Probabilistic Methods for Inverse Problems in Computer Vision

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Inverses problems

- General non linear inverse problem:

$$g(\boldsymbol{s}) = [\mathcal{H} f(\boldsymbol{r})](\boldsymbol{s}) + \epsilon(\boldsymbol{s}), \quad \boldsymbol{r} \in \mathcal{R}, \quad \boldsymbol{s} \in \mathcal{S}$$

- Linear model:

$$g(\boldsymbol{s}) = \int_{\mathcal{R}} f(\boldsymbol{r}) h(\boldsymbol{r}, \boldsymbol{s}) \, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{s})$$

- Discretized version

$$\boldsymbol{g} = \boldsymbol{h}(\boldsymbol{f}) + \boldsymbol{\epsilon} \quad \text{or} \quad \boldsymbol{g} = \boldsymbol{H} \boldsymbol{f} + \boldsymbol{\epsilon}$$

where $\boldsymbol{g} = \{g(\boldsymbol{s}),\ \boldsymbol{s} \in \mathcal{S}\}$, $\boldsymbol{\epsilon} = \{\epsilon(\boldsymbol{s}),\ \boldsymbol{s} \in \mathcal{S}\}$ and $\boldsymbol{f} = \{f(\boldsymbol{r}),\ \boldsymbol{r} \in \mathcal{R}\}$

- Multi sensor imaging

$$\boldsymbol{g}_i = \sum_{j=1}^{N} A_{ij} \boldsymbol{H}_j \boldsymbol{f}_j + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M$$

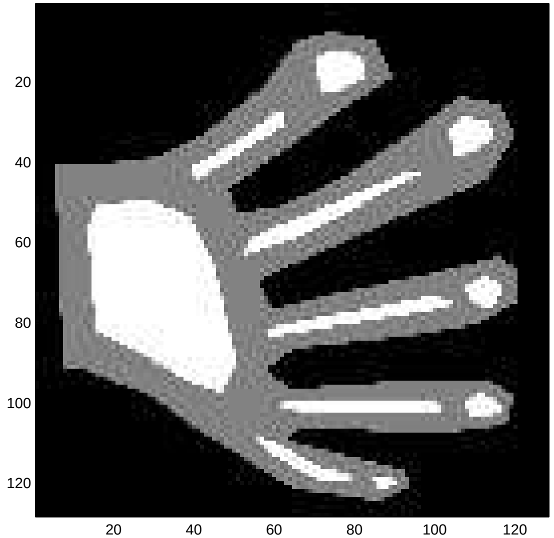
where $\boldsymbol{A} = \{A_{ij}, i = 1, \cdots, M,\ j = 1, \cdots, N\}$ is an unknown mixing matrix.

Fourier synthesis in optical imaging

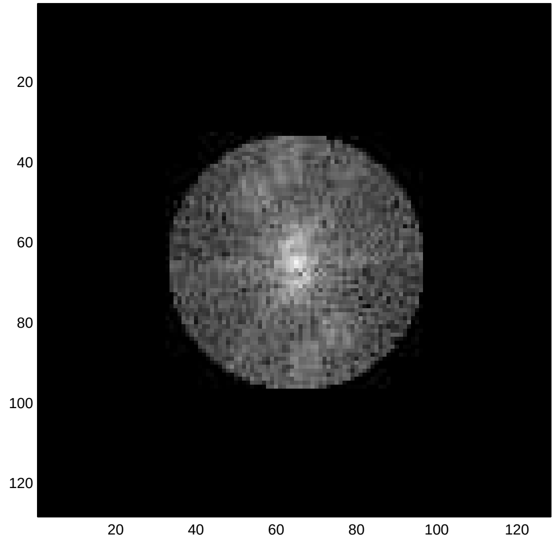
$$g(\boldsymbol{\omega}) = \int f(\boldsymbol{r}) \exp\left[-j\boldsymbol{\omega}^t \boldsymbol{r}\right] \, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{\omega})$$

- Non coherent imaging: $\mathcal{G}(g) = |g| \longrightarrow \boldsymbol{g} = \boldsymbol{h}(\boldsymbol{f}) + \boldsymbol{\epsilon}$
- Coherent imaging: $\mathcal{G}(g) = g \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$

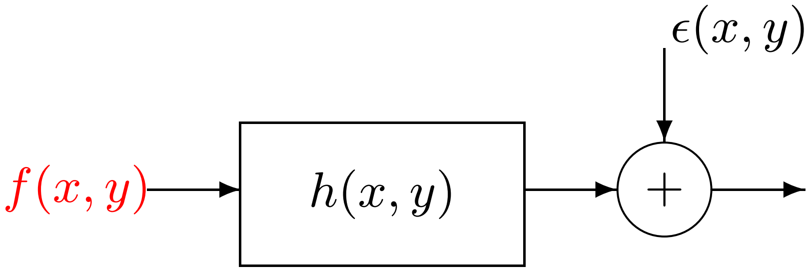
$$\boldsymbol{g} = \{g(\boldsymbol{\omega}),\ \boldsymbol{\omega} \in \Omega\}, \quad \boldsymbol{\epsilon} = \{\epsilon(\boldsymbol{\omega}),\ \boldsymbol{\omega} \in \Omega\} \quad \text{and} \quad \boldsymbol{f} = \{f(\boldsymbol{r}),\ \boldsymbol{r} \in \mathcal{R}\}$$



? $\Longleftarrow$



Single channel image restoration



$$g(x,y) = h(x,y) * f(x,y) + \epsilon(x,y)$$

Observation model : $\boldsymbol{g} = \boldsymbol{H}\,\boldsymbol{f} + \boldsymbol{\epsilon}$

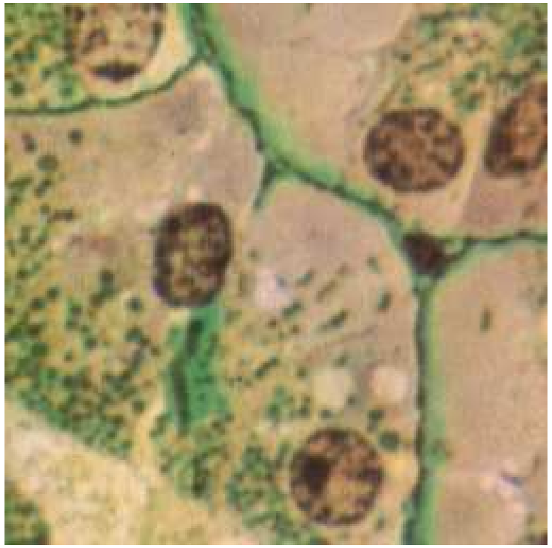
?

$\Longleftarrow$

Color (Multi-spectral) image deconvolution

$$f_i(x,y) \longrightarrow \boxed{h(x,y)} \longrightarrow \bigoplus \xleftarrow{} \epsilon_i(x,y) \longrightarrow g_i(x,y) = h(x,y) * f_i(x,y) + \epsilon_i(x,y)$$

Observation model : $\boldsymbol{g}_i = \boldsymbol{H} \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, 2, 3$



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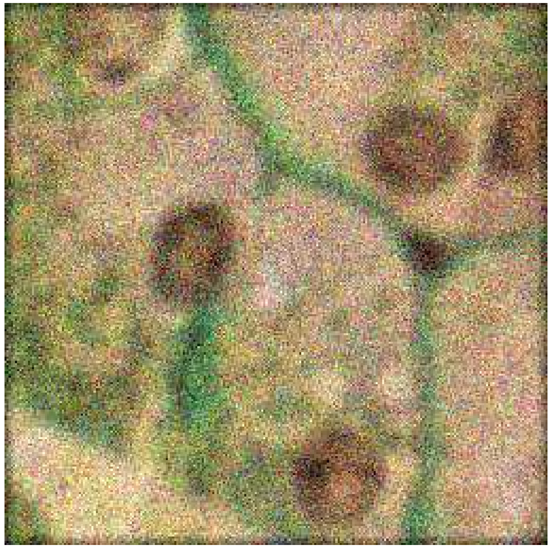
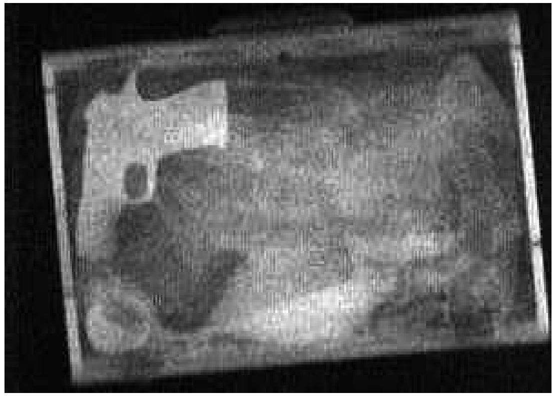


Image fusion and joint segmentation

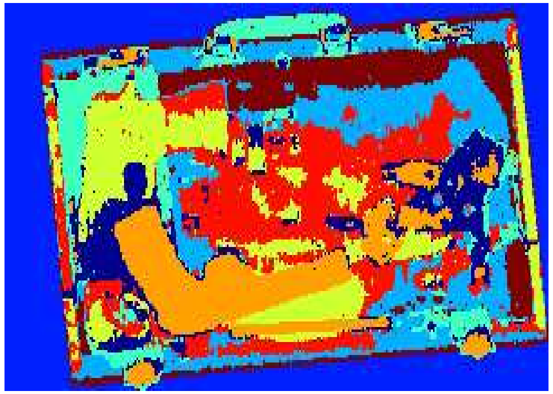
$g_1(\boldsymbol{r})$



$g_2(\boldsymbol{r})$

Fusion ?

$\Longrightarrow$



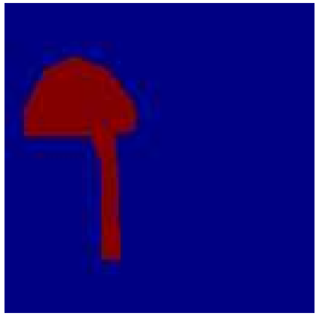
$\boldsymbol{z}$

$$g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}), \quad i = 1, \cdots, M$$

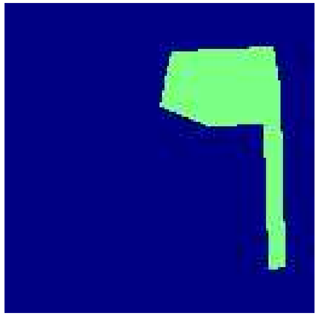
$$\underline{g}(\boldsymbol{r}) = \{g_i(\boldsymbol{r}),\ i = 1, M\}, \qquad \boldsymbol{g}_i = \{g_i(\boldsymbol{r}),\ \boldsymbol{r} \in \mathcal{R}\}, \qquad \underline{\boldsymbol{g}} = \{\boldsymbol{g}_i(\boldsymbol{r}),\ i = 1, M\}$$

$$\underline{g}(\boldsymbol{r}) = \underline{f}(\boldsymbol{r}) + \underline{\epsilon}(\boldsymbol{r}), \qquad \underline{\boldsymbol{g}} = \underline{\boldsymbol{f}} + \underline{\boldsymbol{\epsilon}}$$

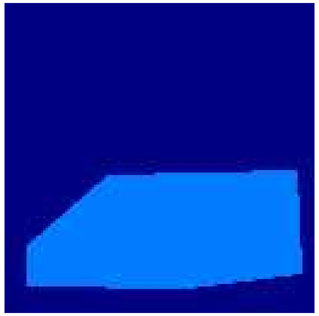
Blind image separation and joint segmentation



$f_1(\boldsymbol{r})$



$f_2(\boldsymbol{r})$

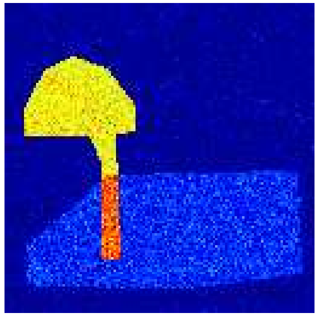


$f_3(\boldsymbol{r})$

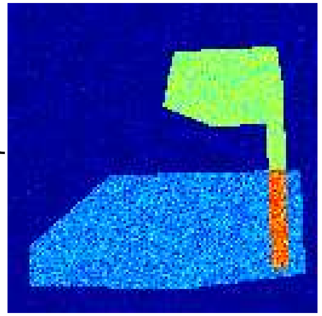
?

Separation

$\Longleftarrow$



$g_1(\boldsymbol{r})$



$g_2(\boldsymbol{r})$

$$g_i(\boldsymbol{r}) = \sum_{j=1}^{N} A_{ij} f_j(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r})$$

$$\underline{g}(\boldsymbol{r}) = \{g_i(\boldsymbol{r}),\ i = 1, M\}$$

$$\underline{g}(\boldsymbol{r}) = \boldsymbol{A}\underline{f}(\boldsymbol{r}) + \underline{\epsilon}(\boldsymbol{r}),$$

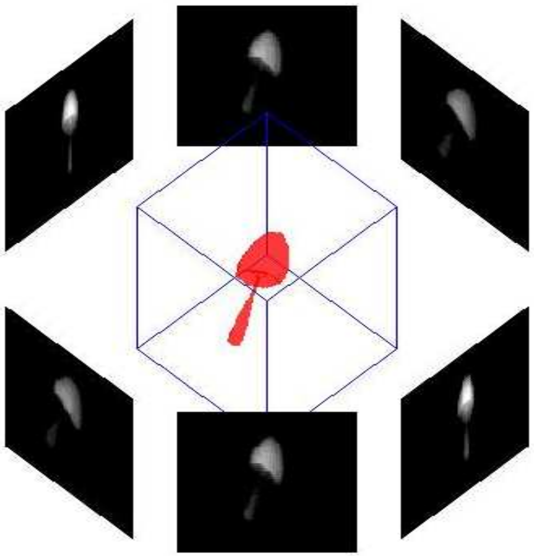
$$\underline{\boldsymbol{g}} = \{\boldsymbol{g}_i(\boldsymbol{r}),\ i = 1, M\}$$

$$\boldsymbol{g}_i = \{g_i(\boldsymbol{r}),\ \boldsymbol{r} \in \mathcal{R}\},$$

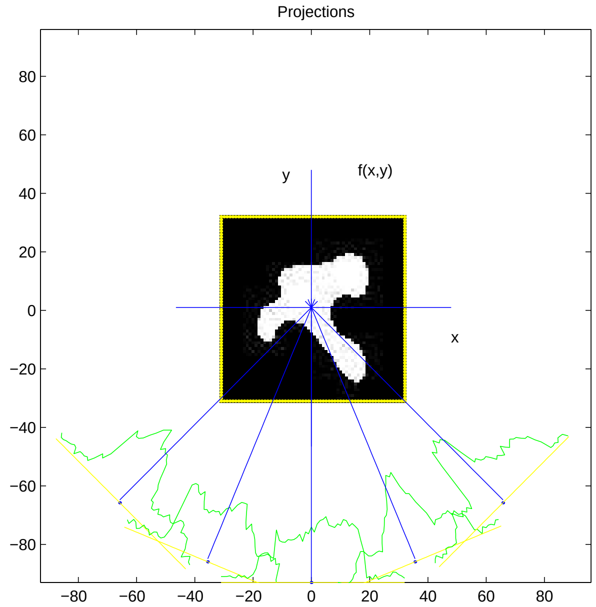
$$\underline{\boldsymbol{g}} = \boldsymbol{A}\underline{\boldsymbol{f}} + \underline{\boldsymbol{\epsilon}}$$

