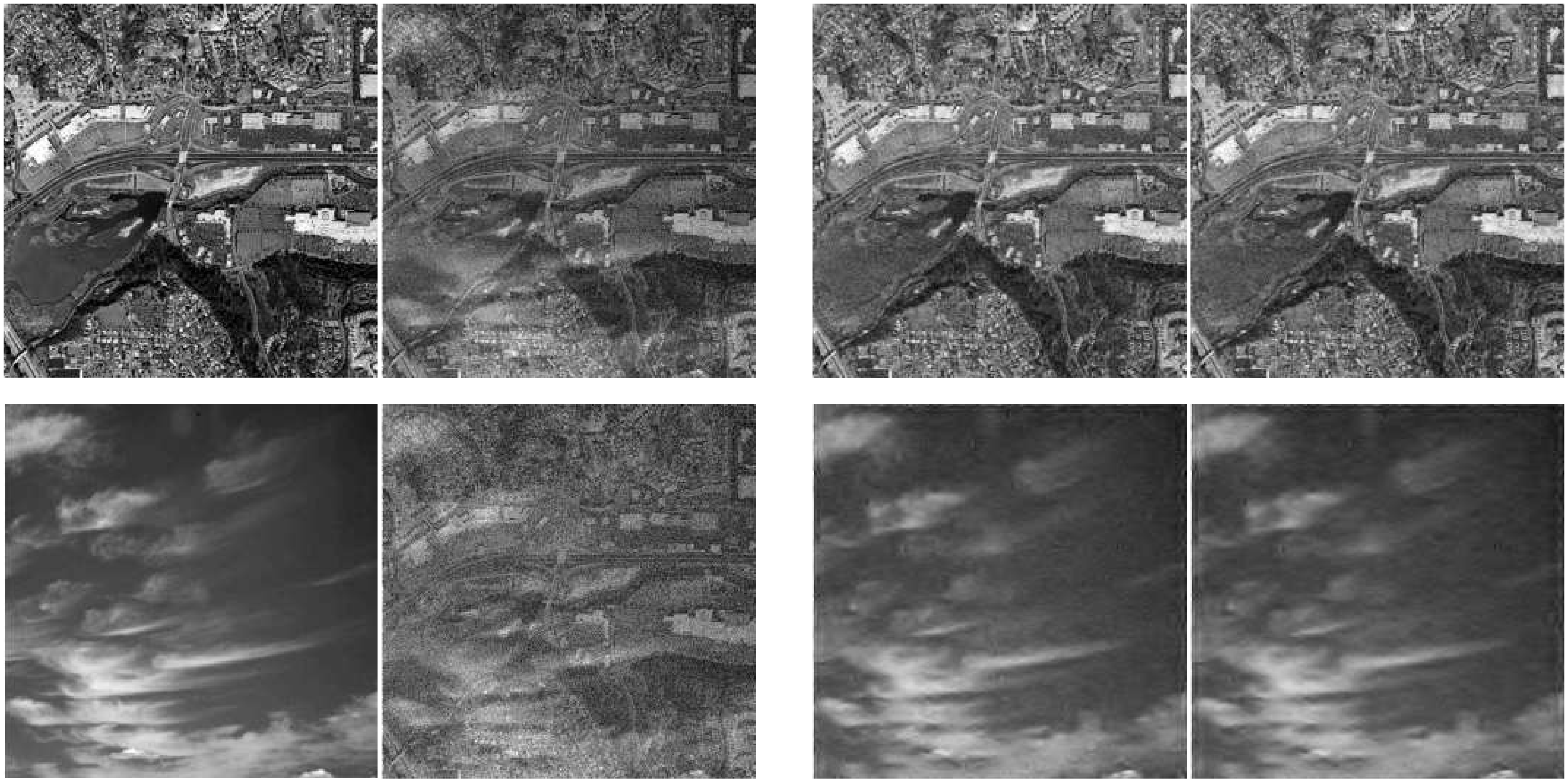
Wavelet domain Bayesian image processing

$$\boldsymbol{g}(\boldsymbol{r}) = \boldsymbol{A}\boldsymbol{f}(\boldsymbol{r}) + \boldsymbol{\epsilon}(\boldsymbol{r}) \longrightarrow \text{WT} \longrightarrow \boldsymbol{g}^j(\boldsymbol{r}) = \boldsymbol{A}\boldsymbol{f}^j(\boldsymbol{r}) + \boldsymbol{\epsilon}^j(\boldsymbol{r})$$

images | Hist. of images | Wavelet coeff. | Hist. of wavelet coeff.



- multi-resolution computation
- Wavelet coefficients can be classified and segmented in only $K = 2$ classes



$\underline{\boldsymbol{f}}$ $\quad$ $\underline{\boldsymbol{g}}$ $\quad$ $\widehat{\underline{\boldsymbol{f}}}$ (GM) $\quad$ $\widehat{\underline{\boldsymbol{f}}}$ (HMM)

Conclusion and works in progress

Bayesian approach & HMM are appropriate tools for many image processing problems

- H. Snoussi : BSS in 1D and 2D either in pixel domain or Fourier transform domain
- M. Ichir : BSS with mixture of Gamma and BSS in wavelet domain
- S. Moussaoui : BSS for non negative sources with application in spectrometry
- O. Féron : Data and image fusion, Fourier synthesis and inverse problems in microwave imaging
- P. Brault : Segmentation of images sequences either directly or in wavelet domain
- A. Mohammadpour : Segmentation of hyper-spectral images,
- Z. Chama : Image recovery from the Fourier phase (Fourier Synthesis)
- F. Humblot : Obtaining a super-resolution image from a set of lower resolution images