X ray Tomography

3D



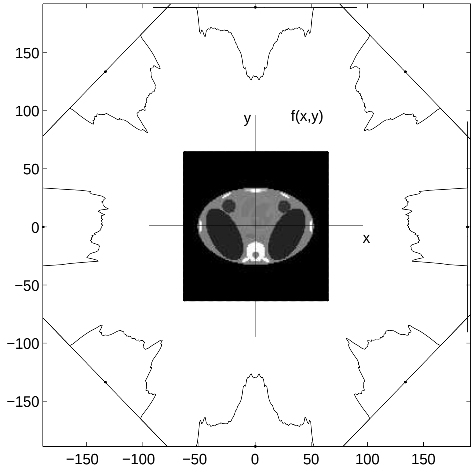
2D



$$g_\phi(r_1,r_2) = \int_{\mathcal{L}_{r_1,r_2,\phi}} f(x,y,z)\,\mathrm{d}l \qquad g_\phi(r) = \int_{\mathcal{L}_{r,\phi}} f(x,y)\,\mathrm{d}l$$

- Forward problem: $f(x,y)$ or $f(x,y,z) \longrightarrow g_\phi(r)$ or $g_{\theta,\phi}(r_1,r_2)$
- Inverse problem: $g_\phi(r)$ or $g_{\phi,\phi}(r_1,r_2) \longrightarrow f(x,y)$ or $f(x,y,z)$

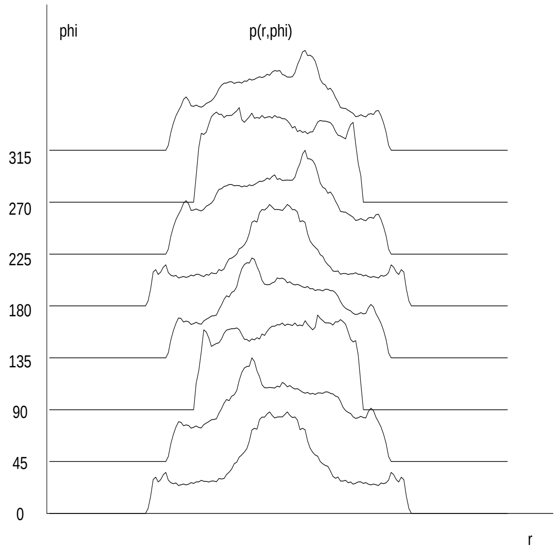
X ray Tomography and Radon Transform



$$f(x, y) \longrightarrow \boxed{\text{TR}} \longrightarrow g(r, \phi)$$

$$g(r, \phi) = \int_{L_{r,\phi}} f(x, y) \, \mathrm{d}l$$

$$g(r, \phi) = \iint_D f(x, y) \, \delta(r - x \cos\phi - y \sin\phi) \, \mathrm{d}x \, \mathrm{d}y$$



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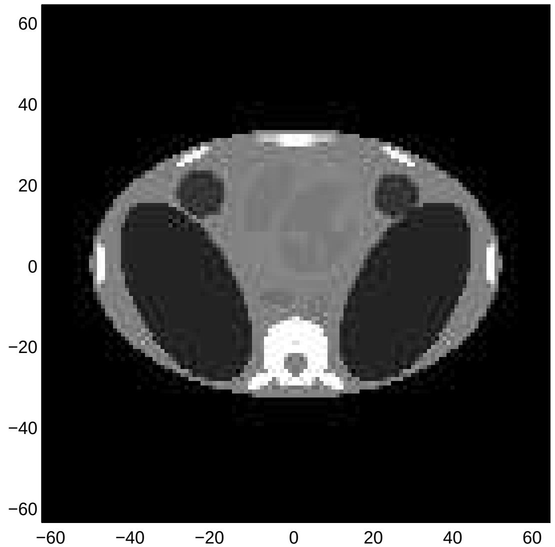
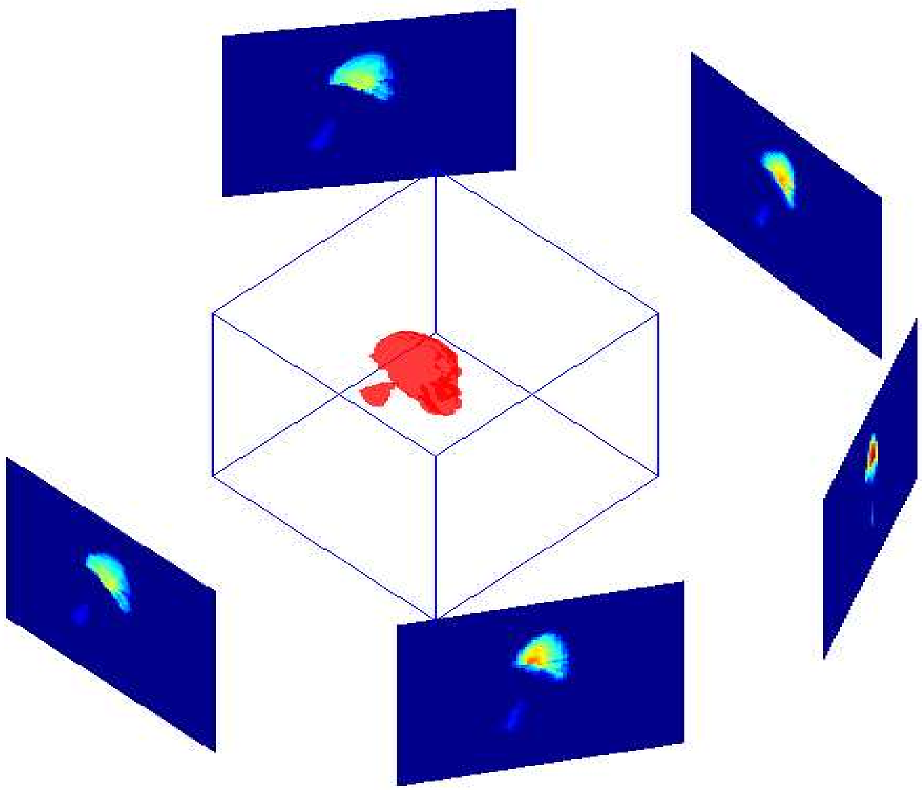
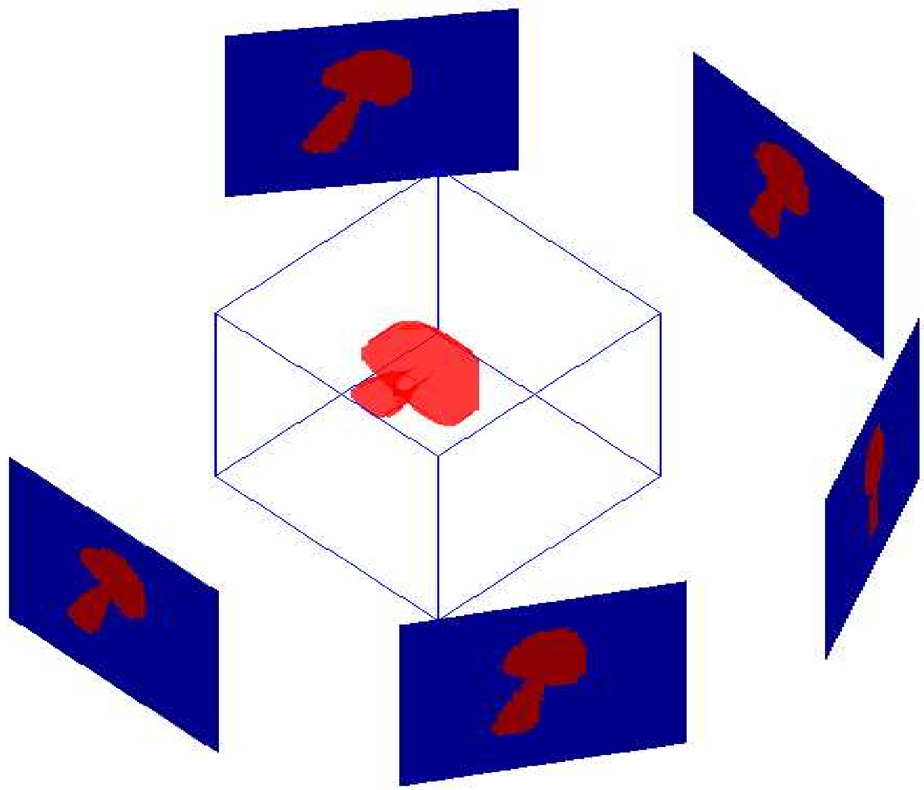


3D Computed Tomography / 3D Shape from shadows

3D Computed Tomography

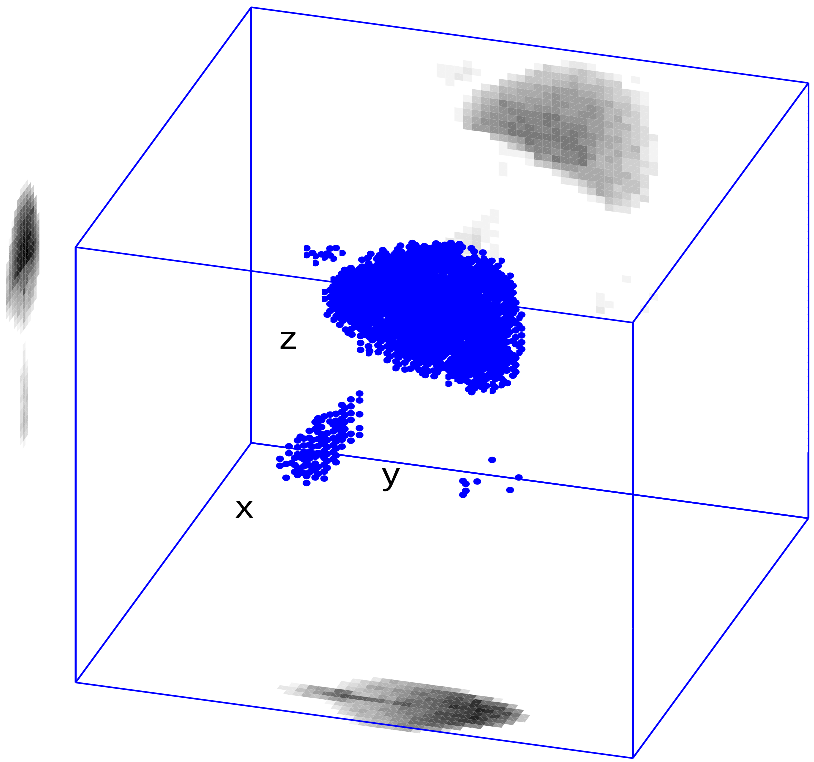


3D Shape from shadows

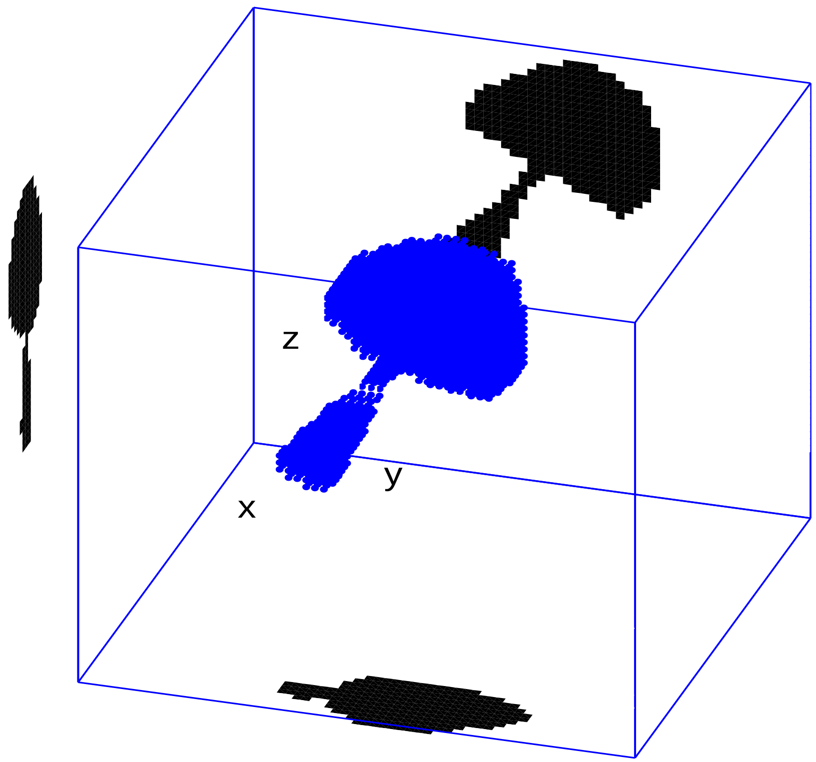


3D Computed Tomography / 3D Shape from shadows

3D Computed Tomography



3D Shape from shadows



Deterministic methods

Data matching

- Observation model $\quad g_i = h_i(\boldsymbol{f}) + \epsilon_i, \quad i = 1, \ldots, M \longrightarrow \boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f})\boldsymbol{\epsilon}$
- Misatch between data and output of the model $\Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f}))$

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{\Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f}))\}$$

- Examples:
  - LS $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \|\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})\|^2 = \sum_i |g_i - h_i(\boldsymbol{f})|^2$
  - $L_p \quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \|\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})\|^p = \sum_i |g_i - h_i(\boldsymbol{f})|^p, \quad 1 < p < 2$
  - KL $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \sum_i g_i \ln \frac{g_i}{h_i(\boldsymbol{f})}$

- In general, does not give satisfactory results for inverse problems.

Regularization theory

Inverse problems = Ill posed problems

$\longrightarrow$ Need of prior information

- Minimum norme LS (MNLS): $J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda||\boldsymbol{f}||^2$
- Classical regularization: $J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda||\boldsymbol{D}\boldsymbol{f}||^2$
- More general regularization:

$$J(\boldsymbol{f}) = \mathcal{Q}(\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})) + \lambda\Phi(\boldsymbol{D}\boldsymbol{f})$$

or

$$J(\boldsymbol{f}) = \Delta_1(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) + \lambda\Delta_2(\boldsymbol{f}, \boldsymbol{f}_\infty)$$

Limitations:

- Errors are considered implicitly white and Gaussian
- Limited prior information on the solution
- Lack of tools for the determination of the hyperparameters

Probabilistic methods

Taking account of errors and uncertainties $\longrightarrow$ Probability theory

- Maximum Likelihood (ML)
- Minimum Inaccuracy (MI)
- Probability Distribution Matching (PDM)
- Maximum Entropy (ME) and Information Theory (IT)

- Bayesian Inference (Bayes)

Advantages:

- Explicit account of the errors and noise
- A large class of priors via explicit or implicit modeling
- A coherent approach to combine information content of the data and priors

Limitations:

- Practical implementation and cost of calculation

Bayesian estimation approach

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Observation model + Hypothesis on the noise $\longrightarrow p(\boldsymbol{g}|\boldsymbol{f}) = p_{\boldsymbol{\epsilon}}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f})$
- A priori information $\quad p(\boldsymbol{f})$
- Bayes : $\quad p(\boldsymbol{f}|\boldsymbol{g}) = \dfrac{p(\boldsymbol{g}|\boldsymbol{f})\, p(\boldsymbol{f})}{p(\boldsymbol{g})}$
- Choice of a point estimator based on $p(\boldsymbol{f}|\boldsymbol{g})$

Link with regularisation :

- Maximum A Posteriori (MAP) :

$$\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \left\{p(\boldsymbol{f}|\boldsymbol{g})\right\} = \arg\max_{\boldsymbol{f}} \left\{p(\boldsymbol{g}|\boldsymbol{f})\; p(\boldsymbol{f})\right\}$$

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \left\{-\ln p(\boldsymbol{g}|\boldsymbol{f}) - \ln p(\boldsymbol{f})\right\}$$

with

$$Q(\boldsymbol{g}, \boldsymbol{H}\boldsymbol{f}) = -\ln p(\boldsymbol{g}|\boldsymbol{f}) \quad \text{and} \quad \lambda\Omega(\boldsymbol{f}) = -\ln p(\boldsymbol{f})$$

Case of linear models and Gaussian priors

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Hypothesis on the noise:
  $\boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{0}, \sigma_\epsilon^2 \boldsymbol{I}) \longrightarrow \boldsymbol{g}|\boldsymbol{f} \sim \mathcal{N}(\boldsymbol{H}\boldsymbol{f}, \sigma_\epsilon^2 \boldsymbol{I}) \longrightarrow p(\boldsymbol{g}|\boldsymbol{f}) \propto \exp\left[-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2\right]$
- Hypothesis on $\boldsymbol{f}$:
  $$\boldsymbol{f} \sim \mathcal{N}(\boldsymbol{f}_0, \sigma_f^2(\boldsymbol{D}^t\boldsymbol{D})^{-1}) \longrightarrow p(\boldsymbol{f}) \propto \exp\left[-\frac{1}{2\sigma_f^2}\|\boldsymbol{D}[\boldsymbol{f} - \boldsymbol{f}_0]\|^2\right]$$
- A posteriori:
  $$p(\boldsymbol{f}|\boldsymbol{g}) \propto \exp\left[-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 \frac{1}{2\sigma_f^2}\|\boldsymbol{D}[\boldsymbol{f} - \boldsymbol{f}_0]\|^2\right]$$
- MAP : $\quad \widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$
  with $\quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}(\boldsymbol{f} - \boldsymbol{f}_0)\|^2, \qquad \lambda = \frac{\sigma_\epsilon^2}{\sigma_f^2}$
- Advantage : caracterization of the solution

$$\boldsymbol{f}|\boldsymbol{g} \sim \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{P}}) \text{ with } \widehat{\boldsymbol{f}} = \widehat{\boldsymbol{P}}\boldsymbol{H}^t(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}_0), \quad \widehat{\boldsymbol{P}} = \left(\boldsymbol{H}^t\boldsymbol{H} + \lambda\boldsymbol{D}^t\boldsymbol{D}\right)^{-1}$$

MAP estimation with other priors:

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{avec} \quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\Omega(\boldsymbol{f})$$

Separable priors:

- Gaussian prior:

$$p(f_j) \propto \exp\left[-\alpha(f_j - m_j)^2\right] \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j (f_j - m_j)^2$$

- Gamma prior:

$$p(f_j) \propto (f_j/m_j)^\alpha \exp\left[-f_j/m_j\right] \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j \ln \frac{f_j}{m_j} + \frac{f_j}{m_j},$$

- Beta prior:

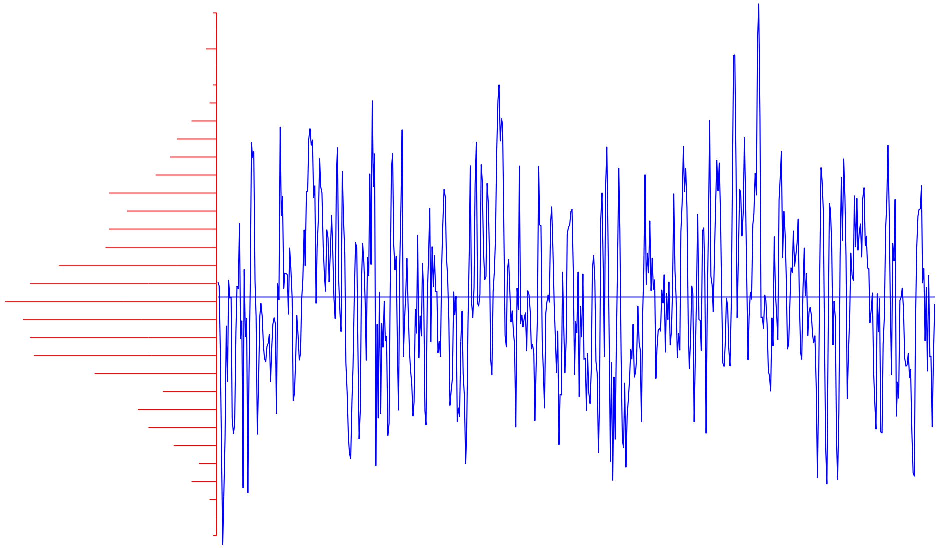
$$p(f_j) \propto f_j^\alpha (1 - f_j)^\beta \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j \ln f_j + \beta \sum_j \ln(1 - f_j),$$

- Generalized gaussienne:

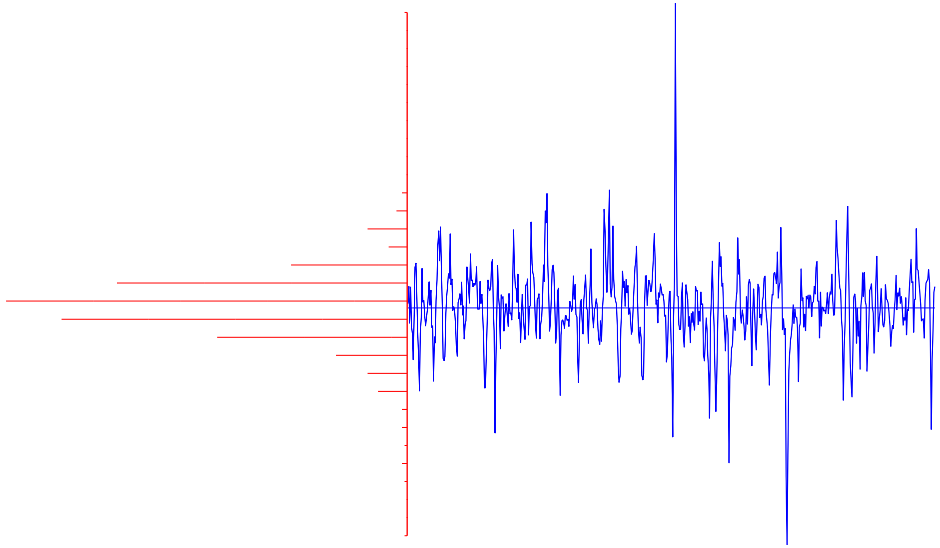
$$p(f_j) \propto \exp\left[-\alpha|f_j - m_j|^p\right], \quad 1 < p < 2 \longrightarrow \quad \Phi(\boldsymbol{f}) = \alpha \sum_j |f_j - m_j|^p,$$

Markovian models:

$$p(f_j|\boldsymbol{f}) \propto \exp\left[-\alpha \sum_{i \in N_j} \phi(f_j, f_i)\right] \longrightarrow \quad \Phi(\boldsymbol{f}) = \alpha \sum_j \sum_{i \in N_j} \phi(f_j, f_i),$$



Gaussian



Generalized Gaussian

$$p(f_j|f_{j-1}) = \mathcal{N}(f_{j-1}, \sigma_f^2)$$

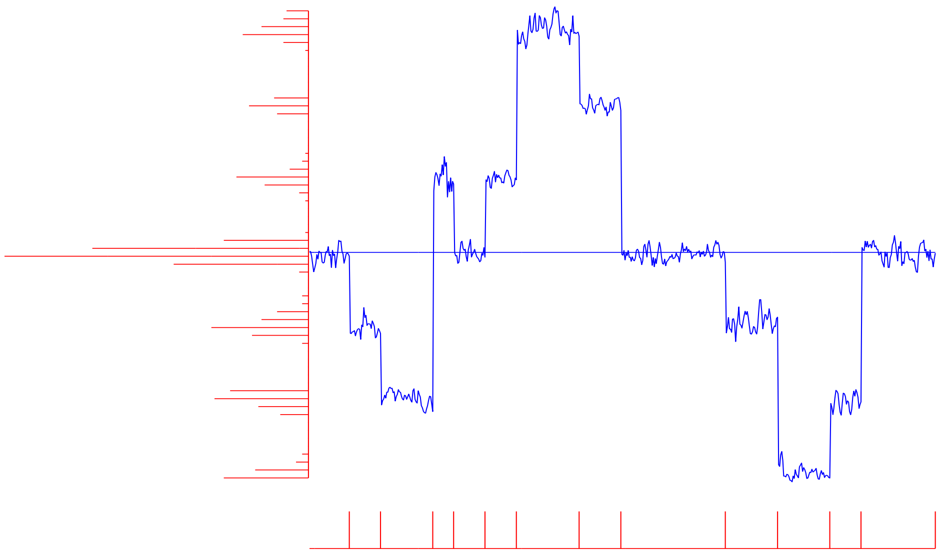
$$p(\boldsymbol{f}) \propto \exp\left[-\alpha \sum_j |f_j - f_{j-1}|^2\right]$$

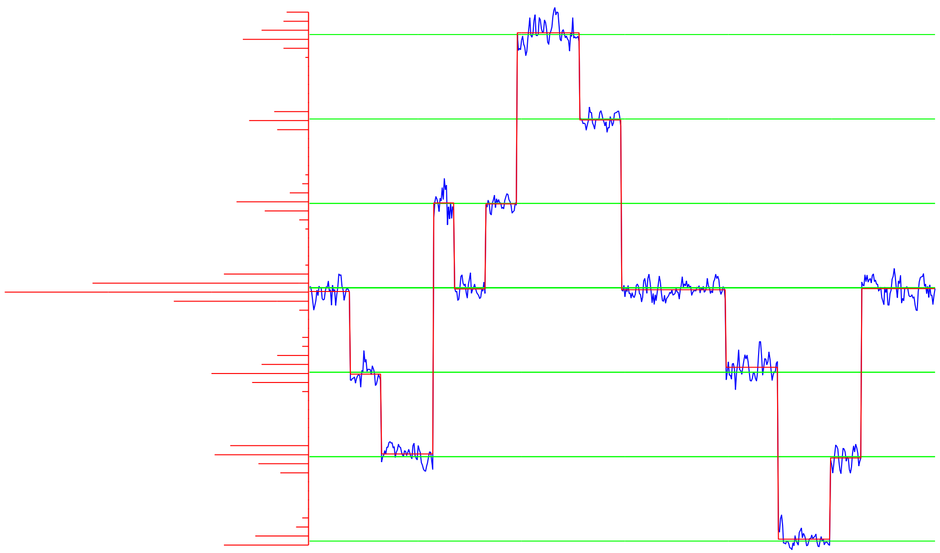
$$p(\boldsymbol{f}) \propto \exp\left[-\alpha \sum_{\boldsymbol{r}\in\mathcal{R}} \left|f(\boldsymbol{r}) - \beta \sum_{\boldsymbol{s}\in\mathcal{V}(\boldsymbol{r})} f(\boldsymbol{s})\right|^2\right]$$

$$p(f_j|f_{j-1}) \propto \exp\left[-\alpha(f_j - f_{j-1})^p\right]$$

$$p(\boldsymbol{f}) \propto \exp\left[-\alpha \sum_j |f_j - f_{j-1}|^p\right]$$

$$p(\boldsymbol{f}) \propto \exp\left[-\alpha \sum_{\boldsymbol{r}\in\mathcal{R}} \left|f(\boldsymbol{r}) - \beta \sum_{\boldsymbol{s}\in\mathcal{V}(\boldsymbol{r})} f(\boldsymbol{s})\right|^p\right]$$





Picewize Gaussian (Line process)

$$p(f_j|q_j, f_{j-1}) = \mathcal{N}\left((1-q_j)f_{j-1}, \sigma_f^2\right)$$

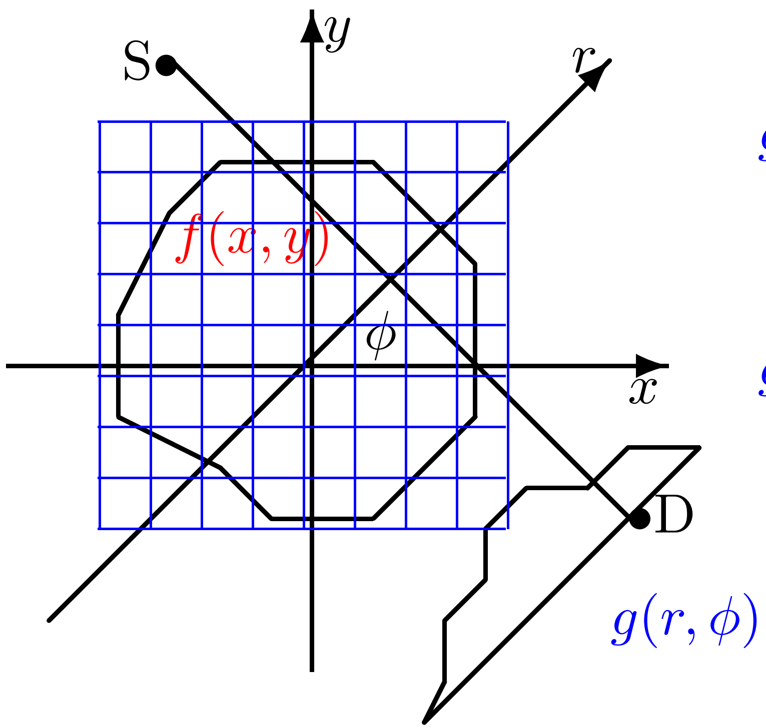
$$p(\boldsymbol{f}|\boldsymbol{q}) \propto \exp\left[-\alpha \sum_j (1-q_j)|f_j - f_{j-1}|^2\right]$$

Mixture of Gaussians (Label process)

$$p(f_j|z_j = k, f_{j-1}) = \mathcal{N}\left(m_k, \sigma_k^2\right)$$

$$p(\boldsymbol{f}|\boldsymbol{z}) \propto \exp\left[-\alpha \sum_k \sum_{j\in\mathcal{R}_k} (f_j - m_k/\sigma_k)^2\right]$$

X ray Tomography and Radon Transform



$$g(r,\phi) = -\ln\left(\frac{I}{I_0}\right) = \int_L f(x,y)\,\mathrm{d}l$$

$$g(r,\phi) = \iint_D f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

RADON:

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right)\int_0^{\pi}\int_{-\infty}^{+\infty}\frac{\frac{\partial}{\partial r}g(r,\phi)}{(r - x\cos\phi - y\sin\phi)}\,\mathrm{d}r\,\mathrm{d}\phi$$

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right) \int_0^{\pi} \int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} g(r,\phi)}{(r - x\cos\phi - y\sin\phi)} \,\mathrm{d}r\,\mathrm{d}\phi$$

$$\text{Derivation } \mathcal{D} : \qquad \bar{g}(r,\phi) = \frac{\partial g(r,\phi)}{\partial r}$$

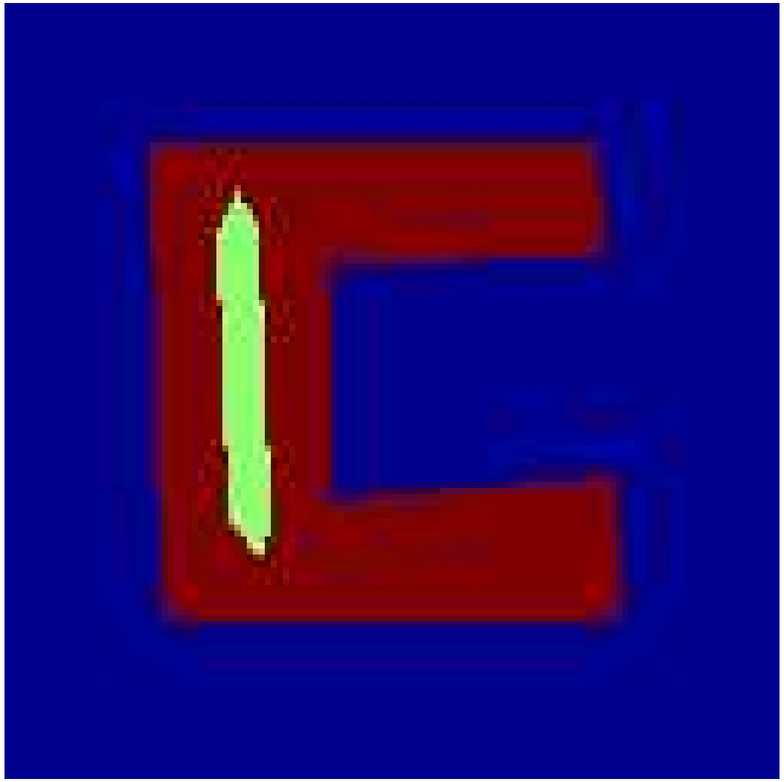
$$\text{Hilbert Transform} \mathcal{H} : \qquad g_1(r',\phi) = \frac{1}{\pi} \int_0^{\infty} \frac{\bar{g}(r,\phi)}{(r - r')} \,\mathrm{d}r$$

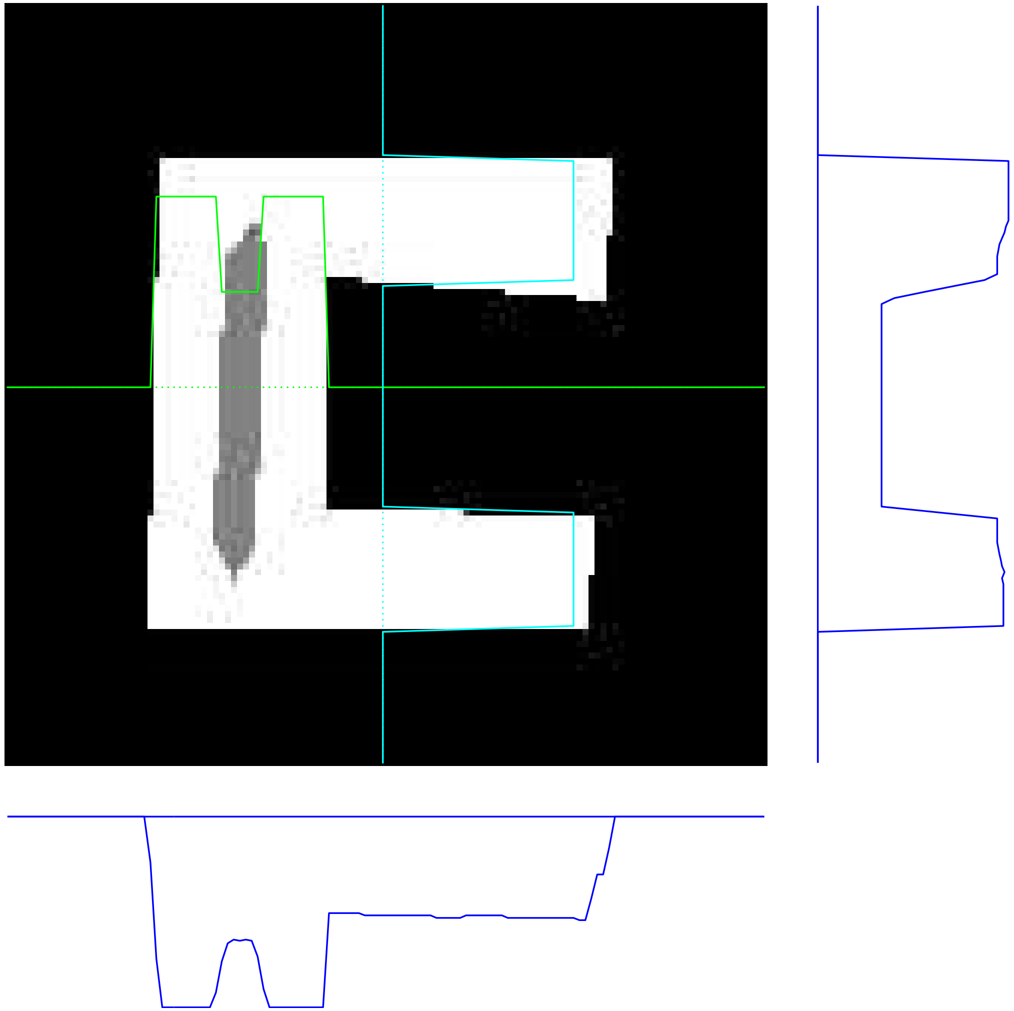
$$\text{Backprojection } \mathcal{B} : \qquad f(x,y) = \frac{1}{2\pi} \int_0^{\pi} g_1(r' = x\cos\phi + y\sin\phi, \phi) \,\mathrm{d}\phi$$

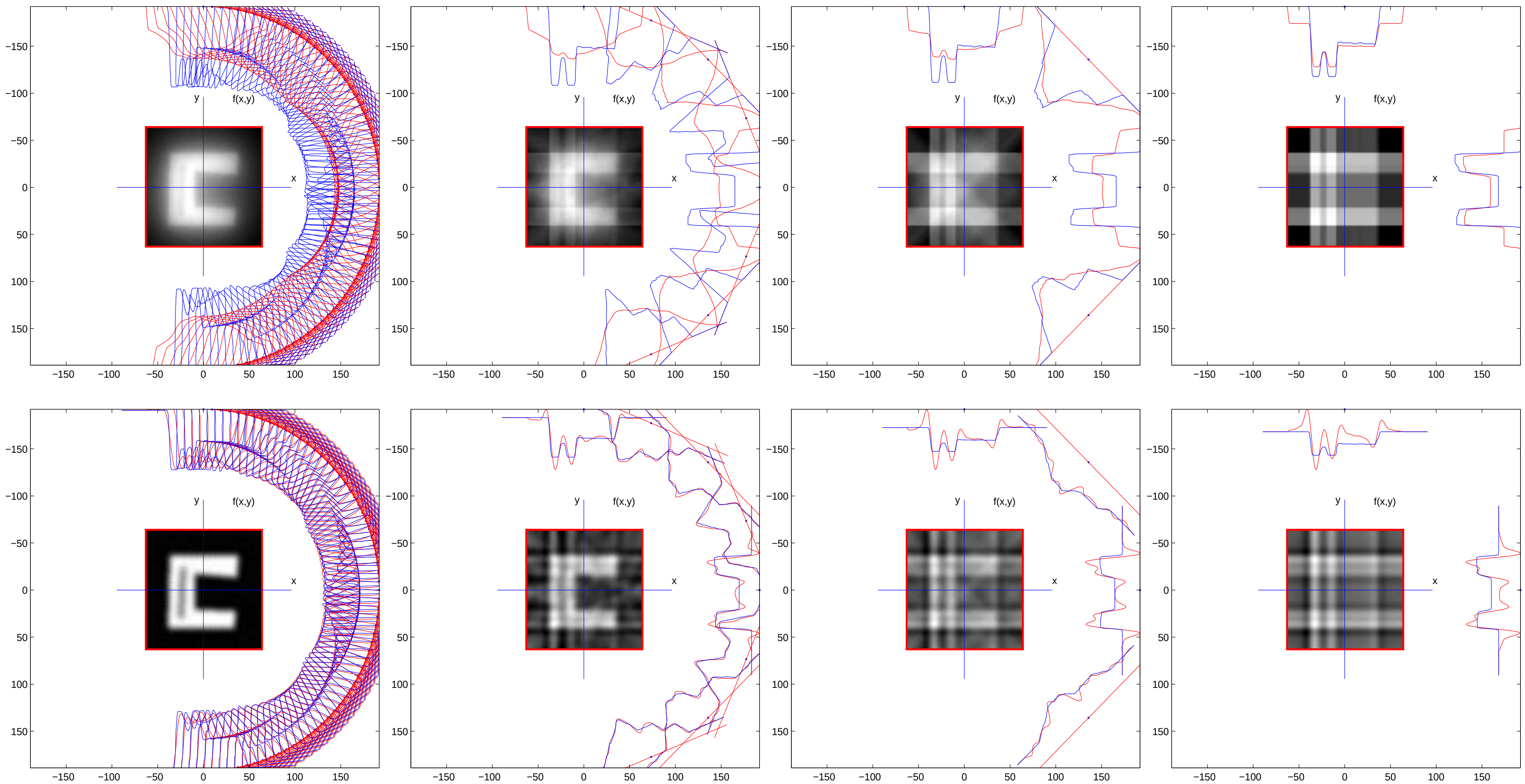
$$f = \mathcal{B}\,\mathcal{H}\,\mathcal{D}\,\mathcal{R}\, f$$

- Backprojection of filtered projections:

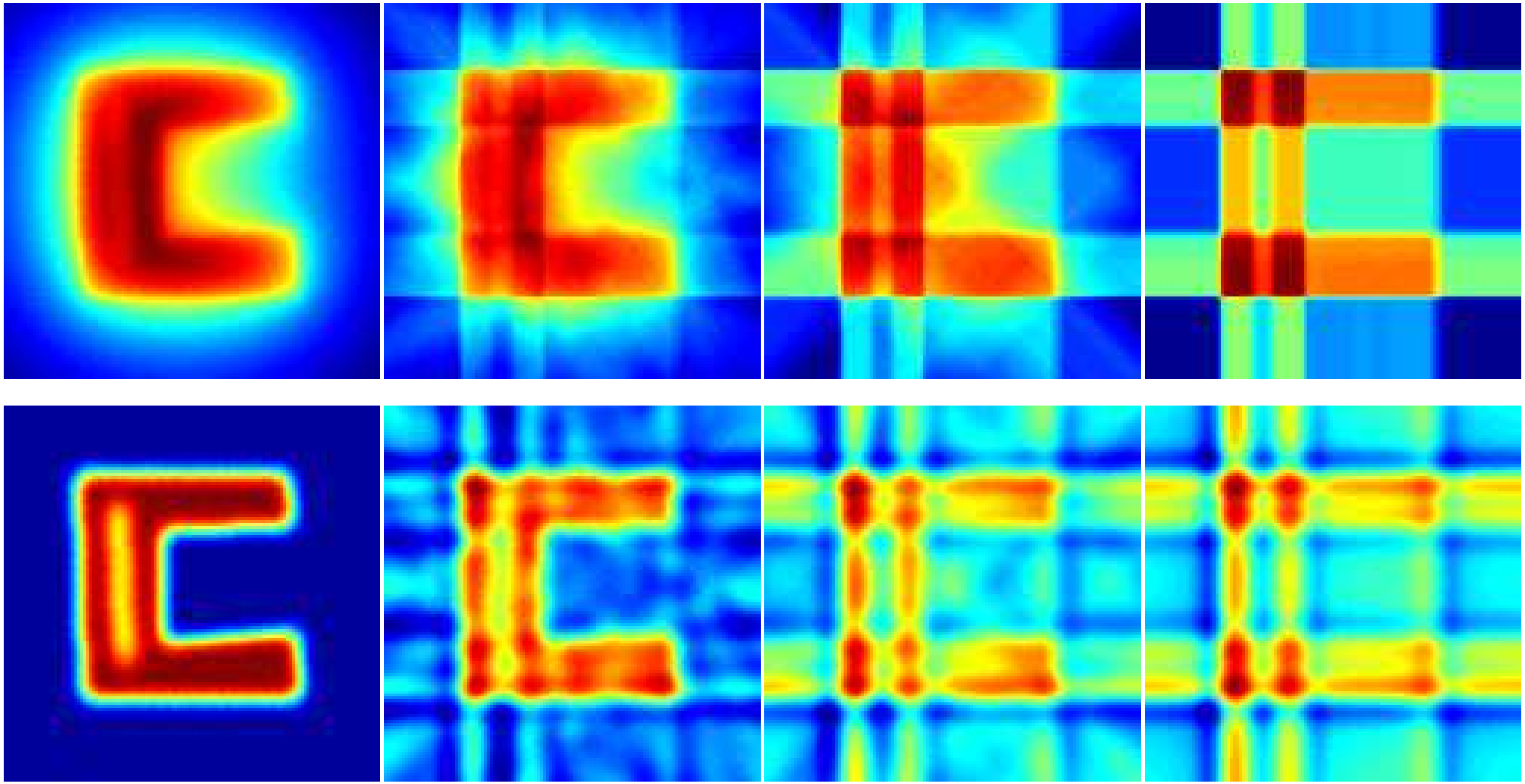
$$\xrightarrow{g(r,\phi)} \boxed{\begin{array}{c}\text{FT}\\ \mathcal{F}_1\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{Filter}\\ |\Omega|\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{IFT}\\ \mathcal{F}_1^{-1}\end{array}} \xrightarrow{g_1(r,\phi)} \boxed{\begin{array}{c}\text{Backprojection}\\ \mathcal{B}\end{array}} \xrightarrow{f(x,y)}$$



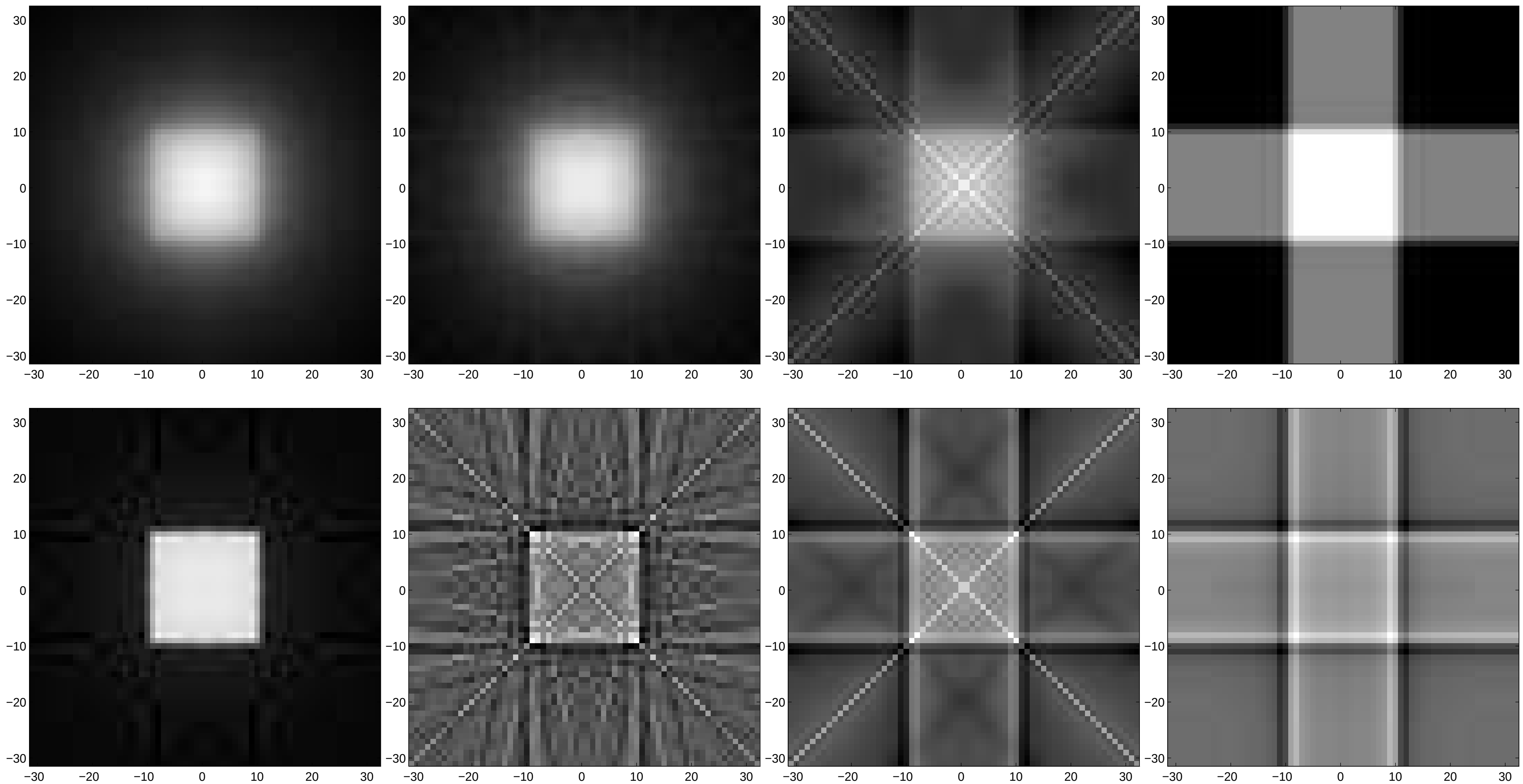




64 projections | 16 projections | 4 projections | 2 projections

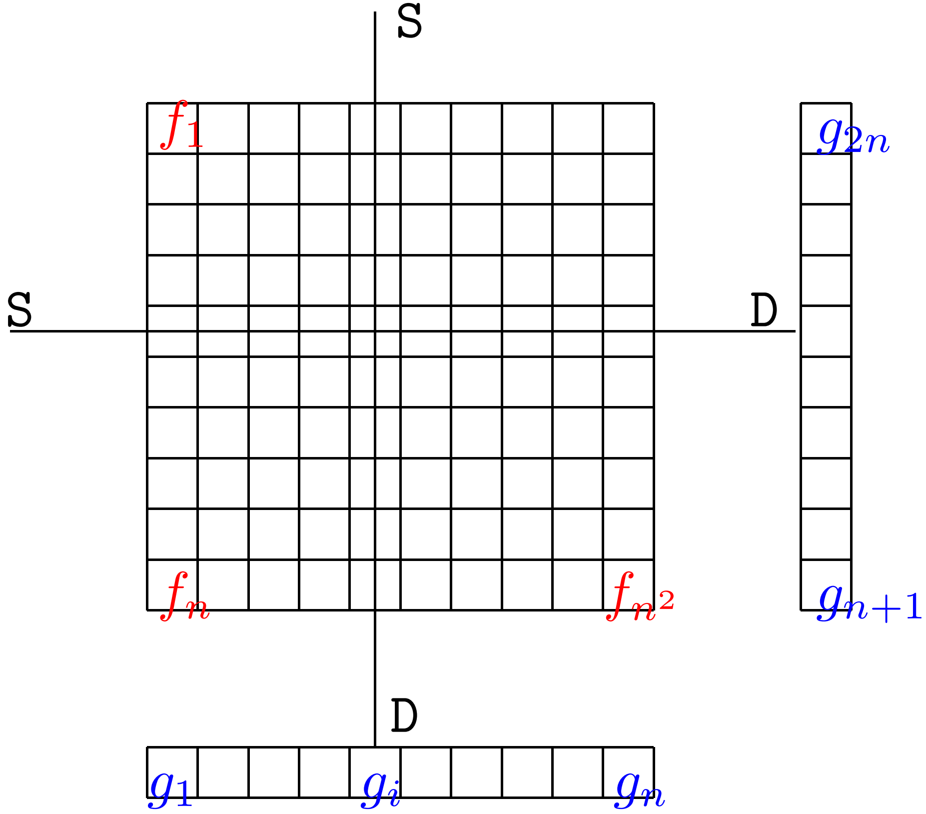


64 projections 16 projections 4 projections 2 projections



64 projections 16 projections 4 projections 2 projections

Image Reconstruction from two projections



$\boldsymbol{g}_1 = \{g_1, \cdots, g_n\}$ horizontal

$\boldsymbol{g}_2 = \{g_{n+1}, \cdots, g_{2n}\}$ vertical

$$\sum_{j=1}^{N=n^2} H_{ij}\, f_j = g_i, \quad i = 1, \cdots M = 2n$$

$H_{ij} = \{0, 1\}$

$\boldsymbol{f} = \{f_1, \cdots, f_N\}$

$\boldsymbol{g} = \{g_1, \cdots, g_M\} = [\boldsymbol{g}_1; \boldsymbol{g}_2]$

$\boldsymbol{g}_1 = \boldsymbol{H}_1 \boldsymbol{f}, \quad \boldsymbol{g}_2 = \boldsymbol{H}_2 \boldsymbol{f}$

$\boldsymbol{g} = \boldsymbol{H} \boldsymbol{f}$

Continuous Signals and Images

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

$f(\boldsymbol{r})$ Continuous Image : Gauss-Markov

$$p(\boldsymbol{f}) = N\left(\boldsymbol{0}, \boldsymbol{\Sigma}_f\right)$$

$$p(f_j | f_i, i \neq j) = \mathcal{N}(\beta f_{j-1}, \sigma_f^2)$$

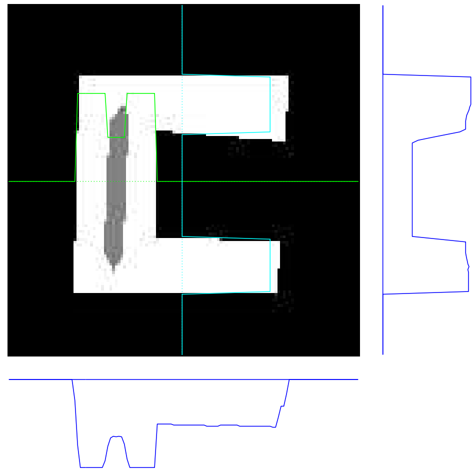
$$p(f(\boldsymbol{r}) | f(\boldsymbol{s})) = \mathcal{N}\left(\beta \sum_{\boldsymbol{s}\in\mathcal{V}(\boldsymbol{r})} f(\boldsymbol{s}), \sigma_f^2\right)$$

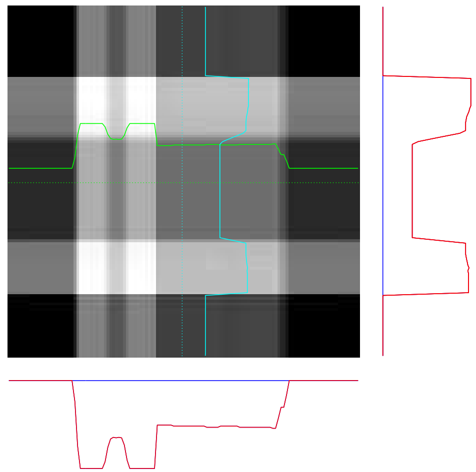
MAP :

$$\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \left\{p(\boldsymbol{f}|\boldsymbol{g})\right\} = \arg\min_{\boldsymbol{f}} \left\{J(\boldsymbol{f})\right\}$$

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \sum_j \left(f_j - \beta f_{j-1}\right)^2$$

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \sum_{\boldsymbol{r}} \left(f(\boldsymbol{r}) - \beta \sum_{\boldsymbol{s}\in\mathcal{V}(\boldsymbol{r})} f(\boldsymbol{s})\right)^2$$





Piecewise continuous signals and images

$\boldsymbol{g} = \boldsymbol{H} \boldsymbol{f} + \boldsymbol{\epsilon}$,

$f(\boldsymbol{r})$ piecewise continuous image :

Composed MRF Intensity-Contours

Hidden contour variable: $q(\boldsymbol{r})$

$$p(f_j | q_j, f_i, i \neq j) = \mathcal{N}(\beta(1-q_j) f_{j-1}, \sigma_f^2)$$

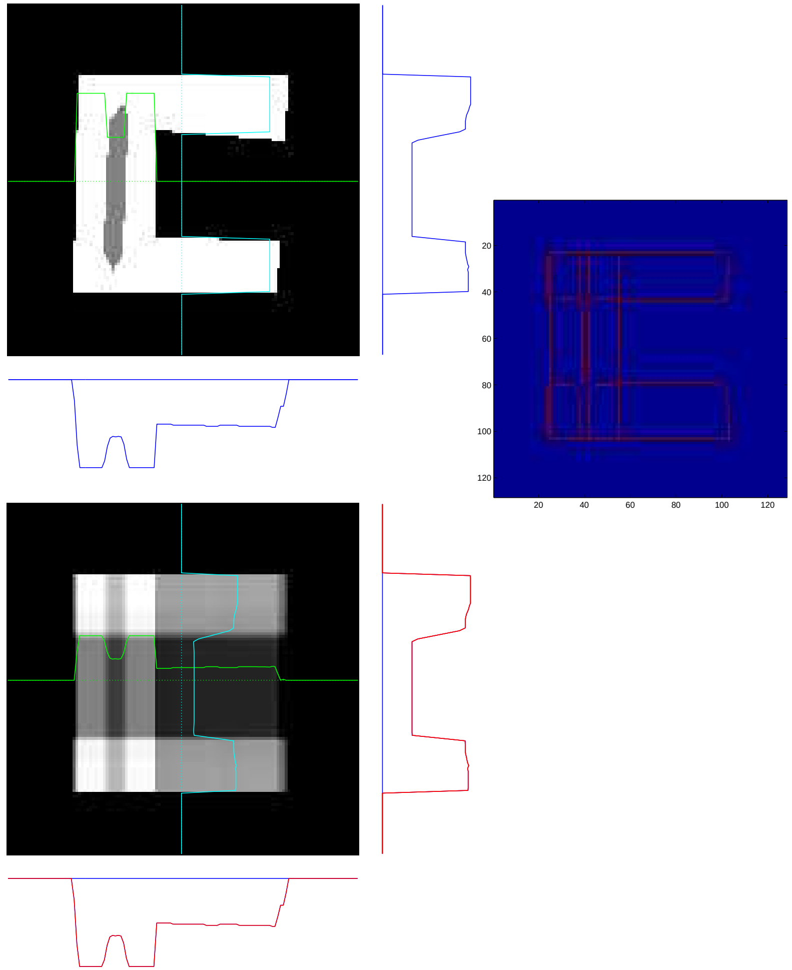
$$p(f(\boldsymbol{r}) | q(\boldsymbol{r}), f(\boldsymbol{s})) \\ = \mathcal{N}\left(\beta(1-q(\boldsymbol{r})) \sum_{\boldsymbol{s} \in \mathcal{V}(\boldsymbol{r})} f(\boldsymbol{s}), \sigma_f^2\right)$$

$$\text{MAP : } (\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{q}}) = \arg\max_{\boldsymbol{f},\boldsymbol{q}} \{p(\boldsymbol{f}, \boldsymbol{q} | \boldsymbol{g})\}$$

$$\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f} | \boldsymbol{g}, \boldsymbol{q})\} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$$

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 \\ + \sum_{\boldsymbol{r}} (1-q(\boldsymbol{r})) \left(f(\boldsymbol{r}) - \beta \sum_{\boldsymbol{s} \in \mathcal{V}(\boldsymbol{r})} f(\boldsymbol{s})\right)^2$$

$$\widehat{\boldsymbol{q}} = \arg\max_{\boldsymbol{q}} \{p(\boldsymbol{q} | \boldsymbol{g})\}$$



Objects composed of a finite number of homogeneous materials

$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$

$\boldsymbol{f}$ represents an image of an object $f(\boldsymbol{r})$

composed of a finite homogeneous matrials

Composed MRF intensity-regions

Introduction of a class label variable $z(\boldsymbol{r})$

$z(\boldsymbol{r}) = k, \quad k = 1, \cdots, K$

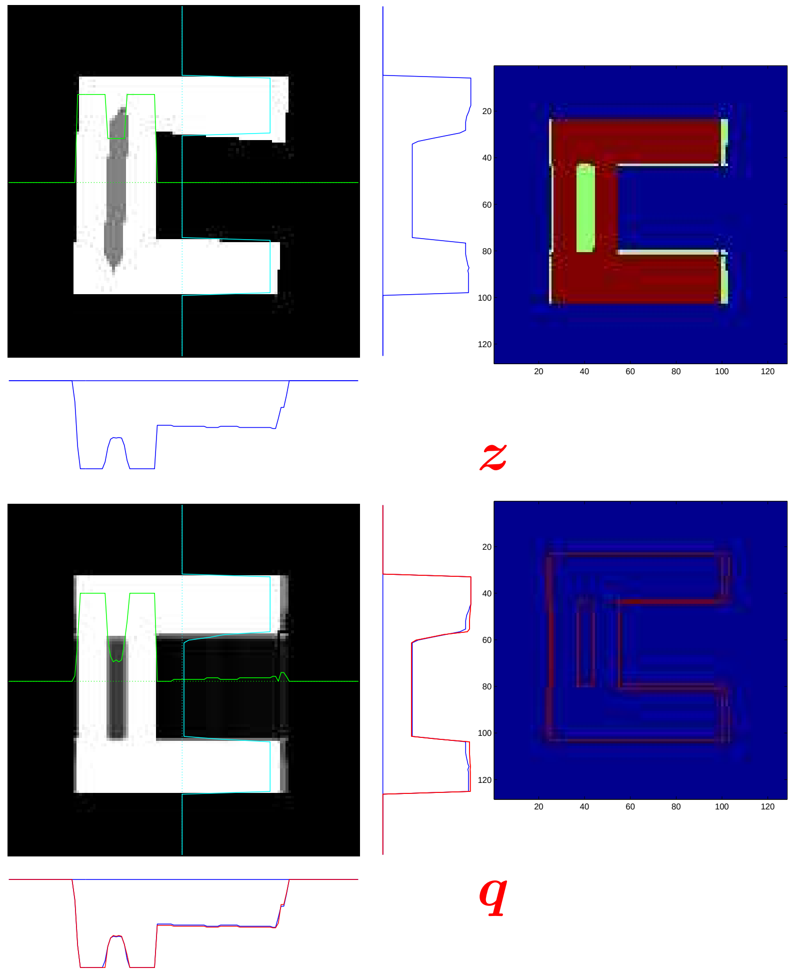
$\mathcal{R}_k = \{\boldsymbol{r} \;:\; z(\boldsymbol{r}) = k\}, \quad \mathcal{R} = \cup_k \mathcal{R}_k$

$p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(f(\boldsymbol{r})|m_k, \sigma_k^2)$

$\boldsymbol{z} = \{z(\boldsymbol{r}), \boldsymbol{r} \in \mathcal{R}\}$ a segmented image

Potts MRF:

$p(\boldsymbol{z}) \propto \exp\left[\alpha \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{s}\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{s}))\right]$



- Mixture of Gaussians model with $\{z(\boldsymbol{r}), \boldsymbol{r} \in \mathcal{R}\}$ i.i.d.

$$p(\boldsymbol{z}) = \prod_{k=1}^{K} p_k \qquad \text{with} \qquad P(z(\boldsymbol{r}) = k) = p_k \quad \text{et} \quad \sum_{k=1}^{K} p_k = 1$$

- Mixture of Gaussians with a Potts MRF for $\boldsymbol{z}$

$$p(\boldsymbol{z}) \propto \exp\left[\alpha \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{s}\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{s}))\right]$$

- Hyperparameters $\boldsymbol{\theta} = \{{\sigma_\epsilon}^2, (m_k, \sigma_k^2), k = 1, \cdots, K\}$:

$$p(m_k) = \mathcal{N}(m_k | {m_k}_0, {\sigma_k}_0^2), \qquad p(\sigma_k^2) = \mathcal{IG}(\sigma_k^2 | \alpha_{k0}, \beta_{k0}),$$
$$p(\boldsymbol{\Sigma}_k) = \mathcal{IW}(\boldsymbol{\Sigma}_k | \alpha_{k0}, \boldsymbol{\Lambda}_{k0}), \quad p({\sigma_{\epsilon_i}}^2) = \mathcal{IG}({\sigma_{\epsilon_i}}^2 | \alpha_0^{\epsilon_i}, \beta_0^{\epsilon_i}).$$

- Joint *a posteriori* law of $\boldsymbol{f}$, $\boldsymbol{z}$ and $\boldsymbol{\theta}$

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta} | \boldsymbol{g}) \propto p(\boldsymbol{g} | \boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f} | \boldsymbol{z}, \boldsymbol{\theta}_2)\, p(\boldsymbol{z} | \boldsymbol{\theta}_2)\, p(\boldsymbol{\theta})$$

First step: $\boldsymbol{\theta}$ and $\boldsymbol{z}$ known:

MAP:

$$\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \left\{ p(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{z},\boldsymbol{\theta}) \right\} = \arg\min_{\boldsymbol{f}} \left\{ \boldsymbol{J}(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{z},\boldsymbol{\theta}) \right\}.$$

- With an i.i.d. model :

$$\begin{aligned} J(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{z},\boldsymbol{\theta}) &= ||\boldsymbol{g}-\boldsymbol{H}\boldsymbol{f}||^2 + \lambda \sum_{k=1}^{K} \frac{||\boldsymbol{f}_k - m_k \mathbf{1}||^2}{\sigma_k^2} \\ &= ||\boldsymbol{g}-\boldsymbol{H}\boldsymbol{f}||^2 + \lambda \sum_{k=1}^{K} \sum_{\boldsymbol{r}\in\mathcal{R}_k} \frac{||f(\boldsymbol{r})-m_k||^2}{\sigma_k^2} \end{aligned}$$

- With a Markovien model :

$$\begin{aligned} J(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{z},\boldsymbol{\theta}) &= ||\boldsymbol{g}-\boldsymbol{H}\boldsymbol{f}||^2 + \lambda \sum_{k=1}^{K} \frac{||\tilde{\boldsymbol{f}}_k - m_k \mathbf{1}||^2}{\sigma_k^2} \\ &= ||\boldsymbol{g}-\boldsymbol{H}\boldsymbol{f}||^2 + \lambda \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \left(\tilde{f}(\boldsymbol{r}) - \beta_{\boldsymbol{r}} \sum_{\boldsymbol{s}\in(\mathcal{V}(\boldsymbol{r})\cap\mathcal{R}_k)} \tilde{f}(\boldsymbol{s}) \right)^2 \end{aligned}$$

where $\tilde{f}(\boldsymbol{r}) = f(\boldsymbol{r}) - m(\boldsymbol{r}), \quad \beta_{\boldsymbol{r}} = \frac{1}{n_{\boldsymbol{r}}}, \quad n_{\boldsymbol{r}} = Card(\mathcal{V}(\boldsymbol{r}) \cap \mathcal{R}_k).$

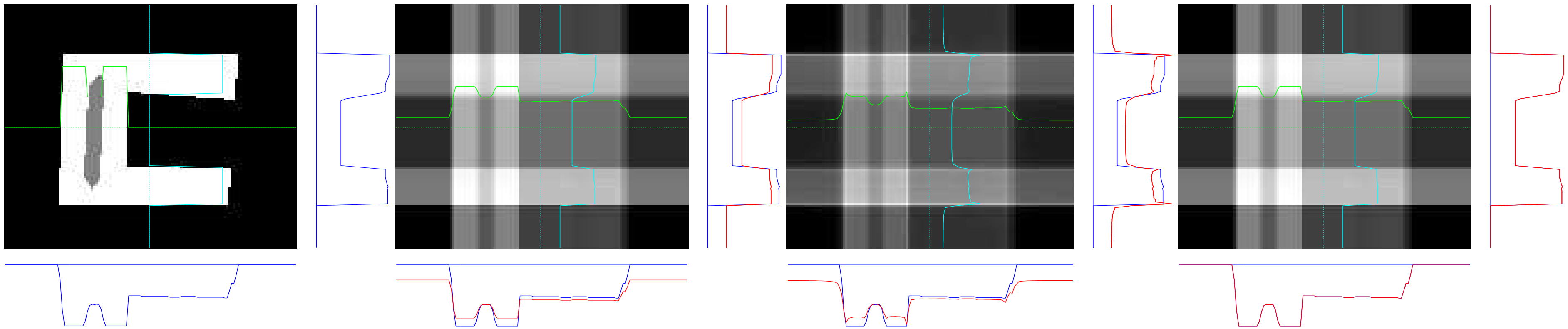
Estimation of $(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta})$ using $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g})$

- MAP (Algorithm 1):

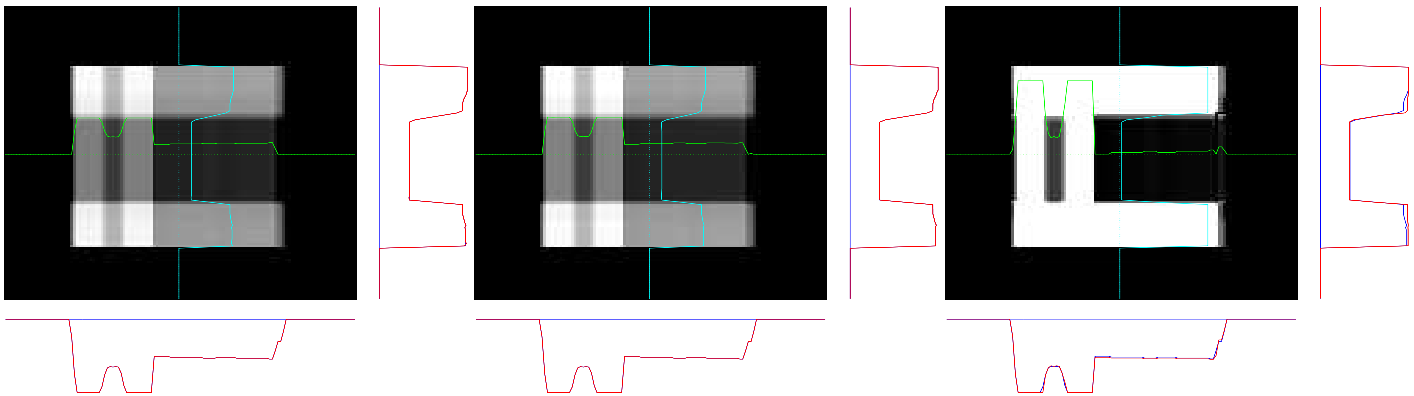
$$\left\{\begin{array}{lll} \widehat{\boldsymbol{f}} & = \arg\max_{\boldsymbol{f}} \left\{p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta},\boldsymbol{g})\right\} & \\ \widehat{\boldsymbol{\theta}} & = \arg\max_{\boldsymbol{\theta}} \left\{p(\boldsymbol{\theta}|\boldsymbol{f},\boldsymbol{z},\boldsymbol{g})\right\} & \text{or} \quad = \arg\max_{\boldsymbol{\theta}} \left\{p(\boldsymbol{\theta}|\boldsymbol{z},\boldsymbol{g})\right\} \\ \widehat{\boldsymbol{z}} & = \arg\max_{\boldsymbol{z}} \left\{p(\boldsymbol{z}|\boldsymbol{f},\boldsymbol{\theta},\boldsymbol{g})\right\} & \text{or} \quad = \arg\max_{\boldsymbol{z}} \left\{p(\boldsymbol{z}|\boldsymbol{\theta},\boldsymbol{g})\right\} \end{array}\right.$$

- MAP-Gibbs (Algorithm 2):

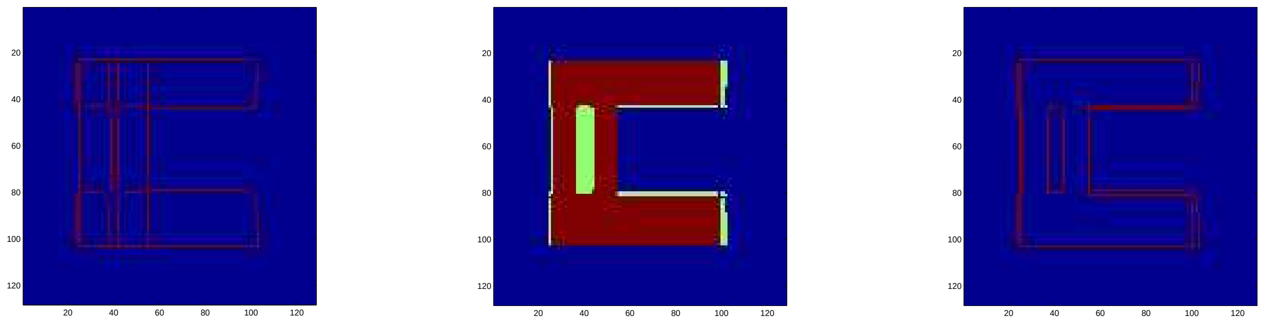
$$\left\{\begin{array}{rlll} & \widehat{\boldsymbol{f}} & = \arg\max_{\boldsymbol{f}} \left\{p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta},\boldsymbol{g})\right\} & \\ \text{sample} & \widehat{\boldsymbol{\theta}} & \text{with} \quad p(\boldsymbol{\theta}|\boldsymbol{f},\boldsymbol{z},\boldsymbol{g}) & \text{or with} \quad p(\boldsymbol{\theta}|\boldsymbol{z},\boldsymbol{g}) \\ \text{sample} & \widehat{\boldsymbol{z}} & \text{with} \quad p(\boldsymbol{z}|\boldsymbol{f},\boldsymbol{\theta},\boldsymbol{g}) & \text{or with} \quad p(\boldsymbol{z}|\boldsymbol{\theta},\boldsymbol{g}) \end{array}\right.$$



Original Backprojection Filtered BP LS



$\boldsymbol{f}$ $\boldsymbol{f}+\boldsymbol{q}$ $\boldsymbol{f}+\boldsymbol{z}$ $+\boldsymbol{q}$



Multi sensor, Fusion, Separation

$$\boldsymbol{g}_i = \sum_{j=1}^{N} A_{ij}\, \boldsymbol{H}_{ij} \boldsymbol{f}_j + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M,$$

- No mixture, No convolution applications:
  - Multi channel image fusion and joint segmentation

    $$\boldsymbol{g}_i = \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M, \qquad \boldsymbol{f}_i | \boldsymbol{z} \;\; \text{independent}$$

  - Hyperspectral image segmentation

    $$\boldsymbol{g}_i = \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M, \qquad \boldsymbol{f}_i | \boldsymbol{z} \;\; \text{dependent}$$

  - Video movie segmentation with motion estimation

    $$\boldsymbol{g}_i = \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M, \qquad \boldsymbol{f}_i | \boldsymbol{z}_i \;\; \text{independent}$$

- Mixture, No convolution applications:
  - Blind source (image) separation (BSS) and joint segmentation

    $$\boldsymbol{g}_i = \sum_{j=1}^{N} A_{ij}\, \boldsymbol{f}_j + \boldsymbol{\epsilon}_i, \quad i = 1, \cdots, M, \quad \boldsymbol{f}_i | \boldsymbol{z}_i \;\; \text{independent}$$

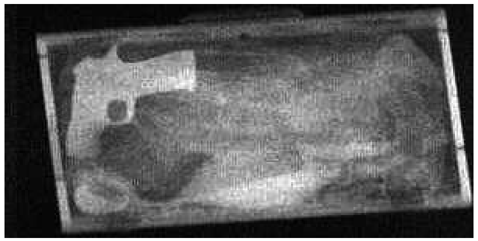
- Convolution but No mixture applications:
  - Fourier synthesis in optical imaging $\quad \boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) + \boldsymbol{\epsilon}, \quad \boldsymbol{f} | \boldsymbol{z}$
  - Single channel image restoration $\quad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}, \quad \boldsymbol{f} | \boldsymbol{z}$

Images fusion and joint segmentation

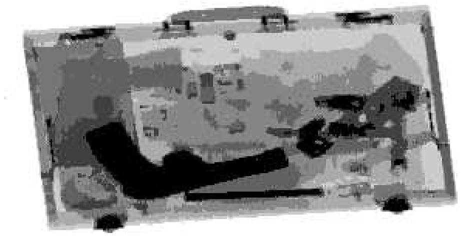
(Olivier FÉRON)

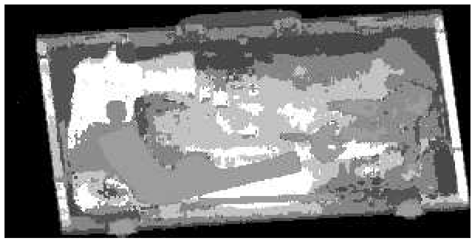
$$\begin{cases} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_{ik}, \sigma_{ik}) \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}) \end{cases}$$

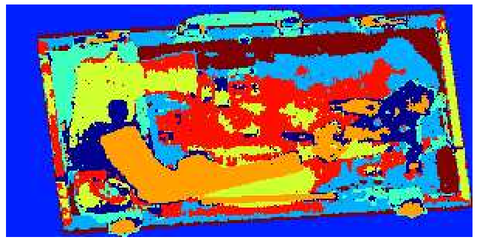
$\boldsymbol{g}_1$ 

$\boldsymbol{g}_2$ 

$\longrightarrow$

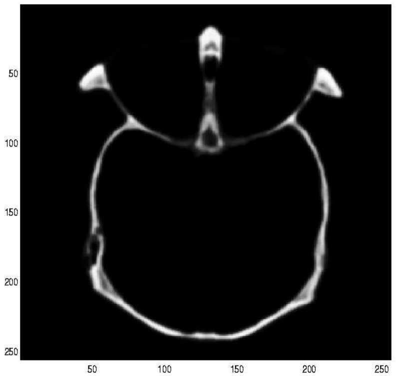
$\widehat{\boldsymbol{f}}_1$ 

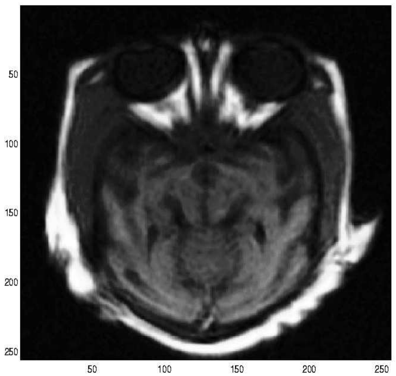
$\widehat{\boldsymbol{f}}_2$ 

$\widehat{\boldsymbol{z}}$ 

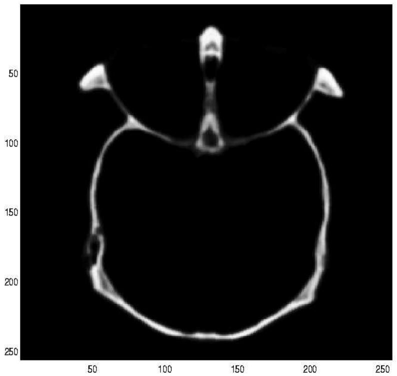
Data fusion in medical imaging (Olivier FÉRON)

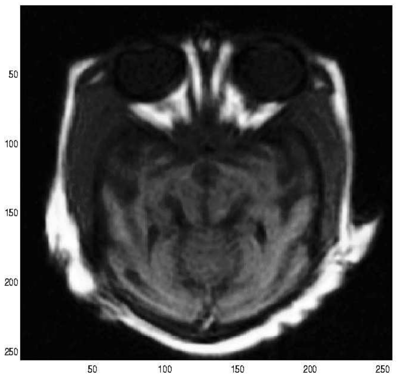
$$\left\{\begin{array}{l} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z(\boldsymbol{r})=k) = \mathcal{N}(m_{i\,k},\sigma_{i\,k}) \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}) \end{array}\right.$$

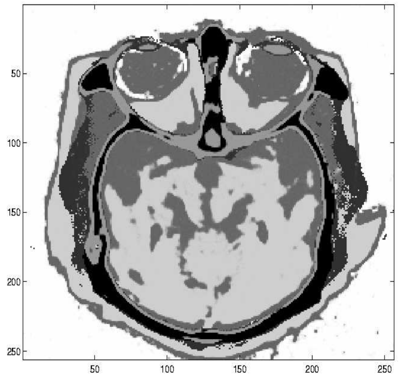
$\boldsymbol{g}_1$ 

$\boldsymbol{g}_2$ 

$\longrightarrow$

$\widehat{\boldsymbol{f}}_1$ 

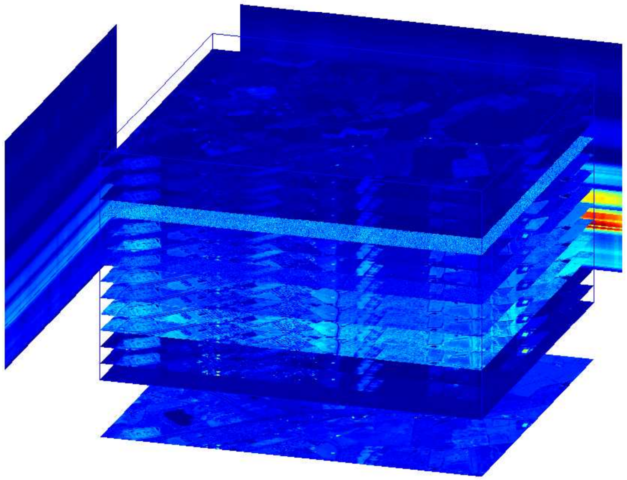
$\widehat{\boldsymbol{f}}_2$ 

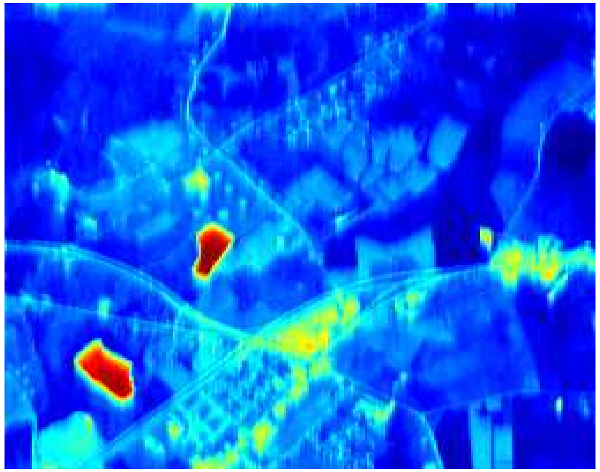
$\widehat{\boldsymbol{z}}$ 

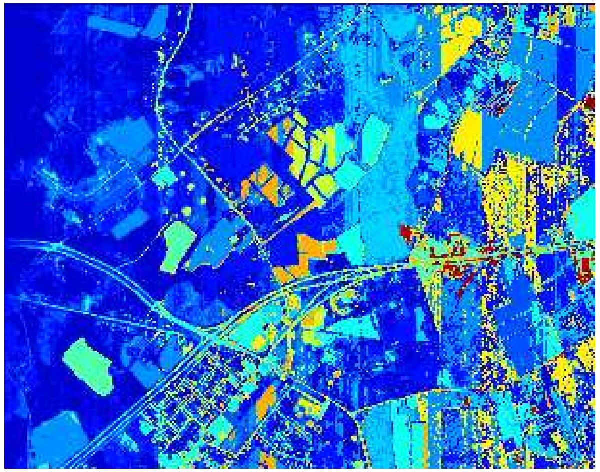
Joint segmentation of hyper-spectral images

(Adel MOHAMMADPOUR)

$$\begin{cases} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z(\boldsymbol{r})=k) = \mathcal{N}(m_{ik}, \sigma_{ik}), \quad k=1,\cdots,K \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}) \\ m_{ik} \text{ follow a Markovian model along the index } i \end{cases}$$



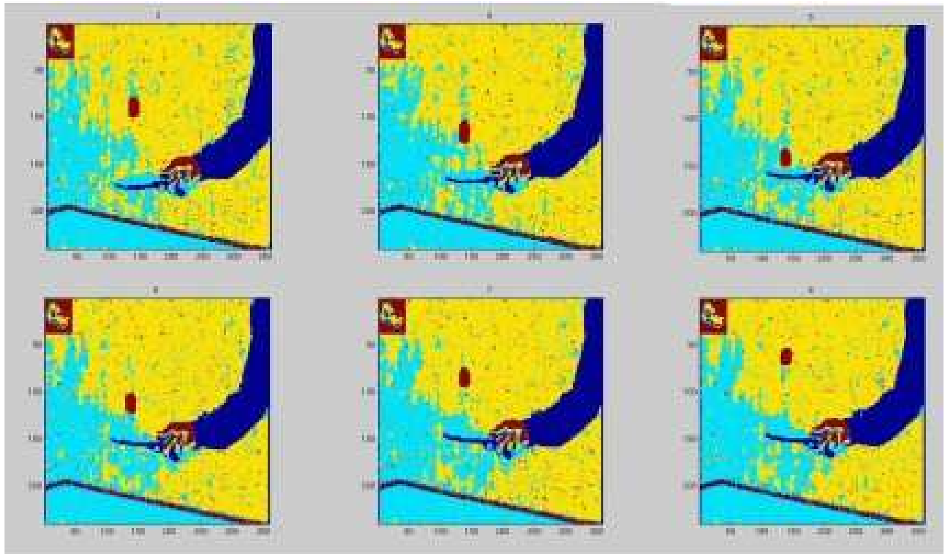


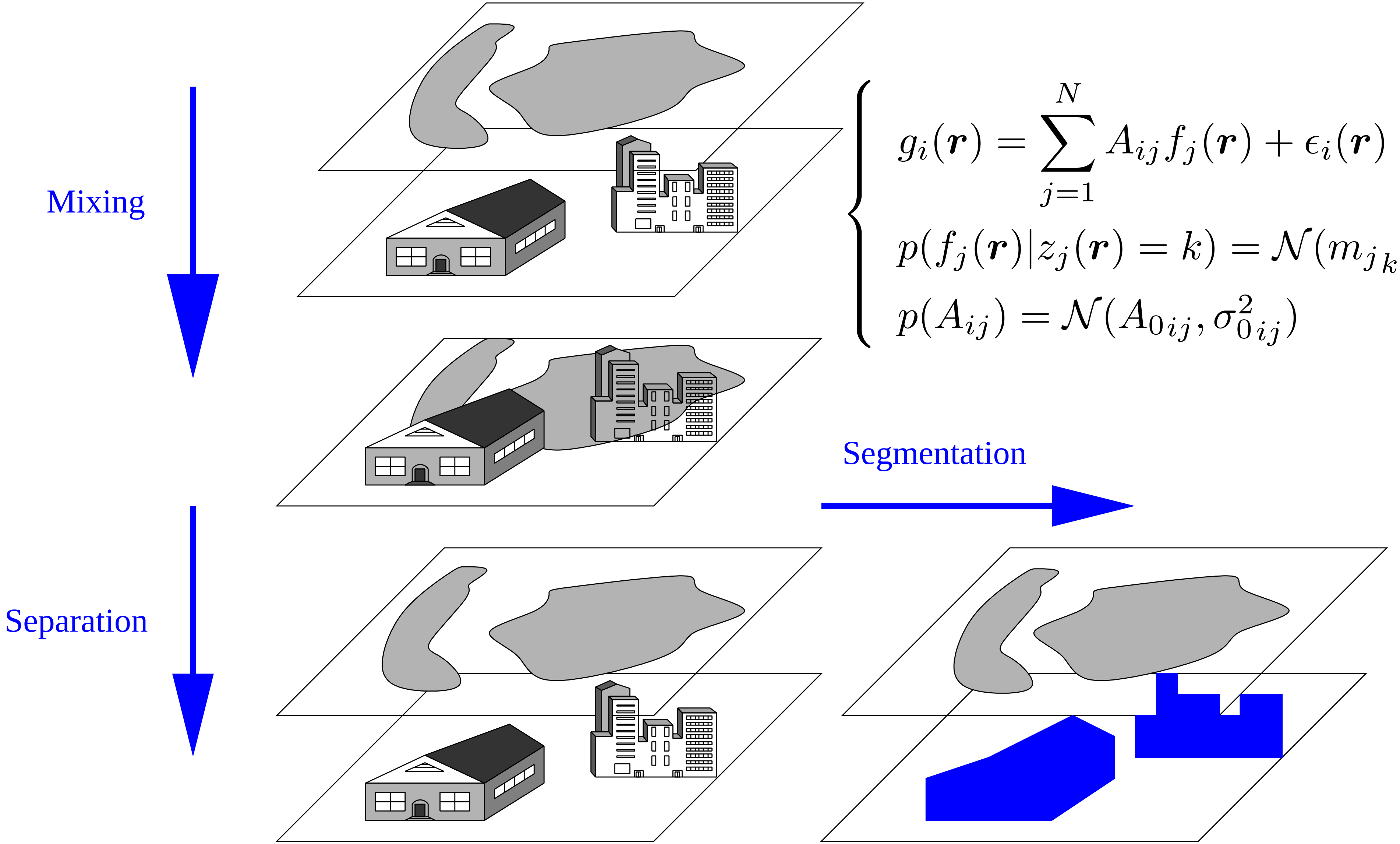


Segmentation of a video sequence of images

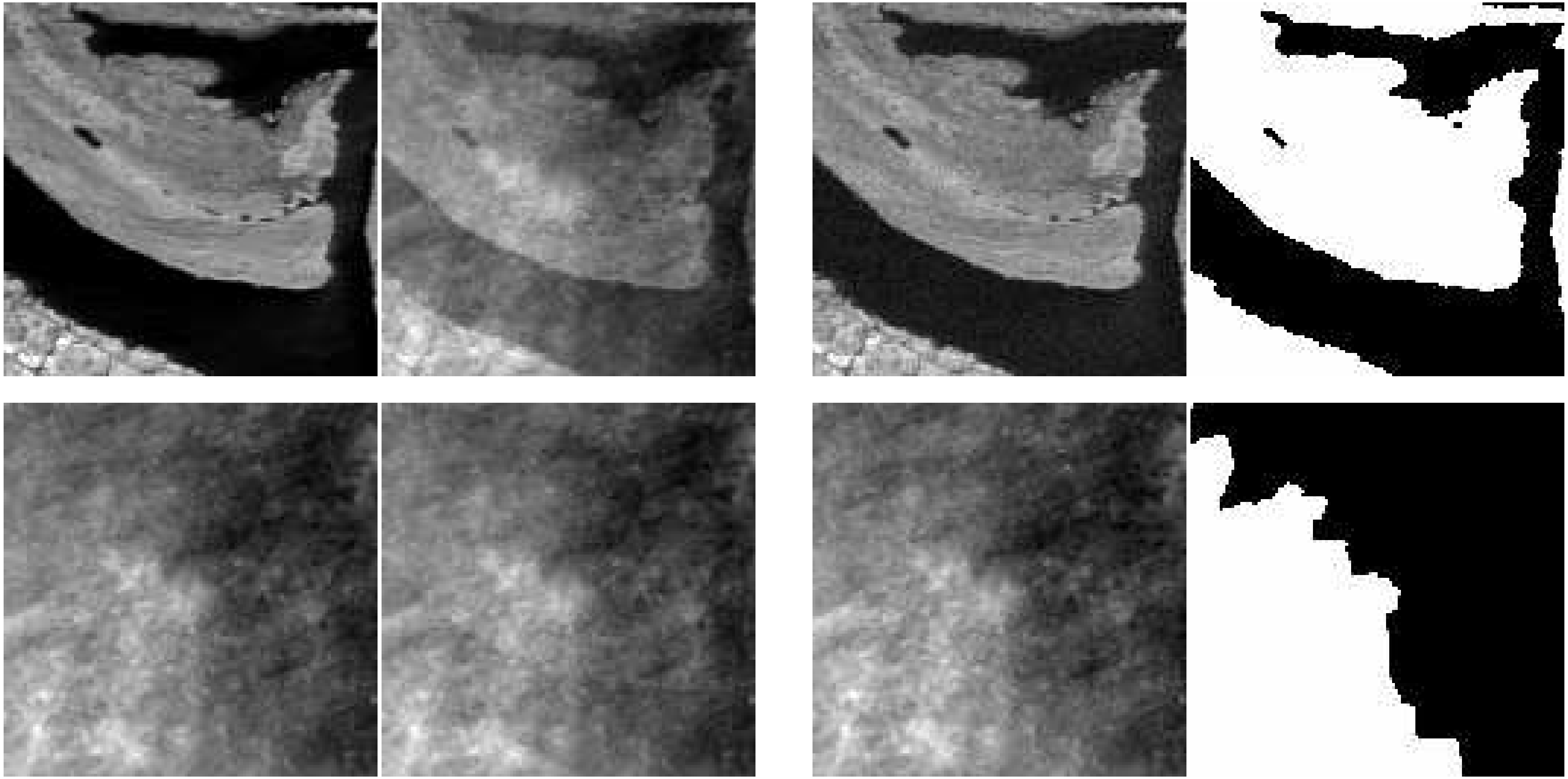
(Patrice BRAULT)

$$\begin{cases} g_i(\boldsymbol{r}) = f_i(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_i(\boldsymbol{r})|z_i(\boldsymbol{r}) = k) = \mathcal{N}(m_{ik}, \sigma_{ik}), \quad k = 1, \cdots, K \\ p(\underline{\boldsymbol{f}}|\boldsymbol{z}) = \prod_i p(\boldsymbol{f}_i|\boldsymbol{z}_i) \\ z_i(\boldsymbol{r}) \text{ follow a Markovian model along the index } i \end{cases}$$





$$\begin{cases} g_i(\boldsymbol{r}) = \sum_{j=1}^{N} A_{ij} f_j(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}) \\ p(f_j(\boldsymbol{r})|z_j(\boldsymbol{r}) = k) = \mathcal{N}(m_{j\,k}, \sigma^2_{j\,k}) \\ p(A_{ij}) = \mathcal{N}(A_{0\,ij}, \sigma^2_{0\,ij}) \end{cases}$$



$\underline{\boldsymbol{f}}$ $\underline{\boldsymbol{g}}$ $\underline{\widehat{\boldsymbol{f}}}$ $\underline{\widehat{\boldsymbol{z}}}$

Single channel image restoration

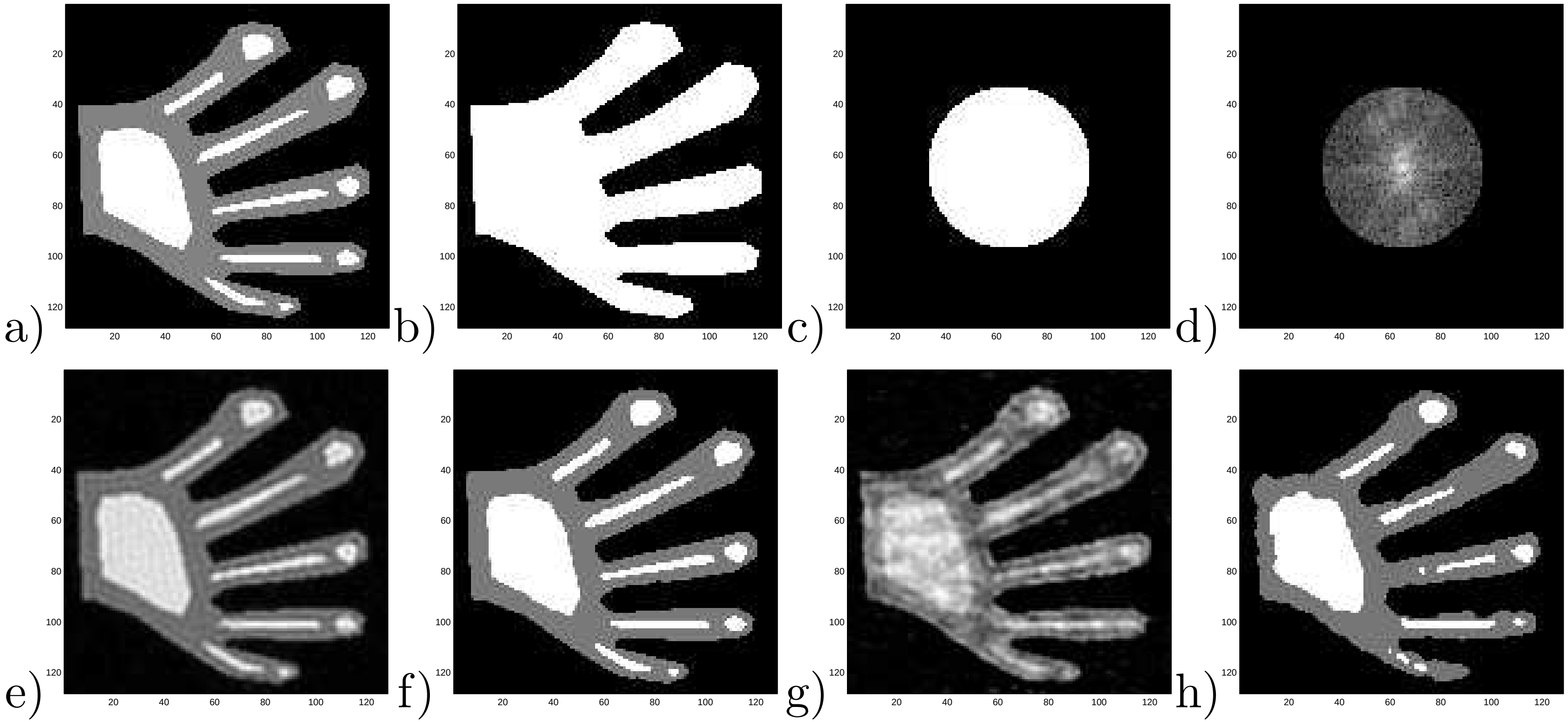
$$g(\boldsymbol{r}') = \int h(\boldsymbol{r}' - \boldsymbol{r})\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{r}') \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \tag{1}$$

Fourier synthesis inverse problem

$$g(\boldsymbol{\omega}) = \int \exp\left[-j(\boldsymbol{\omega}.\boldsymbol{r})\right]\, f(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{\omega}) \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon} \tag{2}$$

$$\left\{\begin{array}{l}
p(\boldsymbol{\epsilon}) = \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}) \longrightarrow p(\boldsymbol{g}|\boldsymbol{f}) = \mathcal{N}(\boldsymbol{H}\boldsymbol{f}, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}) \quad \text{with} \quad \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} = {\sigma_\epsilon}^2 \boldsymbol{I} \\
p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_k, \sigma_k^2), \quad k = 1, \cdots, K \\
p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta}, \boldsymbol{g}) = \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\Sigma}}) \quad \text{with} \\
\widehat{\boldsymbol{\Sigma}} = (\boldsymbol{H}^t {\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}}^{-1} \boldsymbol{H} + {\boldsymbol{\Sigma}_z}^{-1})^{-1} \quad \text{and} \quad \widehat{\boldsymbol{f}} = \widehat{\boldsymbol{\Sigma}} \left(\boldsymbol{H}^t {\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}}^{-1} \boldsymbol{g} + {\boldsymbol{\Sigma}_z}^{-1} \boldsymbol{m}_z\right) \\
\text{Compute} \quad \widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta}, \boldsymbol{g})\} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{with} \\
J(\boldsymbol{f}) = \frac{1}{\sigma_\epsilon^2} \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \sum_k \frac{\|\boldsymbol{f}_k - m_k \boldsymbol{1}\|^2}{\sigma_k^2} \\
p(\boldsymbol{z}|\boldsymbol{g}, \boldsymbol{\theta}) \propto p(\boldsymbol{g}|\boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{z}) \quad \text{with} \\
p(\boldsymbol{g}|\boldsymbol{z}, \boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{H}\boldsymbol{m}_z, \boldsymbol{\Sigma}_{\boldsymbol{g}}) \quad \text{with} \quad \boldsymbol{\Sigma}_{\boldsymbol{g}} = \boldsymbol{H}\boldsymbol{\Sigma}_z \boldsymbol{H}^t + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} \\
\text{Use} \quad p(\boldsymbol{z}|\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{\theta}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{z})
\end{array}\right. \tag{3}$$

Simulation results

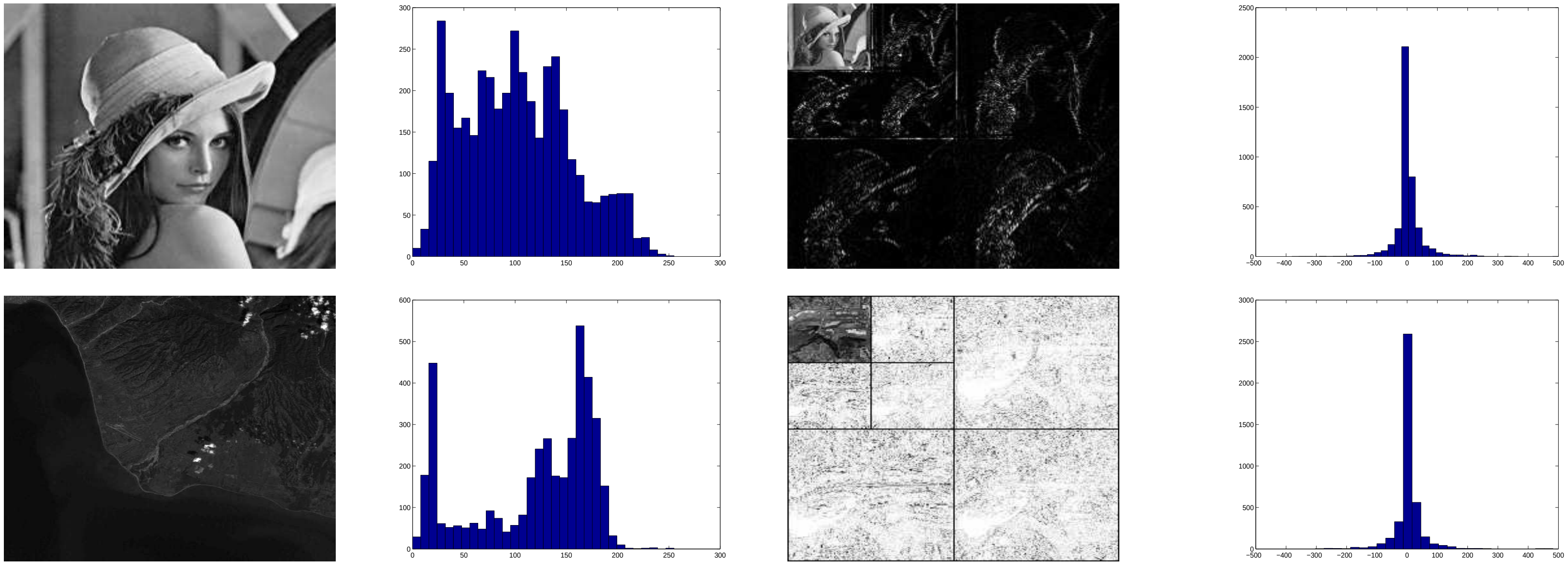


a) object, b) exact known support, c) support of the data, d) measured data, e) and f) Results when phase is measured: e) IFT and f) proposed method, g) and h) Results when the phase is not measured but we know the support of the object: g) by Gerchberg-Saxton h) by the proposed method.

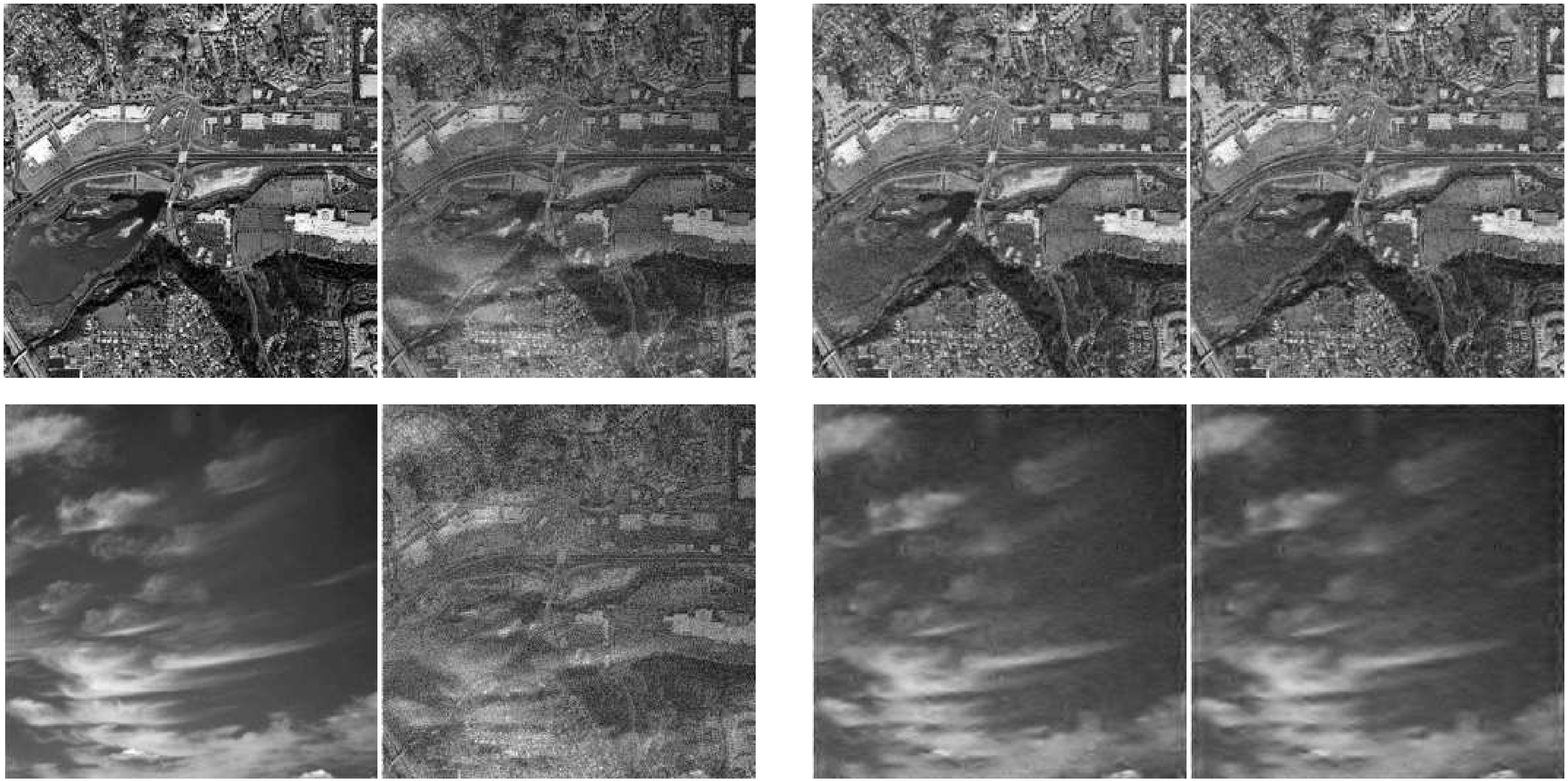
Wavelet domain Bayesian image processing

$$\boldsymbol{g}(\boldsymbol{r}) = \boldsymbol{f}(\boldsymbol{r}) + \boldsymbol{\epsilon}(\boldsymbol{r}) \longrightarrow \text{WT} \longrightarrow \widetilde{\boldsymbol{g}}^j(\boldsymbol{r}) = \widetilde{\boldsymbol{f}}^j(\boldsymbol{r}) + \widetilde{\boldsymbol{\epsilon}}^j(\boldsymbol{r})$$

images | Hist. of images | Wavelet coeff. | Hist. of wavelet coeff.



- multi-resolution computation
- Wavelet coefficients can be classified and segmented in only $K = 2$ classes



$\underline{\boldsymbol{f}}$ $\underline{\boldsymbol{g}}$ $\widehat{\underline{\boldsymbol{f}}}$ (GM) $\widehat{\underline{\boldsymbol{f}}}$ (HMM)

Conclusion and works in progress

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Bayesian approach & HMM are appropriate tools for many image processing problems

- H. Snoussi : BSS in 1D and 2D either in pixel domain or Fourier transform domain
- M. Ichir : BSS with mixture of Gamma and BSS in wavelet domain
- S. Moussaoui : BSS for non negative sources with application in spectrometry
- O. Féron : Data and image fusion and inverse problems in microwave imaging
- P. Brault : Segmentation of images sequences either directly or in wavelet domain
- A. Mohammadpour : Segmentation of hyper-spectral images,
- Z. Chama : Image recovery from the Fourier phase (Fourier Synthesis)
- F. Humblot : Super-resolution from a set of lower resolution images
- N. Bali : Source separation using different Hidden Markov models for images